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RESEARCH ARTICLE

# A New Semilattice of Function Algebras and Its Boolean Form on a Lattice of Groups

**Hassan Sedaghat**

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In this paper we define a new class of algebras of bounded, continuous, complex-valued functions on a semitopological semigroup $S$. Each C*-algebra of *left locally continuous functions* in this class generalizes the familiar C*-algebra $LUC(S)$ of all left uniformly continuous functions on $S$ (Section 1). One of the main goals of this paper is to establish that the inclusion relationships among these algebras of left locally continuous functions can be useful in the study and classification of the semigroup $S$. Thus we show in Section 2 that the class of all C*-algebras of left locally continuous functions on $S$ forms a semilattice with respect to set inclusion. The equivalences that give rise to this semilattice result, among other things, in partitions of $S$ and of $\mathcal{P}(S)$ into classes that are characterized by certain local continuity properties. In fact, whenever translation invariant, each C*-algebra of left locally continuous functions gives rise to a semigroup compactification enjoying a "local joint continuity property" (a local version of the analogous property of the $LUC$-compactification as in 1.5 and 1.6 below).

In order to present a concrete case, in Section 3 we discuss in detail the structure of the semilattice of C*-algebras of left locally continuous functions for the direct product of a family $\{S_\alpha : \alpha \in I\}$ consisting of groups and groups with zeros (we refer to such direct products as *full lattices of groups*). We show that in this case the aforementioned semilattice may be identified with the Boolean lattice of all subsets of $I$ (or $I$ less a singleton) via a lattice isomorphism that is defined with the aid of the idempotents in the direct product (3.10). Moreover, the index set $I$ and the idempotents can be used to obtain a rather simple characterization of each member of the lattice in terms of the C*-algebra of all bounded, continuous functions on an appropriate principal ideal in the direct product (3.6, 3.15). The validity of these conclusions requires no further restrictions on the full lattice of groups than the (sufficient) conditions needed to make Lawson's Joint Continuity Theorem applicable (e.g., local compactness or complete metrizablility; see 3.4 and 3.6 below).

## 0. Preliminary concepts

All the topologies in this paper will be assumed Hausdorff. Also, $S$ will denote a *semitopological semigroup* unless otherwise noted. This means that $S$ is an algebraic semigroup endowed with a topology relative to which multiplication (semigroup operation) is *separately* continuous. If the semigroup multiplication (i.e., the mapping $(s,t) \mapsto st : S \times S \mapsto S$) is continuous, then $S$ is called a *topological semigroup*. Many (though not all) of the semigroups discussed in this paper are actually topological. Currently, there are a number of good treatments of semitopological semigroups available and each adopts a somewhat different point of view. The presentation in [1], for example, contains all the preliminary material required for this paper and we adopt essentially the same terminology. In this section we highlight those concepts that are used here.

As mentioned before, $C(S)$ denotes the set of all bounded, continuous, complex-valued functions on $S$; it is a translation invariant C*-algebra (with complex conjugation as involution) containing all the constant functions on $S$ (an algebra $F$ of functions on $S$ is *translation invariant* if $R_sF \cup L_sF \subset F$ for all $s \in S$, where $R_sf(t) = f(ts)$ and $L_sf(t) = f(st)$, $t \in S$, $f \in F$). Two well-known C*-subalgebras of $C(S)$ will be of special interest to us here: The algebra $LUC(S)$ of all *left uniformly continuous functions* and the algebra $LMC(S)$ of all *left multiplicatively continuous functions* [1]. Of interest is also the C*-algebra $WAP(S)$ of all *weakly almost periodic functions* on $S$. For every semitopological semigroup $S$, all three algebras are translation invariant and we have $WAP(S) \cup LUC(S) \subset LMC(S)$ [1].

And now for a summary of some of the main ideas concerning semigroup compactifications. A *compactification* of a semitopological semigroup $S$ is a pair $(\phi, X)$, where $X$ is a compact, Hausdorff, right topological semigroup ($x \mapsto xy : X \mapsto X$ is continuous for each $y \in X$) and $\phi : S \mapsto X$ is a continuous homomorphism such that $\overline{\phi(S)} = X$ and the mappings $x \mapsto \phi(s)x : X \mapsto X$, $s \in S$, are continuous. A continuous function $\pi$ from a compactification $(\phi, X)$ of $S$ to a compactification $(\psi, Y)$ of $S$ is said to be a *homomorphism* if $\pi \circ \phi = \psi$. Note that such a mapping preserves multiplication and is surjective. A compactification $(\phi, X)$ of $S$ which possesses a certain property $P$ (such as that of being a topological group) is a *universal* compactification with respect to $P$ if for every compactification $(\psi, Y)$ of $S$ which has $P$, there exists a homomorphism from $(\phi, X)$ onto $(\psi, Y)$.

Let $(\phi, X)$ be a compactification of $S$ and let $\phi^* : C(X) \mapsto C(S)$ denote the *dual mapping* $f \mapsto f \circ \phi$. Then the C*-subalgebra $F(S) := \phi^*(C(X))$ is translation invariant and *left m-introverted*; i.e., $T_\mu F(S) \subset F(S)$ for all $\mu$ in the spectrum of $F(S)$, where $T_\mu$ is defined by $T_\mu f(s) = \mu(L_sf)$, $f \in F(S)$, $s \in S$. Conversely, let $F(S)$ be a translation invariant and left m-introverted C*-subalgebra of $C(S)$ containing the constant functions (such an algebra is called *m-admissible*). Let $X$ denote the spectrum of $F(S)$ with the weak* topology, and let $\phi : S \mapsto X$ be the *evaluation mapping* defined by $\phi(s)(f) = f(s)$, $f \in F(S)$, $s \in S$. Then $(\phi, X)$ is a compactification of $S$ such that $F(S) = \phi^*(C(X))$, where multiplication on $X$ is defined by $xy = x \circ T_y$. $(\phi, X)$ is called the *canonical $F(S)$-compactification* of $S$.

The algebras $WAP(S)$, $LUC(S)$, $LMC(S)$ are m-admissible [1]. The $WAP(S)$-compactification is the universal semitopological semigroup compactification of $S$. The $LUC(S)$-compactification $(\phi, X)$ is universal with respect to the property that the mapping $(s, x) \mapsto \phi(s)x : S \times X \mapsto X$ is continuous. Finally, the $LMC(S)$-compactification is the universal (right topological) semigroup compactification of $S$ (i.e., $LMC(S)$ is the largest m-admissible subalgebra of $C(S)$).

## 1. Left locally continuous functions

In this section we define the C*-algebras of left locally continuous functions on semitopological semigroups and discuss the basic properties of each such algebra. We will also introduce the semigroup compactifications associated with these C*-algebras.

**Definition 1.1.** A function $f \in LMC(S)$ is said to be *left locally continuous at $a \in S$* if the mapping

$$s \mapsto L_sf : S \mapsto C(S) \tag{1}$$

is continuous at the point $a$ relative to the uniform topology on $C(S)$. Thus $f$ being left locally continuous is equivalent to the norm quantity

$$\|L_{s_\eta}f - L_a f\| = \sup_{x\in S}|f(s_\eta x) - f(ax)|$$

approaching zero for every net $\{s_\eta\}$ in $S$ that converges to $a$. The set of all functions of this type is denoted by $LLC(S,a)$. Further, if $A$ is a non-empty subset of $S$, we define

$$LLC(S,A) = \bigcap\{LLC(S,a):\ a \in A\}.$$

It is clear that $LUC(S) = LLC(S,S) \subset LLC(S,a)$ for every $a \in S$, and if $A \subset B$, then $LLC(S,B) \subset LLC(S,A)$. It is thus reasonable (and helpful) to define

$$LLC(S,\emptyset) = LMC(S)$$

where $\emptyset$ represents the empty set. Notice that for $f$ to be in $LLC(S,A)$, it is not sufficient that the *restriction* of the mapping in (1) to $A$ be norm continuous (an example of the restriction case appears in [6] in connection with the topology of uniform convergence on compact subsets of $S$).

The set of all *right locally continuous functions at* $a$, which we denote $RLC(S,a)$, is defined similarly via the right translation operator and the algebra $RMC(S)$ ($RMC(S)$ and $RUC(S)$ are the "right analogs" of $LMC(S)$ and $LUC(S)$, respectively; they are defined in, e.g., [1]) The extensions to subsets of $S$ are likewise defined and, of course, $RLC(S,S) = RUC(S)$, $RLC(S,\emptyset) = RMC(S)$. The following lemma lists some of the elementary properties of the algebras of left locally continuous functions that are used in this paper (the right locally continuous analogs are similar). The routine proof is omitted.

**Lemma 1.2.**

*(i)* $LLC(S,\cup\mathcal{F}) = \bigcap\{LLC(S,A):\ A \in \mathcal{F})\}$ *for every non-empty family* $\mathcal{F}$ *of subsets of* $S$.

*(ii)* $L_t(LLC(S,tA)) \subset LLC(S,A)$,
$L_t(LLC(S,A)) \subset LLC(S,t^{-1}A)$, *for every* $t \in S$, *where*

$$t^{-1}A = \{s \in S:\ ts \in A\}.$$

*(iii) For every subset* $A \subset S$, $LLC(S,A)$ *is a right translation invariant* $C^*$*-subalgebra of* $C(S)$. ∎

The next lemma is important with regard to the existence of semigroup compactifications.

**Lemma 1.3.** *For each* $A \subset S$, $LLC(S,A)$ *is m-admissible if and only if it is left translation invariant.*

**Proof.** We need only show that if $LLC(S,A)$ is left translation invariant, then it is left m-introverted. For each $\mu$ in the spectrum of $LLC(S,A)$ there is $\mu'$ in the spectrum of $LMC(S)$ such that $\mu'|_{LLC(S,A)} = \mu$. Since $LMC(S)$ is left m-introverted and for each $s \in S$ and $f \in LLC(S,A)$

$$T_\mu f(s) = \mu(L_s f) = \mu'(L_s f) = T_{\mu'} f(s),$$

we conclude that $T_\mu f \in LMC(S)$. Also since for each $a \in A$ and $f \in LLC(S,A)$,

$$\|L_s T_\mu f - L_a T_\mu f\| = \sup_{x\in S}|\mu(L_{sx}f) - \mu(L_{ax}f)| \le \|L_s f - L_a f\|,$$

it follows that $T_\mu f$ is left locally continuous at each point of $A$. ∎

**Examples 1.4.** By 1.2 and 1.3, for every left ideal $L$ in $S$, $LLC(S, L)$ is m-admissible, and if $S$ is abelian, then $LLC(S, A)$ is m-admissible for every $A \subset S$. On the other hand, if $S = L \times \mathbf{Q}$, where $L = \{a_1, a_2\}$ is a discrete left zero semigroup ($xy = x$ for each $x, y \in L$) and $\mathbf{Q}$ is the group of additive rationals under the usual topology, then $LLC(S, (a_1, 0))$ is not left translation invariant (hence not m-admissible). To see this, note that $LUC(\mathbf{Q}) \neq LMC(\mathbf{Q})$ [5]. Let $f \in LMC(\mathbf{Q}) \backslash LUC(\mathbf{Q})$. Define the function $g \in C(S)$ as

$$g(a_1, y) = 0, \quad g(a_2, y) = f(y), \quad y \in \mathbf{Q}.$$

Using a double-limit criterion it can be shown that $g \in LMC(S)$, after which direct calculation shows that in fact $g \in LLC(S, (a_1, 0)) \backslash LLC(S, (a_2, 0))$ (details may be found in [8]). Now the requirement that $L_{(a,q)} g \in LLC(S, (a_1, 0))$ for every $(a, q) \in S$ is equivalent to the quantity

$$\|L_{(s,t)}(L_{(a,q)}g) - L_{(a_1,0)}(L_{(a,q)}g)\| = \|L_{(a,q+t)}g - L_{(a,q)}g\| \qquad (2)$$

approaching zero as $(s, t) \to (a_1, 0)$. Let $\{q_n\}$ be any sequence in $\mathbf{Q}$ that converges to zero, and let $a = a_2, q = 0$ in (2). Then $\|L_{(a_2,q_n)}g - L_{(a_2,0)}g\| \to 0$ as $n \to \infty$ if and only if $g \in LLC(S, (a_2, 0))$, and it was shown above that this is not the case.

**Definition 1.5.** A semigroup compactification $(\psi, X)$ of $S$ is said to have the *local joint continuity property* with respect to a non-empty subset $A \subset S$ (or $ljcA$) if the map

$$(s, x) \mapsto \psi(s)x \; : \; S \times X \mapsto X$$

is continuous at every point of $A \times X$.

The proof of the first part of the following theorem for the $LUC$ case is given in [1]. With the aid of 1.2 and 1.3, the generalization to the $LLC$ case is straight-forward, as is the proof of the second part of the theorem (or see [8]). The theorem shows that the $LLC$-compactifications are the local versions of the $LUC$-compactification in the obvious way.

**Theorem 1.6.**

*(i) Let $A$ be a non-empty subset of $S$ such that $LLC(S, A)$ is left translation invariant. Then the canonical $LLC(S, A)$-compactification of $S$ is universal with respect to the $ljcA$ property.*

*(ii) The $ljc$ property is invariant under compactification homomorphisms; i.e., if $A$ is a non-empty subset of $S$ such that $LLC(S, A)$ is left translation invariant, and if $(\psi, X)$ is a compactification having the $ljcA$ property, then every factor (homomorphic image) of it also has the $ljcA$ property.* ∎

The next corollary is a useful consequence of 1.6 and Lawson's Joint Continuity Theorem [4]. It gives information about the left local continuity of functions in $LMC(S)$ and will be used in Section 3 below.

**Corollary 1.7.** *Let $S$ be a locally compact or complete metrizable semitopological semigroup with identity 1 and group of units $H(1)$. If $A \subset H(1)$ and $LLC(S, A)$ is left translation invariant, then $LLC(S, A) = LMC(S)$.*

**Proof.** Let $(\phi, X)$ be the canonical $LMC(S)$-compactification of $S$, and define the action $\sigma : S \times X \mapsto X$ by $\sigma(s, x) = \phi(s)x$. By Lawson's Theorem $\sigma$ is continuous at every point of $H(1) \times X$. Let $A \subset H(1)$, and suppose that $LLC(S, A)$ is left translation invariant. Then by 1.3 and 1.6, $(\phi, X)$ is a factor of the canonical $LLC(S, A)$-compactification of $S$. It follows that $LLC(S, A) = LMC(S)$. ∎

We shall also have much use for the next lemma, whose simple proof we omit.

**Lemma 1.8.** *Let $S$ and $T$ be semitopological semigroups, and let $A \subset S$. Also let $\theta : S \mapsto T$ be a continuous homomorphism with $\theta A = \{\theta(a) : a \in A\}$. Then*

$$\theta^* LLC(T, \theta A) \subset LLC(S, A). \qquad \blacksquare$$

**Corollary 1.9.** *If $\theta : S \mapsto T$ is a topological isomorphism, then for every $A \subset S$, $LLC(S, A)$ is isometrically $*$-isomorphic to $LLC(T, \theta A)$ under the dual mapping $\theta^*$. In particular, if $s, t \in S$ and there is a topological automorphism $\theta$ of $S$ such that $\theta(s) = t$, then $LLC(S, s)$ is isometrically $*$-isomorphic to $LLC(S, t)$.* ■

An application of Corollary 1.9 appears in Example 3.13 below. The converse is easily seen to be false. If $G$ is a topological group, then $LLC(G, a) = LLC(G, 1)$ for every $a \in G$ where 1 is the identity of $G$, but for every endomorphism $\theta$ of $G$, $\theta(1) = 1$.

**Remark 1.10.** It should be clear from what has been presented so far that the left locally continuous functions and their semigroup compactifications (in the left translation invariant cases) are natural generalizations of the left uniformly continuous functions and the $LUC$-compactification. As in Theorem 1.6, many of the results that are established in the literature about the algebra $LUC(S)$ generalize (with negligible effort) to the left translation invariant $LLC$ algebras. We mention Theorem 2.10, Chapter 5 in [1] (concerning compactifications of semidirect products) and Theorems 3.2, 3.4, 3.6 and 3.8 in [2] as further examples of such results. These extensions and their consequences are discussed in [8], where one may also consider the roles of left and right locally continuous functions in Theorem 3.13 on the compactifications of projective (or inverse) limits. Items 1.1–1.8 in this section are taken from [8].

## 2. The $LLC$-semilattice

We now define the fundamental order relation and equivalences that are associated with left locally continuous functions and which give rise to the semilattice structure for the C*-algebras discussed in Section 1.

**2.1 Basic definitions and remarks.** In this sub-section we define three basic relations ($\leq$, $\rho$, $\rho_0$) and the semilattice structure on a semigroup $S$, all of which are due to left local continuity in $S$. For each pair of subsets $A$ and $B$ of $S$, define $A \leq B$ if $LLC(S, B) \subset LLC(S, A)$. Then $\leq$ is a preordering on the family $\mathcal{P}(S)$ of all subsets of $S$, and $\leq$ extends the relation $\subset$ (set inclusion). Clearly, $A \leq B$ *and* $B \leq A$ if and only if $LLC(S, A) = LLC(S, B)$. We define $A \rho B$ if this latter condition holds, and note that $\rho$ is an equivalence relation in $\mathcal{P}(S)$. Let $[A]$ represent the $\rho$-cell (or equivalence class) of the subset $A \subset S$. Lemma 1.2(i) implies that $\cup[A] \in [A]$, so we have a unique representation of each $\rho$-cell by its maximum element. Let $\mathcal{E}(S)$ denote the set of all such maximal representatives for $\rho$, and note that $\mathcal{E}(S)$ is non-empty since it always contains $S$.

Observe that the restriction of the canonical ordering $\leq$ to $\mathcal{E}(S)$ coincides with the set inclusion $\subset$. This follows from the fact that if $A \subset S$ and $B \in \mathcal{E}(S)$, then $A \leq B$ if and only if $A \subset B$ (use 1.2(i) to show that $A \cup B \subset B$). Therefore, the family

$$LLC(S) = \{LLC(S, A) : A \in \mathcal{E}(S)\}$$

of all *distinct* spaces of left locally continuous functions on $S$ is partially ordered by set inclusion $\subset$ and as such it is order anti-isomorphic to $\mathcal{E}(S)$. Furthermore, for $A, B \in \mathcal{E}(S)$ we have

$$LLC(S,A) \cap LLC(S,B) = LLC(S, A \cup B) = LLC(S, \cup[A \cup B])$$

and $\cup[A \cup B] \in \mathcal{E}(S)$. Thus $LLC(S)$ forms a lower (or meet) semilattice with respect to set intersection, which we call the $LLC$*-semilattice* of $S$. Note that the C*-algebras $LUC(S)$ and $LMC(S)$ are, respectively, the minimum and the maximum elements of $LLC(S)$.

In the corresponding semilattice $\mathcal{E}(S)$ of subsets of $S$, the semigroup $S$ itself represents the maximum element with respect to the ordering (set inclusion) on $\mathcal{E}(S)$. The minimum element (which corresponds to $LMC(S)$) is obviously the set $\cup[\emptyset]$, namely, the maximum element of the $\rho$-cell of $\emptyset$. $\cup[\emptyset]$ is the largest subset of $S$ where *every* member of $LMC(S)$ is left locally continuous at every point. From, e.g., [1] and [5], it is easy to see that if $S_1$ and $S_2$ are, respectively, the additive groups of real and rational numbers with the usual topology, then $\cup[\emptyset] = \mathbf{R}$ for the group $S_1$ while $\cup[\emptyset] = \emptyset$ for the group $S_2$ (also see 2.3 below).

It should also be noted here that as a consequence of 1.8 if $S$ and $T$ are topologically isomorphic semitopological semigroups under a mapping $\theta : S \mapsto T$, then $LLC(S)$ is semilattice-isomorphic to $LLC(T)$ under the mapping

$$LLC(S,A) \mapsto LLC(T, \cup[\theta A]), \qquad A \in \mathcal{E}(S),$$

and every C*-algebra $LLC(S,A)$ in $LLC(S)$ is isometrically $*$-isomorphic under the dual map $\theta^*$ to $LLC(T, \theta A) = LLC(T, \cup[\theta A])$ in $LLC(T)$.

Now we define a relation in $S$ that arises naturally when considering left local continuity. For each pair of elements $a, b \in S$, we define $a\rho_0 b$ if $LLC(S,a) = LLC(S,b)$. Then $\rho_0$ is an equivalence relation in $S$ and for each $s \in S$, the $\rho_0$-cell $[s]$ is precisely the union of all the singletons in the $\rho$-cell $[\{s\}]$. Hence $[s] \subset \cup[\{s\}]$, and this containment is usually strict (as in the usual multiplicative reals, where $[0]=\{0\}$ while $\cup[\{0\}] = \mathbf{R}$). Since the $\rho_0$-cells partition $S$, if $R$ is any complete set of representatives for $\rho_0$, then as one might expect, $LLC(S,R) = \bigcap\{LLC(S,[r]) : r \in R\} = LUC(S)$. However, unlike $\rho$, in general there is no explicit way of selecting a representative from each $\rho_0$-cell and constructing a complete set of representatives for $\rho_0$ (i.e., constructing the analog of $\mathcal{E}(S)$). This deficiency is to some extent overcome by a remarkable property of $\rho_0$: If $S$ has an identity, then the $\rho_0$-cells are always bounded below by the cells of a *purely algebraic* relation which we proceed to define next.

Let $S$ be a semigroup with an identity element 1. For $a, b \in S$, define $a\rho_1 b$ if $aH(1) = bH(1)$. The relation $\rho_1$ is easily seen to be an equivalence relation (in fact, a left congruence). If $H(1)$ is "normal" in $S$ (in the sense that $sH(1) = H(1)s$ for all $s \in S$) then $\rho_1$ is actually a congruence. Note that $\rho_1$ is the universal relation $S \times S$ if and only if $S$ is a group, and at the other extreme, $\rho_1$ is the identity relation "=" if and only if $H(1) = \{1\}$ (e.g., if $S$ is a band with identity). The following useful lemma establishes the link between $\rho_1$ and $\rho_0$.

**Lemma 2.2.** *Let $S$ be a semitopological semigroup with identity* 1. *Then*

*(i) For each $s \in S$, the $\rho_1$-cell of $s$ is the set $sH(1)$.*

*(ii) $sH(1) \subset [s]$ for every $s \in S$; i.e., if $t\rho_1 s$ then $t\rho_0 s$, $t \in S$.*

**Proof.** (i) $t \in sH(1)$ implies that $tH(1) \subset sH(1)$ and that $s \in tH(1)$. Hence, $sH(1) \subset tH(1)$ also; i.e., $tH(1) = sH(1)$ or $t\rho_1 s$. The converse is clear.

(ii) Let $s \in S$, $u \in H(1)$ and let $f \in LLC(S, su)$. Suppose that $\{s_\eta\}$ is a net that converges to $s$. Then the net $\{s_\eta u\}$ converges to $su$ and for each $\eta$

$$\|L_{s_\eta} f - L_s f\| = \|L_{s_\eta uu^{-1}} f - L_{suu^{-1}} f\| \leq \|L_{s_\eta u} f - L_{su} f\|.$$

It follows that $f \in LLC(S, s)$. Conversely, if $f \in LLC(S, s)$ and if $\{t_\gamma\}$ is a net converging to $su$, then the net $\{t_\gamma u^{-1}\}$ converges to $s$ and an argument similar to the one above shows that $f \in LLC(S, su)$. Hence for each $u \in H(1)$, $LLC(S, su) = LLC(S, s)$; i.e., $su \in [s]$. ∎

The next corollary is immediate from 2.2 and verifies an expected situation for groups. More substantial applications of 2.2 are given in the next section.

**Corollary 2.3.** *For a semitopological group $S$, $LLC(S) = \{LUC(S), LMC(S)\}$ and $\cup[\emptyset]$ is empty unless $LUC(S) = LMC(S)$ (e.g., if $S$ is locally compact), in which case $\cup[\emptyset] = S$.* ∎

Semigroups $S$ for which $LLC(S)$ is a singleton (i.e., when $LUC(S) = LMC(S)$) may be called *$LLC$-trivial.* Hence a locally compact or complete metrizable topological group is $LLC$-trivial, as is a compact topological semigroup, a discrete semigroup or a left zero semigroup. The classification "$LLC$-trivial" is clearly invariant under topological isomorphisms.

## 3. The $LLC$ structure of full lattices of groups

The main purpose of this section is to apply the results of the previous sections to (full) lattices of groups and obtain a complete description of their $LLC$-semilattice and of each of the C*-algebras in the semilattice.

For our purposes, a *group with a zero* is a semitopological semigroup $S$ containing a zero element 0 and an identity element 1 such that the group of units is the complement $S\backslash\{0\}$ which is dense in $S$. The results in [8] indicate that if $S$ is a group with a zero then $LLC(S, A)$ is left translation invariant (hence m-admissible) for every subset $A \subset S$, and if $S$ is locally compact or complete metrizable, then $LLC(S) = \{LUC(S), LMC(S)\}$ with $LUC(S) = LLC(S, 0)$ and $LMC(S) = LLC(S, S\backslash\{0\})$. We now consider the more elaborate situation for locally compact or complete metrizable (arbitrary) direct products of groups with zeros, which are not treated in [8].

**Definition 3.1.** Let $I$ be a non-empty set and let $\{S_\alpha : \alpha \in I\}$ be a family in which $S_\alpha$ is either a group or a group with a zero for each $\alpha \in I$. Then the direct product

$$S(I) = \prod\{S_\alpha : \alpha \in I\} \tag{3}$$

is called a *full lattice of groups.* The topology of $S(I)$ is normally the product topology, and this topology will be assumed throughout this paper. Although used here evidently for the first time, the name "full lattice of groups" is adopted in conformity with the algebraic literature (such as [7] for instance), where the more general "semilattices of groups" are discussed (also known as Clifford semigroups, these are basically subdirect products of the families of groups and groups with zeros).

**Notation 3.2.** Since a direct product of semitopological groups is a semitopological group, for simplicity of notation *we will assume that the family in* (3) *contains at most one semitopological group.* This assumption will not have any significant effect on the results of this section. Thus it is convenient to write the index set as $I' = \{\alpha'\} \cup I$ and assume $S_{\alpha'} = G$ is a semitopological group while $S_\alpha$ is a group with a zero for each $\alpha \in I$. Therefore, $S(I') = G \times S(I)$. For each $\alpha \in I'$, we denote the projection $S(I') \mapsto S_\alpha$ by $p_\alpha$, and denote by $0_\alpha$ and $1_\alpha$ the zero (if $\alpha \neq \alpha'$) and the identity elements, respectively, of $S_\alpha$. Also for each non-empty, proper subset $J \subset I'$, we denote by $1_J$ the element of $S(I')$ satisfying the following:

$$p_\alpha(1_J) = 1_\alpha \text{ if } \alpha \in J, \quad p_\alpha(1_J) = 0_\alpha \text{ if } \alpha \notin J.$$

For consistency and convenience, we also define $1_{I'} = 1$, the identity of $S(I')$. Finally, we use the prime notation for subsets to indicate that the subset contains $\alpha'$. Hence $J' \subset I'$ means that $J'$ is a subset of $I'$ and $J'$ contains $\alpha'$. It is evident that the set $E(S(I')) = \{1_{J'} : J' \subset I'\}$ is the set of *all* idempotents of $S(I')$ and that each idempotent is central.

**Lemma 3.3.** *Let $S = S(I')$ be a full lattice of groups, and let $H(1)$ be the group of units of $S$.*

*(i) For each $J' \subset I'$, $1_{J'}S = \{1_{J'}s : s \in S\}$ is a closed (two-sided) ideal of $S$, called a* principal ideal.

*(ii) $M = S \backslash H(1)$ is the unique maximal ideal of $S$.*

*(iii) $H(1)$ is dense in $S$. If $I'$ is infinite, then $M$ is also dense in $S$.*

*(iv) If $I'$ is finite, then $H(1)$ is open and $M$ is closed.*

**Proof.** (i) This assertion follows from the identity

$$1_{J'}S = \bigcap\{p_\alpha^{-1}(0_\alpha) : \alpha \in I' \backslash J'\}$$

and the fact that $p_\alpha^{-1}(0_\alpha)$ is a closed ideal in $S$.

(ii) Note that $M = \bigcup\{p_\alpha^{-1}(0_\alpha) : \alpha \in I\}$ so that $M$ is an ideal. Uniquness and maximality of $M$ follow from the fact that no proper ideal of $S$ can contain a member of $H(1)$.

(iii) Let $s \in S$, and let $N_s = \bigcap\{p_{\alpha_i}^{-1}(U_{\alpha_i}) : \alpha_i \in I', i = 1, 2, \ldots, n\}$, where $U_{\alpha_i}$ is an open neighborhood of $p_{\alpha_i}(s) = s_{\alpha_i}$ in $S$. Since $S_{\alpha_i} \backslash \{0_{\alpha_i}\}$ is dense in $S_{\alpha_i}$, $U_{\alpha_i}$ contains a point $a_{\alpha_i} \neq 0_{\alpha_i}$, $i = 1, 2, \ldots, n$. Define $a \in S$ such that $p_{\alpha_i}(a) = a_{\alpha_i}$, and $p_\alpha(a) = 1_\alpha$, if $\alpha \neq \alpha_i$, $i = 1, 2, \ldots, n$. Then $a \in H(1) \cap N_s$, implying that $H(1)$ is dense.

If $I'$ is infinite, then for $N_s$ as above, $p_\beta(N_s) = S_\beta$ for some $\beta \in I$. Hence $p_\beta^{-1}(0_\beta) \cap N_s$ is a non-empty subset of $M \cap N_s$, so that $M$ is dense.

(iv) If $I'$ is finite, then $H(1) = \bigcap\{p_\alpha^{-1}(S_\alpha \backslash \{0_\alpha\}) : \alpha \in I\}$ is open and thus $M$ is closed. ∎

Note that each idempotent $1_{J'}$ is the identity for the ideal $1_{J'}S$. This fact is used in the following lemma.

**Lemma 3.4.** *Let $S = S(I')$ be a full lattice of groups and let $J' \subset I'$. If $1_{J'}S$ is locally compact or complete metrizable with group of units $H(1_{J'})$ then*

$$LLC(1_{J'}S, H(1_{J'})) = LMC(1_{J'}S).$$

**Proof.** This lemma is an immediate consequence of Corollary 1.7 provided that we show $LLC(1_{J'}S, H(1_{J'}))$ is left translation invariant. As the following arguments do not depend on the specific choice of the subset $J'$, we only consider the case $J' = I'$ (this will simplify the notation). Hence we must show that $LLC(S, H(1))$ is left translation invariant.

Recall that $S = \prod\{S_\alpha : \alpha \in I'\}$, where $S_\alpha = G$ if $\alpha = \alpha'$, and otherwise $S_\alpha$ is a group with zero $0_\alpha$, $\alpha \in I$. For each $s_\alpha \in S_\alpha$, $s_\alpha \neq 0_\alpha$, let $h_{s_\alpha}$ be the inner automorphism defined by $h_{s_\alpha}(x_\alpha) = s_\alpha x_\alpha s_\alpha^{-1}$, $x_\alpha \in S_\alpha$. Also define $h_{0_\alpha} = h_{1_\alpha}$, the identity function on $S_\alpha$. Note that for $t_\alpha \in S_\alpha$ and non-zero $s_\alpha \in S_\alpha$,

$$s_\alpha h_{t_\alpha}(x_\alpha) = s_\alpha h_{t_\alpha}(x_\alpha) s_\alpha^{-1} s_\alpha = h_{s_\alpha}(h_{t_\alpha}(x_\alpha)) s_\alpha.$$

It follows that

$$s_\alpha h_{t_\alpha}(x_\alpha) = (h_{s_\alpha} \circ h_{t_\alpha})(x_\alpha) s_\alpha$$

even when $s_\alpha = 0_\alpha$. Now for each $s \in S$, define $h_s(x)$ as follows:

$$p_\alpha(h_s(x)) = h_{s_\alpha}(x_\alpha) \qquad \alpha \in I', x \in S$$

where we have written $s_\alpha$ and $x_\alpha$ for $p_\alpha(s)$ and $p_\alpha(x)$, respectively (we shall follow this practice where it is helpful in simplifying the notation). Note that $h_s : S \mapsto S$ is a well-defined function for every $s \in S$ ($h$ is the function whose coordinates consist of the functions $h_{s_\alpha}$). In particular, for $u \in H(1)$ and $x \in S$, $h_u(x) = uxu^{-1}$, so that $h_u$ is a topological automorphism. Further, it is readily verified that for each $s \in S$, $h_s = h_{s'}$, where $s' \in H(1)$ is defined as:

$$p_\alpha(s') = s_\alpha \ \text{ if } \ s_\alpha \neq 0_\alpha, \quad p_\alpha(s') = 1_\alpha \ \text{ if } \ s_\alpha = 0_\alpha.$$

Now suppose that $u \in H(1)$, and let $\{s_\eta\}$ be a net in $S$ converging to $u$. Since $u = uuu^{-1} = h_u(u)$, it follows that if $\mathcal{B}_u$ is a neighborhood basis in $S$ at $u$, then the family $\{h_u N : N \in \mathcal{B}_u\}$ is also a neighborhood basis at $u$. Hence, without loss of generality, we may assume that $\{s_\eta\}$ is in some neighborhood of $u$ of the form $h_u N$. Thus for each $\eta$, $s_\eta = h_u(x_\eta)$ for some $x_\eta \in N$. This implies that for each $\eta$, $x_\eta = u^{-1} s_\eta u = h_{u^{-1}}(s_\eta)$, so the points $x_\eta$ form a net in $N$ that converges to $u$. Therefore, for each $f \in LLC(S, H(1))$ and $s \in S$,

$$\begin{aligned} \|L_{s_\eta}(L_s f) - L_u(L_s f)\| &= \|L_{s h_u(x_\eta)} f - L_{s h_u(u)} f\| \\ &= \|L_{h_s(h_u(x_\eta))s} f - L_{h_s(h_u(u))s} f\| \\ &\leq \|L_{h_{s'u}(x_\eta)} f - L_{h_{s'u}(u)} f\|. \end{aligned}$$

Since $h_{s'u}(u) \in H(1)$ and the net $\{h_{s'u}(x_\eta)\}$ converges to $h_{s'u}(u)$, the last norm quantity above converges to zero, implying that $L_s f \in LLC(S, u)$ for each $u \in H(1)$. It follows that $LLC(S, H(1))$ is left translation invariant. ∎

**Definition 3.5.** Let $S = S(I')$ be a full lattice of groups, and let $K \subset I$ (hence $\alpha' \notin K$). A function $f \in C(S)$ is *$K$-constant* if $f(1_{I' \setminus K} s) = f(s)$ for all $s \in S$. Intuitively, $f$ is $K$-constant if it does not depend on the coordinates in $K$ (also see the paragraph before Lemma 3.14). For convenience, we may extend the defintion to $K = \emptyset$ by defining the set of all $\emptyset$-constant functions to be $C(S)$.

The following is a key result. Corollary 3.15 below provides an alternative characterization in terms of continuous functions on principal ideals.

**Theorem 3.6.** *Let $S = S(I')$ be a full lattice of groups and let $J' \subset I'$. If $f \in LLC(S, 1_{J'})$, then $f$ is $(I' \backslash J')$-constant. Conversely, if $f$ is a $(I' \backslash J')$-constant function in $LMC(S)$ and $1_{J'}S$ is locally compact or complete metrizable, then $f \in LLC(S, 1_{J'})$.*

**Proof.** Assume that $f \in C(S)$ and that $f$ is not $(I' \backslash J')$-constant. Also the case $J' = I'$ being trivial, suppose that $J' \neq I'$. Then there is $s_0 \in S$ such that $f(s_0) \neq f(1_{J'} s_0)$. Since by Lemma 3.3(iii) $H(1)$ is dense in $S$, we may choose a net $\{s_\eta\}$ in $H(1)$ that converges to $1_{J'}$. We further assume that $p_\alpha(s_\eta) = 1_\alpha$ if $\alpha \in J'$. For each $\eta$, define $x_\eta = s_\eta^{-1} s_0$. Then $s_\eta x_\eta = s_0$ and $1_{J'} x_\eta = 1_{J'} s_0$ for every $\eta$. Therefore, for each $\eta$

$$\|L_{s_\eta} f - L_{1_{J'}} f\| \geq |f(s_\eta x_\eta) - f(1_{J'} x_\eta)| = |f(s_0) - f(1_{J'} s_0)| > 0.$$

Hence, $f \notin LLC(S, 1_{J'})$.

Conversely, suppose that $1_{J'}S$ is locally compact or complete metrizable, and let $f$ be an $(I' \backslash J')$-constant function in $LMC(S)$. Note that $g = f|_{1_{J'}S} \in LMC(1_{J'}S)$ (1.8, with $\theta$ the inclusion map and $A$ the empty set). Let $\{s_\eta\}$ be a net in $S$ that converges to $1_{J'}$, and note that

$$\begin{aligned} \|L_{s_\eta} f - L_{1_{J'}} f\| &= \sup_{x \in S} |f(1_{J'} s_\eta 1_{J'} x) - f(1_{J'} x)| \\ &= \sup_{y \in 1_{J'} S} |f(1_{J'} s_\eta y) - f(y)| \\ &= \|L_{1_{J'} s_\eta} g - L_{1_{J'}} g\|. \end{aligned}$$

Since $\{1_{J'} s_\eta\}$ converges to $1_{J'}$ in $1_{J'}S$, Lemma 3.4 implies that the last norm quantity above involving the function $g$ approaches zero, so that $f \in LLC(S, 1_{J'})$. ■

Recall that every closed subspace of a locally compact (respectively, complete metrizable) space is locally compact (respectively, complete metrizable). Hence for a locally compact or complete metrizable full lattice of groups $S(I')$ Theorem 3.6 can be restated as:
"*A function $f$ in $LMC(S(I'))$ is left locally continuous at $1_{J'}$, $J' \subset I'$, if and only if $f$ is $(I' \backslash J')$-constant.*"
The following lemma is needed in the important Corollary 3.8 below.

**Lemma 3.7.** *Let $T$ be a locally compact semitopological group with a zero. Then $WAP(T)$ contains non-constant functions.*

**Proof.** Since $T \backslash \{0\}$ is a locally compact topological group, $C_0(T \backslash \{0\}) \subset WAP(T \backslash \{0\})$ [1]. Let $U$ be a compact neighborhood of 1 in $T \backslash \{0\}$ and let $g$ be a continuous function on $T \backslash \{0\}$ such that $g(1) = 1$, $g(t) = 0$ for $t \notin U$. Thus $g \in WAP(T \backslash \{0\})$. Now define the function $f$ as: $f(t) = g(t)$ if $t \neq 0$ and $f(0) = 0$. Note that $f \in C(T)$ and $f$ is not constant. We apply the double limit criterion of Grothendieck to show that $f \in WAP(T)$. Let $\{s_m\}$ and $\{t_n\}$ be sequences in $T$ such that all of the limits defining the quantities $a$ and $b$ below

$$a = \lim_m \lim_n f(s_m t_n), \qquad b = \lim_n \lim_m f(s_m t_n)$$

exist. If $\{s_m\}$ and $\{t_n\}$ converge to $s$ and $t$ respectively, then $a = f(st) = b$. Otherwise, either $a = 0 = b$ or $f(s_m t_n) = g(s_m t_n)$ for $m$ and $n$ large enough. In the latter case, $a = b$ since $g$ is weakly almost periodic. Thus $a = b$ in all cases, implying that $f \in WAP(T)$. ■

It may be worth noting here that the proof of 3.7 shows that $WAP(T)$ actually contains all continuous functions $f$ with compact support such that $f(0) = 0$ and the support does not contain 0.

Recall that a product space $\prod\{S_\alpha : \alpha \in I\}$ is locally compact if and only if $S_\alpha$ is locally compact for all $\alpha \in I$ and $S_\alpha$ is compact for all but finitely many $\alpha$.

**Corollary 3.8.** *Let $S = S(I')$ be a full lattice of groups.*

*(i) If $S$ is locally compact or complete metrizable, and if $\mathcal{F}$ is a non-empty family of primed subsets of $I'$, then $LLC(S, \{1_{J'} : J' \in \mathcal{F}\}) = LLC(S, 1_F)$, where $F = \cap\mathcal{F}$.*

*(ii) If $S$ is locally compact and $J' \subset K' \subset I'$, then $LLC(S, 1_{J'}) \subset LLC(S, 1'_K)$ with equality holding if and only if $J' = K'$. Hence, $LLC(S, 1_{J'}) \subset LLC(S, 1_{K'})$ if and only if $J' \subset K'$.*

**Proof.** (i) By 3.6 and 1.2(i), $f \in LLC(S, \{1_{J'} : J' \in \mathcal{F}\})$ if and only if $f$ is $(I'\backslash J')$-constant for all $J' \in \mathcal{F}$, if and only if $f$ is $(I'\backslash F)$-constant, if and only if $f \in LLC(S, 1_F)$.

(ii) By Part (i) and 1.2(i)

$$\begin{aligned} LLC(S, 1_{J'}) &= LLC(S, 1_{J' \cap K'}) \\ &= LLC(S, \{1_{J'}, 1_{K'}\}) \\ &= LLC(S, 1_{J'}) \cap LLC(S, 1_{K'}) \end{aligned}$$

which proves the first assertion. To complete the proof, suppose that $J' \neq K'$. Let $f \in LMC(S)$ be $(I'\backslash J')$-constant and choose $\beta \in K'\backslash J'$. By Lemma 3.7 $LMC(S_\beta)$ contains a non-constant function $f_\beta$, so that $p^*_\beta f_\beta = f_\beta \circ p_\beta$ is a non-constant function in $LMC(S)$ (Lemma 1.8). It follows that $f + p^*_\beta f_\beta \in LMC(S)$ is $(I'\backslash K')$-constant but not $(I'\backslash J')$-constant. Hence, by Theorem 3.6 $f + p^*_\beta f_\beta$ is a member of $LLC(S, 1_{K'})\backslash LLC(S, 1_{J'})$. ■

**Corollary 3.9.** *Let $S = S(I') = G \times S(I)$ be a locally compact or complete metrizable full lattice of groups. Then $LUC(S)$ is isometrically $*$-isomorphic to $LUC(G)$.*

**Proof.** Since $\bigcap\{J' : J' \subset I'\} = \{\alpha'\}$ Corollary 3.8(i) implies that $LUC(S) = LLC(S, 1_{\{\alpha'\}})$. Hence, by Theorem 3.6, $LUC(S)$ consists of the $I$-constant functions in $LMC(S)$. Note that since $[1_{\{\alpha'\}}] = G \times \{0\}$, it follows that $LLC(S, 1_{\{\alpha'\}}) = LLC(S, G \times \{0\})$, where 0 is the zero element in $S(I)$. Let $p_{\alpha'} : S \mapsto G$ be the projection of $S$ onto $G$, and let $q_{\alpha'}$ denote the embedding $x \mapsto (x, 0) : G \mapsto S$. For every $f \in LUC(S)$, due to $I$-constancy, $f = f \circ q_{\alpha'} \circ p_{\alpha'} = (q_{\alpha'} \circ p_{\alpha'})^* f$, so that $LUC(S) = (q_{\alpha'} \circ p_{\alpha'})^* LUC(S)$. Thus

$$LUC(S) = p^*_{\alpha'} q^*_{\alpha'} LLC(S, G \times \{0\}) \subset p^*_{\alpha'} LUC(G) \subset LUC(S),$$

where the inclusions follow from Lemma 1.8. In particular, $p^*_{\alpha'} LUC(G) = LUC(S)$, with $p^*_{\alpha'}$ an isometric $*$-isomorphism onto $LUC(S)$. ■

We are now ready for the second main theorem of this section.

**Theorem 3.10.** *If $S = S(I')$ is a locally compact full lattice of groups, then $LLC(S)$ is isomorphic to the Boolean lattice $(\mathcal{P}(I), \cap, \cup)$ of all subsets of $I = I'\backslash\{\alpha'\}$ with respect to set intersection and union.*

**Proof.** Corollary 3.8 implies that the partially ordered set

$$(\mathcal{L}, \subset) = (\{LLC(S, 1_{J'}) : J' \subset I'\}, \subset)$$

is isomorphic to the partially ordered set $(\mathcal{P}(I), \subset)$ (in the sense that the bijection $J \mapsto LLC(S, 1_{J'}) : \mathcal{P}(I) \mapsto \mathcal{L}$ and its inverse are both order-preserving). Thus $(\mathcal{P}(I), \cap, \cup)$ is lattice-isomorphic to $\mathcal{L}$ under set intersection $\cap$ and a "join" (least upper bound) operation $\vee$ defined by

$$LLC(S, 1_{J'}) \vee LLC(S, 1_{K'}) = LLC(S, 1_{J' \cup K'}).$$

We now show that $LLC(S)$ is isomorphic to the lattice $\mathcal{L}$. This clearly follows if we show that for every subset $A \subset S$, there exists $J' \subset I'$ such that $LLC(S, A) = LLC(S, 1_{J'})$. If $A$ is empty, then by Lemma 3.4 $LLC(S, A) = LMC(S) = LLC(S, 1)$, and we may set $J' = I'$. Now suppose that $A \neq \emptyset$ and let $E(S)$ be the set of all idempotents in $S$. We first establish that $E(S)H(1) = S$. Let $s \in S$ and define $K_s = \{\alpha \in I' : p_\alpha(s) \neq 0_\alpha\}$. If $K_s = I'$, then $s \in H(1)$, so that $s = 1s \in E(S)H(1)$. Otherwise, define $t_s \in H(1)$ as follows:

$$p_\alpha(t_s) = p_\alpha(s) \ \text{ if } \ \alpha \in K_s, \qquad p_\alpha(t_s) = 1_\alpha \ \text{ if } \ \alpha \notin K_s.$$

Then $s = 1_{K_s} t_s \in E(S)H(1)$, and it follows that $S = E(S)H(1)$. Lemma 2.2 now implies that $S = \bigcup\{[1_{K'}] : K' \subset I'\}$, so that for each $a \in A$ there is a primed $J_a \subset I'$ such that $a \in [1_{J_a}]$; i.e., $LLC(S, a) = LLC(S, 1_{J_a})$. Hence, from 3.8(i) and 1.2(i) it follows that $LLC(S, A) = \bigcap\{LLC(S, 1_{J_a}) : a \in A\} = LLC(S, 1_{J'})$, where $J' = \bigcap\{J_a : a \in A\}$. ∎

**Corollary 3.11.** *Let $S = S(I')$ be a full lattice of groups.*

*(i) If $S$ is locally compact or complete metrizable, then $E(S)$ is a complete set of representatives for the relation $\rho_0$, $[s] = sH(1)$ for all $s \in S$, and $H(1)$ is normal. Hence, $\rho_0$ is a congruence and the quotient $S/\rho_0$ is algebraically isomorphic to $E(S)$.*

*(ii) If $S$ is locally compact and $E_{J'}(S) = \{1_{K'} : K' \supset J'\}$, then*

$$\mathcal{E}(S) = \{E_{J'}(S)H(1) : J' \subset I'\}.$$

**Proof.** (i) That $E(S)$ is a complete set of representatives for $\rho_0$ is established in the proof of Theorem 3.10 where we show that $\mathcal{L} = LLC(S)$. Further, $S = \bigcup\{1_{J'}H(1) : J' \subset I'\} = \bigcup\{[1_{J'}] : J' \subset I'\}$, also as in the proof of 3.10. Since the above unions are disjoint, Lemma 2.2 implies that $[1_{J'}] = 1_{J'}H(1)$. Now, for each $s \in S$ there is $J' \subset I'$ such that $[s] = [1_{J'}]$. This means that there is $u \in H(1)$ such that $su = 1_{J'}$. Hence, $1_{J'}H(1) \subset sH(1)$, implying that $[s] = sH(1)$.

Furthermore, for each $s \in S$, $sH(1) = 1_{J'}H(1)$ for some $J' \subset I'$. Hence there is $w \in H(1)$ such that for each $u \in H(1)$

$$su = 1_{J'}wu = 1_{J'}wuw^{-1}w = wuw^{-1}1_{J'}w = (wuw^{-1})s \in H(1)s.$$

A similar argument implies that $us \in sH(1)$ for each $u \in H(1)$. Therefore, $sH(1) = H(1)s$; i.e., $H(1)$ is normal. The statement about $S/\rho_0$ is now clear.

(ii) Suppose $J' \subset K' \subset I'$. Then $LLC(S, 1_{J'}) \subset LLC(S, 1_{K'})$ and $LLC(S, 1_{K'}H(1)) = LLC(S, 1_{K'})$. Thus

$$\begin{aligned} LLC(S, E_{J'}(S)H(1)) &= \bigcap\{LLC(S, 1_{K'}) : K' \supset J'\} \\ &= LLC(S, 1_{J'}) \\ &= LLC(S, \cup[\{1_{J'}\}]). \end{aligned}$$

Therefore, $E_{J'}(S)H(1) \subset \cup[\{1_{J'}\}]$. On the other hand, if $a \in \cup[\{1_{J'}\}]$, then by Part (i) $a \in [1_{K'}] = 1_{K'}H(1)$ for some $K' \subset I'$. Note that

$$LLC(S,1_{J'}) = LLC(S,\cup[\{1_{J'}\}]) \subset LLC(S,a) = LLC(S,1_{K'}).$$

Now Corollary 3.8 implies that $J' \subset K'$, so that $a \in E_{J'}(S)H(1)$. Hence, for each $J' \subset I'$, $\cup[\{1_{J'}\}] = E_{J'}(S)H(1)$. The assertion about $\mathcal{E}(S)$ now follows from Theorem 3.10. ∎

Note that when $S(I')$ is locally compact (see also 3.12 below), Corollary 3.11 identifies the $LLC$ equivalence relation $\rho_0$ with the natural semilattice congruence on $S(I')$ and it also identifies the $LLC$ lattice with the lattice $E(S(I'))$ under the canonical idempotent ordering [7].

The assumption of local compactness is not a necessary condition in Theorem 3.10, as the next example demonstrates.

**Example 3.12.** (A metric, non-locally compact case.) Let $S = \prod_{n=1}^{\infty} S_n$, where $S_n$ is a complete metrizable group with a zero for each $n = 1,2,3,\ldots$. Since all metric spaces are homeomorphic to metric spaces of diameter one [3], we may assume that $S_n$ admits a metric $d_n$ such that $d_n(x,y) \leq 1$ for $x,y \in S_n$, $n = 1,2,3,\ldots$. If we define $d : S \times S \mapsto [0,\infty)$ by

$$d(s,t) = \sum_{n=1}^{\infty} 2^{-n} d_n(s_n,t_n), \quad s_n = p_n(s), t_n = p_n(t), \quad n = 1,2,3,\ldots$$

then $d$ is a metric for the space $S$ (the familiar "product metric") and $(S,d)$ is a complete metric space of diameter one whose metric topology coincides with the product topology [3]. Although 3.10 cannot be directly applied to this example, all of the results preceeding 3.10 (except 3.7) do apply if $LMC(S_n)$ contains non-constant functions for all $n \geq 1$. This last condition is, in particular, satisfied by Lemma 3.7 if for each $n \geq 1$, $S_n$ is locally compact. Note that even in this latter case, if $S_n$ is non-compact for infinitely many $n$ then $S$ is not locally compact, although $LLC(S)$ has the same properties stated in 3.10.

In certain cases, it may happen that while distinct, some members of $LLC(S)$ are isomorphic to each other. This is illustrated in the following example.

**Example 3.13.** Let $I$ be a non-empty set and let $T$ be a group with a zero. Then the direct product $T^I$ is the space of all functions $x : I \mapsto T$. With the topology of pointwise convergence $T^I$ is a full lattice of groups and for each $J \subset I$, the idempotent $1_J$ is just the characteristic function of $J$. Now suppose that one of the following restrictions holds:

(i) $T$ is compact;

(ii) $T$ is locally compact and complete metrizable, and $I$ is countable;

(iii) $T$ is locally compact and $I$ is finite;

Then by Theorem 3.10 (for (ii) use Example 3.12 with $S_n = T$ for all $n$) $LLC(T^I)$ is Boolean and lattice isomorphic to $(\mathcal{P}(I),\cap,\cup)$. Let $J,K \subset I$ and $|J| = |K|$. Let $b : I \mapsto I$ be a bijection with $b(K) = J$, and define $\theta_b : T^I \mapsto T^I$ by $\theta_b(x) = x \circ b$, $x \in T^I$. Then $\theta_b$ is a topological automorphism with $\theta_b(1_J) = 1_K$. Hence Corollary 1.9 implies that $LLC(T^I,1_J)$ is isometrically $*$-isomorphic to $LLC(T^I,1_K)$. Thus if $LLC(T^I,1_J), LLC(T^I,1_K) \in LLC(T^I)$ with $|J| \neq |K|$, then by identifying isomorphic copies we may assume that

$J \subset K$, or equivalently (3.8), that $LLC(T^I, 1_J) \subset LLC(T^I, 1_K)$. It follows that $LLC(T^I)$ is linear up to isomorphisms. We note here that for compact $T$, $T^I$ is also compact. Hence in case (i) above, instead of 1.9 it would be easier to use Corollary 3.16 below.

We close this section by characterizing the set of all continuous $(I'\backslash J')$-constant functions (hence also $LLC(S, 1_{J'})$ ) in terms of $C(1_{J'}S)$, $J' \subset I'$. As remarked in Example 3.13, these characterizations can be helpful in settling isomorphism questions about $LLC$ algebras. Let $I$ be a non-empty set and let $\{S_\alpha : \alpha \in I\}$ be a family of topological spaces. For each $J \subset I$, $J \neq \emptyset, I$, let $S_J = \prod\{S_\alpha : \alpha \in J\}$, and let $p_J : S \mapsto S_J$ be the projection of $S$ onto $S_J$. Then the dual map $p_J^* : C(S_J) \mapsto C(S)$ is a linear isometry and a conjugate-preserving monomorphism, as may be verified directly. We also extend Definition 3.5 to continuous functions defined on topological spaces: If $K \neq \emptyset, I$, then $f \in C(S)$ is $K$-constant on the product $S$, if $f(s) = f(t)$ for every pair $s, t \in S$ satisfying $p_{I\backslash K}(s) = p_{I\backslash K}(t)$.

**Lemma 3.14.** *Let $S = \prod\{S_\alpha : \alpha \in I\}$ be a direct product of topological spaces $S_\alpha$, and let $J \subset I$, $J \neq \emptyset, I$. Then $p_J^* C(S_J)$ is the set of all $(I\backslash J)$-constant functions in $C(S)$.*

**Proof.** For each $g \in C(S_J)$, $p_J^* g = g \circ p_J$ is continuous on $S$ and $(I\backslash J)$-constant. Conversely, let $f$ be an $(I\backslash J)$-constant function in $C(S)$. Let $t$ be a fixed element of $S_{I\backslash J}$ and for each $s \in S_J$, define $x_s \in S$ as follows: $p_J(x_s) = s$, $p_{I\backslash J}(x_s) = t$. Now define $h(s) = f(x_s)$, $s \in S_J$. Then $h$ is uniquely defined by $f$ and it is easily verified that $h \in C(S_J)$. Furthermore, due to $I\backslash J$-constancy, $p_J^* h = f$, and the lemma follows. ∎

As a consequence of Lemma 3.14, we may assert that *the set of all $(I\backslash J)$-constant functions in $C(S)$ is isometrically $*$-isomorphic to $C(S_J)$*. In the next corollary we resume the use of Notations 3.2.

**Corollary 3.15.** *Let $S = S(I')$ be a locally compact or complete metrizable full lattice of groups, and let $J' \subset I'$, $J' \neq I'$. Then there exists a surjective continuous homomorphism $\pi : S \mapsto 1_{J'}S$ such that*

$$LLC(S, 1_{J'}) = \pi^* C(1_{J'}S) \cap LMC(S).$$

**Proof.** From Theorem 3.6 and Lemma 3.14 we may infer that $LLC(S, 1_{J'}) = p_{J'}^* C(S_{J'}) \cap LMC(S)$. Also $1_{J'}S$ is topologically isomorphic to $S_{J'}$ under the obvious identification of elements which we denote by $\theta_{J'} : S_{J'} \mapsto 1_{J'}S$ ($\theta_{J'}$ is the mapping that inserts all the zero coordinates outside $J'$). Define $\pi = \theta_{J'} \circ p_{J'}$, and note that $\pi$ has the stated properties. ∎

**Corollary 3.16.** *Let $S = S(I')$ be a compact full lattice of groups, and let $J', K'$ be proper subsets of $I'$.*

*(i) If the ideal $1_{J'}S$ is topologically isomorphic to the ideal $1_{K'}S$, then $LLC(S, 1_{J'})$ is isometrically $*$-isomorphic to $LLC(S, 1_{K'})$.*

*(ii) $C(S)|_{1_{J'}S} = LLC(S, 1_{J'})|_{1_{J'}S} = C(1_{J'}S)$.*

**Proof.** (i) Since $LMC(S) = C(S)$, Corollary 3.15 implies that $LLC(S, 1_L) = \pi_L^* C(1_L S)$, $L = J', K'$, with $\pi_L : S \mapsto 1_L S$ as given in 3.15. Hence, if $\tau : 1_{J'}S \mapsto 1_{K'}S$ is a topological isomorphism, then

$$LLC(S, 1_{J'}) = \pi_{J'}^* \tau^* \pi_{K'}^{*-1} LLC(S, 1_{K'})$$

and Part (i) follows.

(ii) Note that if $\pi$ is the mapping in 3.15, then $\pi(s) = s$ for every $s \in 1_{J'}S$. Hence for every $g \in C(1_{J'}S)$, $(\pi^*g)|_{1_{J'}S} = (g \circ \theta_{J'} \circ p_{J'})|_{1_{J'}S} = g$, and Corollary 3.15 implies that

$$C(S)|_{1_{J'}S} \subset C(1_{J'}S) = LLC(S, 1_{J'})|_{1_{J'}S} \subset C(S)|_{1_{J'}S}. \qquad \blacksquare$$

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Department of Mathematical Sciences
Virginia Commonwealth University
Richmond, VA 23284-2014

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