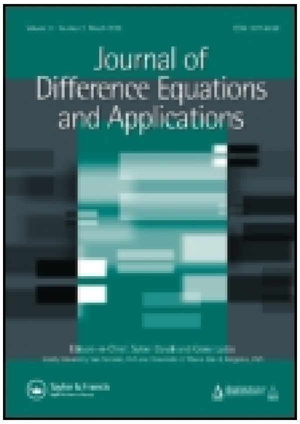

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Semiconjugates of One-dimensional Maps

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Semiconjugates of One-dimensional Maps

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$\mathbb{R}$-semiconjugate maps are defined as a natural means of relating a map F of $\mathbb{R}^m$ to a mapping ϕ of the interval via a link map H. Invariants are seen to be special types of semiconjugate links where ϕ is the identity. Basic relationships between the dynamical behaviors of ϕ and F are established, and conditions under which a link map H is a Liapunov function are obtained. Examples and applications involving concepts from stability to chaos are discussed.

Keywords: $\mathbb{R}$-semiconjugate; Link; Factor; Fiber; Invariant; Liapunov function; stability; sensitivity; chaos

AMS Subject Classification: 39A11

1 INTRODUCTION

Given that the dynamics of maps of the real line $\mathbb{R}$ are considerably better understood than the higher dimensional maps, it is natural to ask if we can use the theory of maps on $\mathbb{R}$ to study higher dimensional maps in a *systematic* way. In this paper, we look at a special type of semiconjugacy

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where a map of $\mathbb{R}^m$ is linked to a map defined on a subset of $\mathbb{R}$ (usually an interval). In particular, we consider the relationships between semiconjugacy and such familiar concepts as Liapunov functions and invariants. The central idea is the following.

DEFINITION 1 Let $F \in C(D)$, where $D \subset \mathbb{R}^m$ is nonempty and $C(D) = C(D, D)$ is the space of all continuous self maps of D. If there is a nonconstant function $H \in C(D, \mathbb{R})$ such that

$$H \circ F = \phi \circ H$$

on D for some $\phi \in C(H(D), \mathbb{R})$, then F is a (D, H, ϕ)-*semiconjugate* map of $\mathbb{R}^m$. The mapping H is called a *link map* and ϕ is the (topological) *factor map* (or the *real factor* of F). Where there is no confusion, we may also use the term $\mathbb{R}$-*semiconjugate* in referring to F. For each $t \in H(D)$, the level set $H^{-1}(t) \cap D$, denoted H_t^{-1} for short, is called a *fiber* of H in D.

Within the context of *self maps* of a topological space, semiconjugacy is a natural extension of topological conjugacy, although this is not an interpretation that we dwell on in this paper. Also, we may profitably define semiconjugates relative to sets other than $\mathbb{R}$ (e.g. the circle) as long as the dynamics of the reference set is well understood. The next result lists some of the elementary properties of $\mathbb{R}$-semiconjugates.

LEMMA 1 *Let F be a (D, H, ϕ)-semiconjugate map.*

(a) *If G is (D, H, ϕ')-semiconjugate, then $F \circ G$ is $(D, H, \phi \circ \phi')$-semiconjugate. In particular, for each positive integer n, $H \circ F^n = \phi^n \circ H$, and the iterate F^n is (D, H, ϕ^n)-semiconjugate.*
(b) *H is a Liapunov function for F on D if and only if $\phi(t) \leq t$ for all $t \in H(D)$.*
(c) *$H(D)$ is invariant under ϕ, i.e. $\phi(H(D)) \subset H(D)$.*
(d) *$F(H_t^{-1}) \subset H_{\phi(t)}^{-1}$ for all $t \in H(D)$.*
(e) *If $D_1 \subset D$ is a nonempty subset, then $F_1 \doteq F|_{D_1}$ is (D_1, H_1, ϕ_1)-semiconjugate, where $\phi_1 \doteq \phi|_{H(D_1)}$, provided that $H_1 \doteq H|_{D_1}$ is not constant.*
(f) *If ϕ is an odd function, then F is also $(D, -H, \phi)$-semiconjugate. If ϕ is even, then F is also $(D, -H, -\phi)$-semiconjugate.*
(g) *If H is linear and G is (D, H, ϕ')-semiconjugate, then the sum $F + G$ is $(D, H, \phi + \phi')$-semiconjugate and the scalar multiple aF is $(D, H, a\phi)$-semiconjugate.*

Part (a) is a crucial property, since dynamics would not be possible without it. Regarding Part (b), a more detailed discussion of Liapunov functions in the context of $\mathbb{R}$-semiconjugate maps appears later in this paper. In the special case where ϕ is the identity function $\phi(t) = t$, the link map H is called an *invariant*, since for every x, we have $H(F(x)) = H(x)$. In particular, by Part (d) of Lemma 1, $F(H_t^{-1}) \subset H_t^{-1}$ for each t, so that each fiber of H will retain a trajectory that starts in it. There is currently a sizable literature on invariants; see, e.g. Refs. [1–3,8–10,17] for theoretical discussions, additional references and some applications of this concept.

Invariant links are somewhat exceptional types of links, and they have properties that are not generally shared by semiconjugate link maps; for instance, if H is an invariant for F, then so is $g \circ H$ for each $g \in C(\mathbb{R})$. On the other hand, many mappings of interest in applications have only trivial invariants. For instance, if F has a globally attracting fixed point $p \in D$, then for every $x \in D$, $H(x) = H(p)$; i.e. H is constant on D. At the other extreme, if F has a *dense* trajectory in D (as is common among chaotic maps), then the constancy of an invariant H on such a trajectory clearly implies that H is constant on D.

The first case above in particular excludes linear mappings whose spectral radius is less than 1. In fact, for a linear mapping $F(x) = Ax$ to have a nonconstant invariant, it is necessary and sufficient that either the matrix A has an eigenvalue of unit modulus, or that A has eigenvalues λ, μ with $0 < |\lambda| < 1 < |\mu|$; see Ref. [1]. However, if we allow the factor ϕ to be any linear mapping of the line (not just the identity), then it can be established with a little effort that all linear maps of $\mathbb{R}^m$ are $\mathbb{R}$-semiconjugates with either linear or quadratic link maps (see Ref. [18]).

Part (d) of Lemma 1 states that F must map each fiber of H into another fiber. This is a characteristic property of $\mathbb{R}$-semiconjugate maps. Since fibers of H are nonintersecting manifolds, to find an H that works with a given F we look for families of nonmutually intersecting surfaces that satisfy Lemma 1(d) and which can be fibers of some mapping H. This is not generally easy, but there exist interesting and nontrivial examples of such nonintersecting manifolds, namely, the level sets of norms, or "spheres." Let $\|\cdot\|$ denote any norm on $\mathbb{R}^m$, and note that if some function F satisfies the identity

$$\|F(x)\| = \phi(\|x\|) \tag{1}$$

for some $\phi \in C([0, \infty)$, then F is semiconjugate to ϕ with the norm as the link map. A mapping F that satisfies Eq. (1), also satisfies Lemma 1(d), since the norm of $F(x)$ depends only on the norm of x for every x; i.e. the sphere that contains x is mapped into the sphere that contains $F(x)$. The mapping in Example 3 below is essentially of this type; for more examples satisfying Eq. (1), see Ref. [18].

SOME BASIC THEORY

In this section, we consider how the dynamics of the one-dimensional factor map influences the dynamics of the original higher dimensional map. We also examine the relationship between semiconjugate links and Liapunov functions. The proofs of Theorems 1–3 and related results below are straightforward; see Ref. [18] for more details.

LEMMA 2 *Let F be a (D,H, ϕ)-semiconjugate map.*

(a) *If $\bar{x} \in D$ is a fixed point of F, then $\bar{t} = H(\bar{x})$ is a fixed point of ϕ.*
(b) *If C is an invariant subset of $H(D)$ under ϕ, then $H^{-1}(C) \cap D = H_C^{-1}$ is invariant under F.*
(c) *$\bar{t} \in H(D)$ is a fixed point of ϕ if and only if the corresponding fiber $H_{\bar{t}}^{-1}$ is invariant under F.*

Fibers of link maps may, in general, be "thick,", i.e. they may contain open sets. To avoid certain undesirable consequences, we assume the following about semiconjugate links in the sequel:

ASSUMPTION Every fiber H_t^{-1} of a link map H has an empty interior.

A link map H satisfying the above assumption may be called *everywhere bending*, since its graph is not flat over any open set. With this intuitively agreeable restriction on links, the following is not hard to prove.

THEOREM 1 (Stability and instability) *Let $\bar{x}$ be a fixed point of a (D, H, ϕ)-semiconjugate mapping F. If we set $\bar{t} = H(\bar{x})$, then the following are true:*

(a) *If $\bar{x}$ is stable under F, then $\bar{t}$ is a stable fixed point of ϕ.*
(b) *If $\bar{x}$ is asymptotically stable under F, then $\bar{t}$ is asymptotically stable under ϕ.*
(c) *If $\bar{t}$ is unstable under ϕ, then $\bar{x}$ is unstable under F.*

COROLLARY 1 *Let F be a(D, H, ϕ)-semiconjugate map, with $D \subset \mathbb{R}^m$. Then:*

(a) *If ϕ has no stable cycles, then neither does F.*
(b) *If ϕ has no asymptotically stable cycles, then neither does F.*
(c) *If ϕ has a trajectory that does not converge to a cycle, then so does F.*

Next, we want to derive suitable restrictions on H that ensure the truth of converse statements (a) and (b) in Theorem 1. Let us begin with a boundedness result that is reminiscent of a similar result in Liapunov theory (see, e.g. Ref. [12, p.9]) and the proof is also similar.

THEOREM 2 (Boundedness) *Let F be a(D, H, ϕ)-semiconjugate map*

(a) *Assume that $|H(x)| \to \infty$ if $\|x\| \to \infty$. If the sequence $\{\phi^n(t_0)\}$ is bounded for some $t_0 \in H(D)$, then every trajectory $\{F^n(x_0)\}$ with $x_0 \in H_{t_0}^{-1}$ is bounded.*
(b) *If $|H(x)| \to \infty$ for $x \in D$ with $\|x\| \to \infty$ and either $H(D)$ is a bounded set or ϕ is a bounded function, then every trajectory of F is bounded in D.*

Example 1 Some condition on H like the one in Theorem 2(a) is necessary for boundedness, even if ϕ is a bounded function. An example is provided by the difference equation [14]

$$x_{n+1} = \frac{a}{x_n} + bx_{n-1}, \quad a > 0, \quad b \geq 1, \quad x_0, x_{-1} > 0. \tag{2}$$

Let us rewrite this equation as

$$x_{n+1} = \frac{a}{x_n} + \frac{b}{y_n} \quad y_{n+1} = \frac{1}{x_n}$$

which corresponds to the mapping $F(x, y) = [a/x + b/y, 1/x]$. Define $H(x, y) \doteq y/x$ and note that

$$H(F(x, y)) = \frac{1}{a + b(x/y)} = \frac{1}{a + b/H(x, y)} = \frac{H(x, y)}{aH(x, y) + b}$$

So if $\phi(t) \doteq t/(at+b)$ then F is (D, H, ϕ)-semiconjugate. In this case, ϕ is a monotonically increasing function on $H(D) = (0, \infty)$, it is bounded above by the constant $1/a$, and $\phi(t) < t$ with $\phi(t) \to 0$ as $t \to 0$. Suppose that the trajectory $(x_n, y_n) = F^n(x_0, y_0)$ is bounded for some (x_0, y_0) in the positive quadrant. Since

$$\frac{y_{n+1}}{x_{n+1}} = H(F^n(x_0, y_0) = \phi^n(H(x_0, y_0) = \phi^n\left(\frac{y_0}{x_0}\right)$$

it follows that the ratio $y_n/x_n \to 0$ as $n \to \infty$. Since x_n is bounded above by our assumption and $y_n = 1/x_{n-1}$, it follows that y_n is bounded away from zero. But this contradicts the fact that y_n/x_n approaches zero. Therefore, every solution of Eq. (2) is unbounded.

The following is a $\mathbb{R}$-semiconjugate analog of LaSalle's invariance principle [12, p.9] (or [6, p.188]). As with LaSalle's result, here too the smaller the fibers of H, the stronger the conclusions.

THEOREM 3 (Attractivity of invariant fibers) *Suppose that F is a (D, H, ϕ)-semiconjugate map, and let $\bar{t}$ be an isolated fixed point of ϕ which attracts all points in an interval $I \subset H(D)$. If $x_0 \in D \cap H^{-1}(I)$ and $\{F^n(x_0)\}$ is a bounded trajectory, then all limit points of $\{F^n(x_0)\}$ are contained in $H_{\bar{t}}^{-1}$; i.e. the F-trajectory converges to the invariant fiber.*

COROLLARY 2 (Asymptotic stability) *Let F be $a(D, H, \phi)$-semiconjugate map, with $|H(x)| \to \infty$ as $\|x\| \to \infty$. Assume that F has a fixed point $\bar{x} \in D$ at which H has either a local minimum or a local maximum. If $\bar{t} = H(\bar{x})$ is asymptotically stable under ϕ, then $\bar{x}$ is asymptotically stable under F.*

We now consider $\mathbb{R}$-semiconjugate links as Liapunov functions. We assume that D is closed to reduce technical details, and make references to the following sets:

$$E = \{x \in D : H(F(x)) = H(x)\}$$

$$L = \{t \in H(D) : \phi(t) \leq t\}.$$

Also we define $S \doteq H^{-1}(I) \cap D$ where I is the *largest* invariant (under ϕ) subset of L. The following shows in particular that as long as E is nonempty, the link map H is always a Liapunov function for F on *some* invariant subset of D.

LEMMA 3 *Let F be a (D, H, ϕ)-semiconjugate map and assume that E is nonempty. Then:*

(a) *$H(E) \subset I$ is the set of all fixed points of ϕ and $F(E) \subset E$.*
(b) *S is closed and contains E, $F(S) \subset S$, and H is a Liapunov function for F on S.*
(c) *If ϕ has a fixed point $\bar{t}$ such that $\bar{t} \leq \phi(t) \leq t$ for $t \in (\bar{t}, \bar{t} + \delta)$ for some $\delta > 0$, then S has nonempty interior in the relative topology of D.*

Proof (a) We note that for each $x \in E$,

$$\phi(H(x)) = H(F(x)) = H(x)$$

so that $H(x)$ is a fixed point of ϕ. Conversely, if $t \in H(D)$ is a fixed point of ϕ, then $t = H(x)$ for some $x \in D$ and thus

$$H(F(x)) = \phi(t) = t = H(x)$$

so $x \in E$. It follows that $H(E)$ is the set of all fixed points of ϕ and thus, the maximality of I implies that $H(E) \subset I$. Further, if $x \in E$, then for $Y = F(x)$,

$$H(F(Y)) = \phi(\phi(H(x))) = \phi(H(x)) = H(Y)$$

from which it follows that $F(x) = Y \in E$.

(b) Since I is closed, so is S, and from Part (a),

$$E \subset H^{-1}(H(E)) \cap D \subset S.$$

Further, if $x \in S$, then $H(x) \in I$ so by the invariance of I it is true that $\phi(H(x)) \in I$. This means that $H(F(x)) \in I$, i.e. $F(x) \in S$, and furthermore,

$$H(F(x)) = \phi(H(x) \leq H(x)$$

so that H is a Liapunov function on S.

(c) The condition on ϕ ensures that the interval $(\bar{t}, \bar{t} + \delta)$ is contained in I. Therefore, S contains the D-open set $H^{-1}((\bar{t}, \bar{t} + \delta) \cap D$. $\square$

If one knows that a mapping F is $\mathbb{R}$-semiconjugate, then it is usually not necessary to invoke Liapunov theory. However, there are circumstances, as in the next theorem, when the two concepts can be used together in a nontrivial way.

THEOREM 4 *Let $G \in C(D, \mathbb{R}^m)$ and assume that there is a (D, H, ϕ)-semiconjugate map F and a nonempty subset $T \subset S$ such that $G(T) \subset T$ and*

$$H(G(x)) \leq H(F(x)), \quad x \in T.$$

Then the following are true:

(a) *H is a Liapunov function for G on T. Further, if $S = D$, then we may take $T = D$.*

(b) *Assume that G has an isolated fixed point $\bar{x}$ in T and that H is locally minimized at $\bar{x}$. Then the function*

$$V(x) \doteq H(x) - H(\bar{x}), \quad x \in S$$

is a positive definite Liapunov function relative to $\bar{x}$. In particular, $\bar{x}$ is a stable fixed point of F.

(c) *If H is locally maximized at $\bar{x}$ and ϕ is an odd function, then*

$$V(x) \doteq -H(x) + H(\bar{x}), \quad x \in S$$

is a positive definite Liapunov function relative to $\bar{x}$. In particular, $\bar{x}$ is a stable fixed point of F.

Proof

(a) This is clear from Lemma 3 and the definition of a Liapunov function.
(b) There is an open ball $B_\delta(\bar{x})$ such that $V(\bar{x}) = 0$ and $V(x) > 0$ for all other $x \in B_\delta(\bar{x})$. Further, since H is Liapunov, so is V, and it follows that V is a positive definite Liapunov function. The stability of $\bar{x}$ now follows (see Ref. [12, p.8] or [6, p185]).
(c) The argument is the same as that in Part (b), since F is $(D, -H, \phi)$-semiconjugate by Lemma 1. Thus $-H$ is Liapunov on S. □

Remark Theorem 4 applies in particular when H is an invariant. This special case appeared in Ref. [10] together with some interesting examples of its use. These examples, like their generalization in Theorem 4, do not pertain to *asymptotic* stability, a topic that was the subject of earlier discussion (e.g. Theorem 3 and Corollary 2 above).

SENSITIVITY AND CHAOS

In this section, we consider chaotic $\mathbb{R}$-semiconjugate maps. Here, the existence of a link to a one-dimensional map allows certain types of chaotic behavior for the associated map of the interval to be carried up to the higher dimensional mapping. In particular, certain conclusions of the Li-Yorke Theorem [13] are seen to be directly useful in this way. For convenience, we quote the statement of that theorem here as a lemma.

LEMMA 4 (Li-Yorke) *Let J be an interval and let $\phi \in C(J)$. Assume that there is a point $a \in J$ such that*

$$\phi^3(a) \leq a < \phi(a) < \phi^2(a),$$

or

$$\phi^3(a) \geq a > \phi(a) > \phi^2(a).$$

Then the following are true:

(a) *For each positive integer k there is a periodic point in J with period k.*
(b) *There is an uncountable set $S \subset J$ such that S contains no periodic points of ϕ and S satisfies the following conditions:*
  2.1. (b1) *For every $p, q \in S$ with $p \neq q$,*

$$\limsup_{n\to\infty} |\phi^n(p) - \phi^n(q)| > 0 \tag{3}$$

$$\liminf_{n\to\infty} |\phi^n(p) - \phi^n(q)| = 0.$$

  2.2. *For every $p \in S$ and periodic $q \in J$,*
$$\limsup_{n\to\infty} |\phi^n(p) - \phi^n(q)| > 0.$$

The set S above is called the *scrambled set* of ϕ.

THEOREM 5 *Let F be a (D, H, ϕ)-semiconjugate map, with $H \in C^1(D, \mathbb{R})$ and $D \subset \mathbb{R}^m$ a compact and convex set. If there are $p, q \in H(D)$ satisfying Eq. (3), then for each $x \in H_p^{-1}$ and $y \in H_q^{-1}$,*

$$\limsup_{n\to\infty} \|F^n(x) - F^n(y)\| > 0. \tag{4}$$

In particular, if ϕ has a scrambled set S, then trajectories starting in H_S^{-1} cannot converge to periodic points of F.

Proof By the mean value theorem [11, p.314], for all $u, v \in D$ and every $n \geq 1$,

$$|H(F^n(u) - H(F^n(v))| \leq \|F^n(u) - F^n(v)\| \sup_{w \in L(u,v)} \|\nabla H(w)\|$$

where $L(u, v)$ is the line segment that joins $F^n(u)$ to $F^n(v)$. Since D is convex, $L(u, v) \subset D$, so that

$$\sup_{w \in L(u,v)} \|\nabla H(w)\| \leq \sigma \doteq \sup_{z \in D} \|\nabla H(z)\|.$$

Note that $0 < \sigma < \infty$. Now for each pair x, y as in the statement of the theorem, $H(x) = p$ and $H(y) = q$, so we obtain

$$\|F^n(x) - F^n(y)\| \geq \frac{1}{\sigma} |H(F^n(x)) - H(F^n(y))| = \frac{1}{\sigma} |\phi^n(p) - \phi^n(q)|.$$

The proof of Eq. (4) is complete upon taking limit supremum and using Eq. (3) for ϕ.

Next, suppose that ϕ has a scrambled set $S \subset H(D)$. If y is a periodic point of F, then there is a positive integer k such that

$$\phi^{\mathrm{k}}(H(y) = H(F^k(y) = H(y)$$

Thus, $H(y)$ is a periodic point of ϕ, i.e. $H(y) \notin S$ and the preceding results apply. □

We show next that a factor's property of sensitivity to initial conditions is preserved by semiconjugate link maps under reasonable restrictions.

Corollary 3 *Let F be a (D, H, ϕ)-semiconjugate map, with $H \in C^1(D, \mathbb{R})$ and $D \subset \mathbb{R}^m$ a compact and convex set. If ϕ has sensitive dependence on initial conditions, then so does F.*

Proof Suppose that $x \in D$ and $\varepsilon > 0$, and let $t \doteq H(x) \in H(D)$. The set $H(B_\varepsilon(x) \cap D)$ is a nontrivial subinterval of $H(D)$ containing t. Due to the sensitivity of ϕ, there is $s \in H(B_\varepsilon(x) \cap D)$ such that

$$\mu \doteq \limsup_{n \to \infty} |\phi^n(t) - \phi^n(s)| > 0.$$

Now if we choose $y \in H_s^{-1} \cap B_\varepsilon(x)$, then Theorem 7 implies that

$$\limsup_{n\to\infty} \|F^n(x) - F^n(y)\| \geq \frac{\mu}{\sigma}$$

which proves that F has sensitive dependence on initial conditions. □

Notably absent from Theorem 5 are the periodic points that are so prominent in Lemma 4 and similar results for chaotic maps of the interval [4]. The next example shows what may happen in a compact set in $\mathbb{R}^2$.

Example 2 Let $D = \{(\rho, \theta) : \rho \in [0, 1], \theta \in \mathbb{R}\}$ be the unit disk in $\mathbb{R}^2$ and define $F \in C(D)$ as

$$F(\rho, \theta) \doteq [a\rho(1 - \rho),\ \theta + \alpha], \quad 2 < a < 4, \quad 0 \leq \alpha < 2\pi.$$

This map is semiconjugate to the logistic map $f(\rho) \doteq a\rho(1 - \rho)$ with a link $H(\rho, \theta) \doteq \rho$. If α/π is irrational, then it is not hard to prove that:

(a) Except for the fixed point at the origin, F has no periodic points in D;
(b) If the factor $f(\rho)$ is sensitive to initial conditions, then so is F (Corollary 3) and for each point (ρ_0, θ_0) in the compact annulus

$$\mathrm{D}_1 \doteq \{(\rho, \theta) : \rho \in [\mu, \gamma], \theta \in R\}, \quad \gamma \doteq \frac{a}{4}, \quad \mu \doteq a\gamma(1 - \gamma),$$

the trajectory $\{F^n(\rho_0, \theta_0)\}$ is dense in D_1 if $\{f^n(\rho_0)\}$ is dense in $[\mu, \gamma]$.

Each phase space trajectory of F in D_1 is, in effect, a time series of the logistic map that wraps around the annulus at a rate determined by α. In particular, periodic trajectories of f turn into almost periodic trajectories for F; the latter can be periodic (with a typically larger period than f) only if α/π is rational.

Example 3 This example illustrates the application of some of the preceding theory to a complicated map of the plane. Consider the function

$$F(x, y) \doteq \left(\sqrt{a^2x^2 + y^2} - a^2x^2 - y^2\right)[\cos(b(x + y), a\sin(b(x + y))]$$

where we assume that $0 < a \leq 4$ and $b > 0$. Set $H(x, y) \doteq \sqrt{a^2x^2 + y^2}$ and note that

$$H(F(x, y) = a\sqrt{a^2x^2 + y^2}|1 - \sqrt{\mathrm{a}^2x^2 + y^2}|$$

If $\phi(t) = at(1-t)$, then $\phi([0,1]) \subset [0,1]$ for $a \in (0,4]$, so that on the compact elliptical region

$$D \doteq \{(x,y) : a^2x^2 + y^2 \leq 1\}$$

F is (D, H, ϕ)-semiconjugate. With $t = \sqrt{a^2x^2+y^2}$, the fibers H_t^{-1} are concentric ellipses within D for $t \in (0,1]$, while $H_0^{-1} = \{(0,0)\}$ is just the origin. The behavior of ϕ in [0,1] is very familiar, and we use this knowledge to detail some of the dynamical properties of F in D.

Case (i) $0 < a \leq 1$. In this case the only fixed point of ϕ is 0, so by Lemma 2(a) the origin has to be the only fixed point of F in D. Further, the origin is clearly an isolated minimum for H, so by Corollary 2 the origin is asymptotically stable, attracting every point in D. Note that the stability, though not attractive, of the origin could also be infered from Theorem 4(b).

Case (ii): $1 < a \leq 3$. In this situation ϕ has an asymptotically stable fixed point $p \doteq 1 - 1/a$. Since p attracts every point of ϕ in (0,1), by Theorem 3 the ellipse H_p^{-1} attracts every point of $D_0 \doteq D^{\circ} - \{(0,0)\}$, where D° is the interior of D. It may also be noted that H_p^{-1} bifurcates from the origin as the parameter a crosses 1, in a manner that is entirely analogous to the Hopf bifurcation. Clearly the origin in D is unstable in this case; this fact also follows from Theorem 1(c).

The asymptotic behavior of F in this case is determined by the behavior of the restriction F_p of F to the attracting ellipse H_p^{-1}. If $(u,v) \in H_p^{-1}$, then $a^2u^2 + v^2 = p^2$ and it follows that

$$F_p(u,v) = p(1-p)[\cos(b(u+v)), a\sin(b(u+v))]$$

$$= \frac{p}{a}(b(u+v)), a\sin(b(u+v)].$$

Set $s \doteq b(u+v) \doteq G(u,v)$ to get

$$G(F_p(u,v) = bp\left(\frac{\cos s}{a} + \sin s\right) \doteq \psi(s) = \psi(G(u,v)).$$

Therefore, F_p is (H_p^{-1}, G, ψ)-semiconjugate where $s \in \mathbb{R}$. The fibers are the sets $G_s^{-1} \cap H_p^{-1}$, i.e. the intersections of lines $u + v = s/b$ with the ellipse H_p^{-1}. These fibers consist of one of two points each (if nonempty) so it is relatively simple to translate the dynamics of ψ into the dynamics of F_p in H_p^{-1}.

As for ψ, finding the maximum and the minimum of G on the ellipse H_p^{-1} is easily done using the Lagrange multiplier method; we get

$$|G(u,v)| \leq b\left(\frac{p}{a\sqrt{a^2+1}} + \frac{ap}{\sqrt{a^2+1}}\right) = \frac{b(a-1)\sqrt{a^2+1}}{a^2} \doteq r.$$

Thus, $|s \leq r|$. Also, using elementary calculus, we find that for all s,

$$|\psi(s)| \leq \psi(\arctan a) = r,$$

since $\cos(\arctan a) = 1/\sqrt{a^2+1}$ and $\sin(\arctan a) = a/\sqrt{a^2+1}$. Inequalities above imply that $\psi([-r,r]) = [-r,r]$.

Let us consider the case $a = 2$ for illustration. Then $p = 1/2$ and $r = b\sqrt{5}/4$, so that, e.g. the interval $I \doteq [-4,4]$ is invariant under ψ if $b \leq 16/\sqrt{5} < 7.16$. The map ψ has at least one and at most three fixed points in I. As b increases towards its upper bound, a complex sequence of fixed points and cycles of ψ appear in I, and these in turn imply a similar behavior for F_p on the fiber $H_{1/2}^{-1}$, i.e. the ellipse $4x^2 + y^2 = 1/4$. For b sufficiently close to 7 (e.g. $b > 6.2$) ψ is chaotic in I with a dense trajectory; hence, orbits of F in D_0 will approach a limit set that is dense in $H_{1/2}^{-1}$. Of course, chaos in a portion of $H_{1/2}^{-1}$ arises at even smaller values of b, e.g. $b > 4.85$ where a period 3 orbit for ψ exists in I and Lemma 4 applies.

Case (iii): $3 < a \leq 4$. In this case, the positive fixed point p becomes unstable and cycles emerge for ϕ in [0,1] in the familiar period doubling fashion. For each limit cycle of ϕ of length k, the behavior noted in Case (ii) above occurs for F^k (Theorem 3) on a particular invariant ellipse in D_0, or equivalently, for F on k concentric ellipses in D_0. For a sufficiently close to 4 (e.g. $a > 3.84$ which results in the appearance of a period 3 orbit) chaotic behavior occurs for ϕ. In particular, since D is convex, Corollary 3 implies that F is sensitively dependent on initial values. In fact, orbits of F eventually become dense in open sets in D_0 as a approaches 4 and as b gets larger.

Remark It is worth noting in the previous example that the intra-fiber semiconjugacy of F_p is a secondary semiconjugacy that proved useful. Since F_p is not injective on H_p^{-1}, it would be more diffcult to look at the intra-fiber situation in H_p^{-1} in terms of a mapping of a non-Euclidean manifold. We may refer to the secondary semiconjugacy of F_p on H_p^{-1} as an *imbedded semiconjugacy*.

A MODEL FROM ECONOMICS

A model of consumer behavior is represented by a bounded, smooth map $F \in C^1([0,\infty)^m)$ that is defined as follows:

$$F(x_1,\ldots,x_m) \doteq \begin{bmatrix} x_1 \exp\left(\alpha_1 - \sum_{j=1}^m c_{ij}x_j\right) \\ \vdots \\ x_m \exp\left(\alpha_m - \sum_{j=1}^m c_{ij}x_j\right) \end{bmatrix} \tag{5}$$

This is derived in Ref. [5], where it is also shown that if all the α_i are equal, then under certain conditions on the matrix $C = [c_{ij}]$, the unique positive equilibrium of F is a snap-back repeller and thus, the iterates of F are chaotic [15]; also see Ref. [18] for a more comprehensive treatment of this model.

The mapping in Eq. (5) can exhibit chaotic behavior even when there are no snap-back repellers and the α_i are not equal. Suppose that all rows of the matrix C are identical, i.e.

$$c_{ij} = c_i > 0, \quad i,j = 1,\ldots,m. \tag{6}$$

In this case, demand is attenuated by the same factor $\exp[-\sum_{j=1}^m c_j x_j]$ for each good i, and F takes the form

$$F(x_1,\ldots,x_m) = \mathrm{e}^{-c_1x_1-\cdots-c_mx_m}[\mathrm{e}^{\alpha_1}x_1,\ldots,\mathrm{e}^{\alpha_m}x_m].$$

This is economically feasible if the first m goods are similar to (and can be substituted for) each other. In particular, such comparable goods may compete for the consumer's attention through prices and other means; see the Remarks following the next theorem. We now give a complete description of the dynamics of Eq. (5) under conditions (6), which are complementary to those in Ref. [5].

THEOREM 6 *Assume that C satisfies Eq. (6). Then the following are true:*

(a) *The map F is $\mathbb{R}$-semiconjugate on $(0,\infty)^m$ to a linear mapping $\phi(t) = \omega t$ with $\omega \geq 1$ and $t > 0$. If $\omega > 1$, then each trajectory $\{F^n(x_0)\}$ approaches a subspace of $(0,\infty)^m$ obtained by setting one of the coordinates equal to zero.*

(b) *If none of the α_i are equal, and α_k is the largest among them, then $\{F^n(x_0)\}$ approaches a subset of the positive k-th axis. On the latter axis, F is topologically conjugate to the map $h(t) \doteq t\exp(\alpha_k - c_k t)$. Thus, if h is periodic or chaotic, then all positive trajectories $\{F^n(x_0)\}$ converge to, respectively, a periodic or chaotic attractor on the positive k-th axis.*

(c) *If $\alpha_i = \alpha > 0$ for $i = 1, \ldots, m$, then F is radial and $\mathbb{R}$-semiconjugate to the function $g(t) \doteq t\exp(\alpha - t)$. Further, for each vector x_0 of initial values, the restriction of F to the ray $R_{x_0} = \{rx_0 : r \geq 0\}$ is topologically conjugate to g. In particular, if α is large enough that g is chaotic (e.g. $\alpha \geq 3.13$) then F is chaotic.*

Proof We may suppose without loss of generality that α_1 is the least among α_i. Then, in particular, $(m-1)\alpha_1 \leq \alpha_2 + \cdots + \alpha_m$. Define

$$H(x_1, \ldots, x_m) \doteq \frac{x_2 x_3 \cdots x_m}{x_1^{m-1}}, \quad \omega \doteq \mathrm{e}^{\alpha_2 + \cdots + \alpha_m - (m-1)\alpha_1}.$$

Note that $\omega \geq 1$. Semiconjugacy to ϕ with H as link is readily verified, since

$$H(F(x_1, \ldots, x_m)) = \frac{e^{\alpha_2 + \cdots + \alpha_m}}{e^{(m-1)\alpha_1}} \frac{x_2 x_3 \cdots x_m}{x_1^{m-1}} = \omega H(x_1, \ldots, x_m).$$

Let x_0 be any point in $(0, \infty)^m$. Since all c_i are positive, F is bounded on $(0, \infty)^m$. In particular, there is $0 < \mu \leq b$ such that each component $F_i(x_i) \leq \mu$ for $x_i > 0$. It follows that the trajectory $\{F^n(x_0)\}$ is in $(0, \mu]^m$ for all $n \geq 1$. Writing $F^n(x_0) = (x_{1,n}, \ldots, x_{m,n})$, we note that

$$\frac{x_{2,n} x_{3,n} \cdots x_{m,n}}{x_{1,n}^{m-1}} = H(F^n(x_0)) = \phi^n(H(x_0)) = \frac{x_{2,0} x_{3,0} \cdots x_{m,0}}{x_{1,0}^{m-1}} \omega^n.$$

If $\omega > 1$, then $H(F^n(x_0)) \to \infty$ as $n \to \infty$, although the product $x_{2,n} x_{3,n} \cdots x_{m,n} \leq \mu^{m-1}$ is bounded. It follows that $x_{1,n} \to 0$ as $n \to \infty$ and therefore, $\{F^n(x_0)\}$ approaches the subspace $x_1 = 0$, as claimed in the statement of the theorem.

(b) We may suppose that $0 < \alpha_1 < \cdots < \alpha_m$ (so $k = m$). By applying Part (a) repeatedly to maps

$$\mathrm{e}^{-c_j x_j - \cdots - c_m x_m}[0, \ldots, 0, \mathrm{e}^{\alpha_j} x_j, \ldots, \mathrm{e}^{\alpha_m} x_m]$$

where $1 \leq j \leq m$, we observe that the only coordinate that does not vanish asymptotically is m. Further, the mapping $e^{-c_m x_m}[0,\ldots,0,e^{\alpha_m x_m}]$ is topologically conjugate to the map h. So if h is periodic or chaotic, then $\{F^n(x_0)\}$ exhibits the same type of behavior.

(c) In this case, F takes the following form:

$$F(x_1,\ldots,x_m) = \exp\left(\alpha - \sum_{j=1}^{m} c_j x_j\right)[x_1,\ldots,x_m]$$

which is obviously radial (i.e. trajectories are confined to rays through the origin in $[0, \infty)^m$ that contain the initial point). Defining $H(x_1,\ldots,x_m) \doteq \sum_{j=1}^{m} c_j x_j$, it is easy to see that

$$H(F(x_1,\ldots,x_m) = H(x_1,\ldots,x_m)e^{\alpha - H(x_1,\ldots,x_m)} = g(H(x_1,\ldots,x_m)$$

which shows F on $[0, \infty)^m$ to be $\mathbb{R}$-semiconjugate to g on $[0, \infty)$. The H-fibers are the parts of hyperplanes

$$\sum_{j=1}^{m} c_j x_j = t \geq 0$$

that are contained in the cone $[0, \infty)^m$ (clearly, all such fibers are compact). To complete the proof, note that for each x_0 the ray $\mathbb{R}_{x_0}$ is homeomorphic to $[0, \infty)$ and the restriction of F to $\mathbb{R}_{x_0}$ is topologically conjugate to g. The latter satisfies the conditions of Lemma 4 if α is sufficiently large (e.g. $\alpha \geq 3.13$) and is thus chaotic. $\square$

Remarks

1. Since by Lemma 4, the mappings g, h in Theorem 6 exhibit sensitivity to initial values for large enough α values, Corollary 3 immediately implies the same for Eq. (5) under the conditions of Theorem 6. However, in Theorem 6 we were able to use the specific form of the function F and its radial trajectories under hypothesis (6) to obtain information beyond sensitivity.
2. (Competition among similar goods) We argued above that Eq. (6) is economically feasible if the m goods are similar enough to be substitutable for each other. In such a case, the consumer may choose one among them and ignore the rest. According to Theorem 6, the consumer chooses the good with the largest α_i value (in the exceptional

case that two or more α_i have the same highest value, then the consumer chooses a mix of these latter goods, with the proportions in the mix arbitrarily determined by x_0).

In Ref. [5], the parameters α_i are defined as $\alpha_i = \ln(b/p_i) + \kappa_i$, where b is the budget and the κ_i are structural constants. Thus $\alpha_i < \alpha_j$ if and only if

$$\ln\left(\frac{p_j}{p_i}\right) < \kappa_j - \kappa_i \tag{7}$$

In particular, if goods i and j are viewed equally by the consumer (e.g. neither is a "brand name" or particularly prefered for some reason), then $\kappa_j - \kappa_i = 0$, so Eq. (7) implies that $p_j < p_i$. Thus, as might be expected, the consumer buys the lower priced good when all else is equal.

3. The mapping F is entirely similar to those used in Refs. [16,19] to model biological populations. Therefore, some of the results here apply to these population models. On the other hand, the exclusion aspect of Theorem 6 (i.e. goods with lower α_i value are dropped) holds under weaker conditions than those stated above. In Ref. [7], the problem of exclusion in a general class of competition models is discussed which include Eq. (5) as a special case.

CONCLUSION

The preceding study shows that $\mathbb{R}$-semiconjugate maps are abundant and include many familiar examples. Additional examples and related topics appear in Ref. [18] that may be helpful in starting further pursuits in this area.

Explicit link maps for arbitrary F are not generally easy to find, and it may be difficult to establish whether a given function is $\mathbb{R}$-semiconjugate (or not). In this respect, there is a similarity between $\mathbb{R}$-semiconjugates and Liapunov functions and invariants; however, semiconjugacy is a more flexible concept that is likely to have more immediate applications to models in social and natural sciences. The appeal is easy to understand: Semiconjugate relations translate higher dimensional problems into one-dimensional ones, thus covering a broad range of issues from basic stability to the occurrence of chaos and complicated behavior.

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