Periodic and Chaotic Behavior in a Class of Second Order Difference Equations

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Abstract. We discuss the occurrence and nature of periodic and chaotic behavior in a class of nonlinear second order difference equations and present criteria for the attractivity of the periodic solutions.

Consider the second order difference equation

$$x_{n+1} = ax_n + bx_{n-1} + f_n(x_n - cx_{n-1}) \tag{1}$$

where $f_n : \mathbb{R} \to \mathbb{R}$ is a given sequence of functions and a, b, c are real constants satisfying the conditions

$$c \neq 0, \ \ b + c(a - c) = 0. \tag{2}$$

Equations bearing some similarity to the autonomous version of (1) have been considered in the literature; see, e.g. [4, Sec.2.5]. These studies have concentrated on the global stability of equilibrium. Equation (1) also generalizes equations introduced in some of the heuristic business cycle models in macroeconomics. For example, special cases of (1) includes Hicks' model (its 2nd order case) and Goodwin's model (its discrete version) as well as Puu's discrete second order equation. A detailed comparative analysis of the mathematics behind these classical models appears in [7].

In this note we take a close look at Equation (1) and obtain criteria for the occurrence of attracting periodic solutions as well as conditions that imply the occurrence of chaotic behavior. We use the fact that under conditions (2), Equation (1) decomposes into a weakly coupled system of first order difference equations. Such decompositions are instances of semiconjugacy; see [7] for some background on this subject.

1 Main Results.

Equation (1) may be restated as

$$x_{n+1} - cx_n = (a - c)x_n + bx_{n-1} + f_n(x_n - cx_{n-1}). \tag{3}$$

If $a-c\neq 0$ then we may factor out $a-c$ and using (2) transform (3) into

$$t_{n+1} = (a-c)t_n + f_n(t_n) \doteq g_n(t_n) \tag{4a}$$

$$x_{n+1} = t_{n+1} + cx_n \tag{4b}$$

Note that if $a=c$ then $b=0$ by (2) so we may still obtain (4a) directly from Equation (1) with $g_n = f_n$.

Equations (4) define a triangular system of first order equations in the sense that the first equation is independent of the second. A general result on the structure of periodic solutions of (4) in terms of the periodic orbits of its two first order equations appears in [1] for the autonomous case, i.e. when $f_n = f_0$ for all n. Here, since the system (4) is specific in its second equation, we derive the needed relationships directly with f_n variable and also establish attractivity when $|c| < 1$.

For a given sequence of real numbers $\{t_n\}$, the general solution of (4b) is

$$x_n = c^n x_0 + \sum_{j=1}^{n} c^{n-j} t_j, \quad n \geq 1. \tag{5}$$

The sum in (5) is of convolution type but here the sequence $\{t_n\}$ is rarely given explicitly.

Lemma 1. *Assume that* $|c| \neq 1$.

(a) *Let* $\{t_n\}$ *be a periodic sequence of real numbers with period* p. *If* $\{\tau_0, \ldots, \tau_{p-1}\}$ *is one cycle of* $\{t_n\}$ *and*

$$\xi_i = \frac{1}{1-c^p} \sum_{j=0}^{p-1} c^{p-j-1} \tau_{(i+j) \bmod p} \quad i = 0, 1, \ldots, p-1 \tag{6}$$

then the solution $\{x_n\}$ *of Eq.(4b) with* $x_0 = \xi_0$ *and* $t_1 = \tau_0$ *has period* p *and* $\{\xi_0, \ldots, \xi_{p-1}\}$ *is a cycle of* $\{x_n\}$.

(b) *If for a given sequence* $\{t_n\}$ *of real numbers Eq.(4b) has a solution* $\{x_n\}$ *of period* p *then* $\{t_n\}$ *is periodic with period* p.

Proof. (a) With $x_0 = \xi_0$ and $t_1 = \tau_0$ we get $x_1 = cx_0 + t_1 = c\xi_0 + \tau_0$. Using (6) for ξ_0 gives

$$x_1 = \frac{c}{1-c^p}\left(\sum_{j=0}^{p-1} c^{p-j-1}\tau_j\right) + \tau_0 = \frac{1}{1-c^p}\left(\sum_{j=0}^{p-2} c^{p-j-1}\tau_{j+1} + \tau_0\right) = \xi_1$$

Proceeding in an inductive fashion, we show in this way that $x_i = \xi_i$ for $i = 0, \ldots, p-1$. Next, we show that $x_p = x_0$. Using (5) we have

$$x_p = c^p \xi_0 + \sum_{j=0}^{p-1} c^{p-j-1}\tau_j = \frac{c^p}{1-c^p}\sum_{j=0}^{p-1} c^{p-j-1}\tau_j + \sum_{j=0}^{p-1} c^{p-j-1}\tau_j = \xi_0 = x_0.$$

Hence $\{x_n\}$ is a solution of (4b) with period p, as claimed.

(b) Suppose that for a given sequence $\{t_n\}$ of real numbers, the corresponding solution of (4b) is periodic with period p. Let $t_1 = x_1 - cx_0$ and from (4b) obtain

$$t_{p+1} = x_{p+1} - cx_p = x_1 - cx_0 = t_1.$$

It follows that $\{t_n\}$ is periodic with period p.

Theorem 1. (periodic solutions and limit cycles)

(a) *Assume that $|c| \neq 1$ and let $\{t_n\}$ be a periodic solution of the first order equation (4a) with prime period p. If $\{\tau_0, \ldots, \tau_{p-1}\}$ is one cycle of $\{t_n\}$ then (1) has a solution $\{x_n\}$ of prime period p with a cycle $\{\xi_0, \ldots, \xi_{p-1}\}$ given by (6).*

(b) *Assume that the functions f_n are continuous. If $|c| < 1$ and $\{t_n\}$ is an attracting periodic solution of (4a) then $\{x_n\}$ is an attracting periodic solution of (1).*

Proof. (a) In light of Lemma 1(a) we only need to show that p is the prime or minimal period for $\{x_n\}$. Let q be the prime period of $\{x_n\}$ so that $q \leq p$. Then by Lemma 1(b) $\{t_n\}$ has period $q \geq p$ since p is the prime period for $\{t_n\}$. Therefore, $q = p$.

(b) Let $\{\tau_0, \ldots, \tau_{p-1}\}$ be an attracting cycle for (4a) with

$$\lim_{n\to\infty} t_{pn+i} = \tau_{i-1}, \quad i = 1, 2, \ldots, p.$$

Let $s_n = \sum_{j=1}^{n} c^{n-j} t_j$. Then by rearranging terms in the summation we find that

$$\begin{aligned} s_{pn} &= c^{pn-1}t_1 + c^{pn-2}t_2 \cdots + c^{pn-p}t_p \\ &\quad + c^{pn-p-1}t_{p+1} + c^{pn-p-2}t_{p+2} \cdots + c^{pn-2p}t_{2p} \\ &\quad + \cdots \\ &\quad + c^{p-1}t_{p(n-1)+1} + c^{p-2}t_{p(n-1)+2} \cdots + c^{pn-pn}t_{p(n-1)+p} \end{aligned}$$

$$\begin{aligned} &= c^{p-1}(c^{pn-p}t_1 + c^{pn-2p}t_{p+1} + \cdots + t_{pn-p+1}) \\ &\quad + c^{p-2}(c^{pn-p}t_2 + c^{pn-2p}t_{p+2} + \cdots + t_{pn-p+2}) \\ &\quad + \cdots \\ &\quad + c^{pn-p}t_p + c^{pn-2p}t_{2p} + \cdots + t_{pn} \\ &= \sum_{i=1}^{p} c^{p-i} \sum_{k=0}^{n-1} (c^p)^{n-k-1} t_{pk+i}. \end{aligned}$$

Now for $i = 1, 2, \ldots, p$ define

$$\sigma_n^i = \sum_{k=0}^{n-1} (c^p)^{n-k-1} t_{pk+i} \quad \text{and} \quad \gamma_n^i = \sum_{k=0}^{n-1} (c^p)^{n-k-1} \tau_{i-1} = \tau_{i-1} \sum_{k=0}^{n-1} c^{pk}.$$

Notice that

$$\begin{aligned}\left|\sigma_n^i - \frac{\tau_{i-1}}{1-c^p}\right| &\le \left|\sigma_n^i - \gamma_n^i\right| + \left|\gamma_n^i - \frac{\tau_{i-1}}{1-c^p}\right| \\ &\le \sum_{k=0}^{n-1} |c^p|^{n-k-1} |t_{pk+i} - \tau_{i-1}| + \left|\gamma_n^i - \frac{\tau_{i-1}}{1-c^p}\right|\end{aligned}$$

Clearly the second term on the right hand side approachs 0 as $n \to \infty$. As for the first term, let $m \ge 1$ and define

$$\delta = \max_{1\le i\le p}\left\{\sup_{k\ge 1} |t_{pk+i} - \tau_{i-1}|\right\} < \infty, \quad \delta_m^i = \sup_{k\ge m} |t_{pk+i} - \tau_{i-1}|$$

and observe that for $m < n$

$$\begin{aligned}\sum_{k=0}^{n-1} |c^p|^{n-k-1}|t_{pk+i} - \tau_{i-1}| &= \sum_{k=0}^{m-1} |c^p|^{n-k-1}|t_{pk+i} - \tau_{i-1}| + \sum_{k=m}^{n} |c^p|^{n-k-1}|t_{pk+i} - \tau_{i-1}| \\ &\le |c^p|^{n-m}\,\delta \sum_{k=0}^{m-1} |c^p|^k + \delta_m^i \sum_{k=m}^{n} |c^p|^{n-k-1}\end{aligned}$$

By taking n and m sufficiently large, each of the last two terms above can be made arbitrarily small. Therefore,

$$\lim_{n\to\infty} \sigma_n^i = \frac{\tau_{i-1}}{1-c^p} \quad i = 1, \ldots, p.$$

It follows that

$$\lim_{n\to\infty} x_{pn} = \lim_{n\to\infty} s_{pn} = \sum_{i=1}^{p} \frac{c^{p-i}\tau_{i-1}}{1-c^p} = \frac{c^{p-1}\tau_0 + \cdots + c\tau_{p-2} + \tau_{p-1}}{1-c^p}.$$

Therefore, $x_{pn} \to \xi_0$ as $n \to \infty$ with ξ_0 as in Lemma 1. From this and (4b) we obtain

$$\lim_{n\to\infty} x_{pn+1} = \lim_{n\to\infty} (t_{pn+1} + cx_{pn}) = \tau_0 + c\xi_0 = \xi_1.$$

Inductively, we find that $x_{pn+i} \to \xi_i$ for $i = 0, 1, \ldots, p-1$. This implies that $\{x_n\}$ is an attracting periodic solution of (1).

Examples. 1. Consider the difference equation

$$x_{n+1} = cx_n + \alpha_n(x_n - cx_{n-1})^q \tag{7a}$$

$$0 < |c|, |q| < 1, \;\; \alpha_{2m} = \alpha_0 > 0, \;\; \alpha_{2m+1} = \alpha_1 > 0, \;\; m = 0, 1, \ldots \tag{7b}$$

In this case, $g_n(t) = \alpha_n t^q$ for $t > 0$ and straightforward calculations show that all positive solutions of the first order equation $t_{n+1} = \alpha_n t_n^q$ converge to the 2-cycle

$$t_{2n} \to \alpha_0^{q/(1-q^2)} \alpha_1^{1/(1-q^2)} = \tau_0, \quad t_{2n+1} \to \alpha_0^{1/(1-q^2)} \alpha_1^{q/(1-q^2)} = \tau_1$$

Therefore, every solution of (7a) in the invariant region $\{(x,y) : x > cy\}$ converges to the attracting cycle

$$\xi_0 = \frac{c\tau_0 + \tau_1}{1-c^2}, \quad \xi_1 = \frac{c\tau_1 + \tau_0}{1-c^2}.$$

Note that in this example α_n can be taken as a sequence with any period p with slightly more calculating effort.

2. For the difference equation

$$x_{n+1} = c^2 x_{n-1} + \alpha(x_n - cx_{n-1})^q \tag{8}$$
$$\alpha > 0,\ 0 < |c| < 1,\ q = 2j/(2k+1) < 1,\ k \geq j \geq 1$$

we have $g_n(t) = g(t) = -ct + \alpha t^q$ for all real t and straightforward calculations show that all solutions of the first order equation $t_{n+1} = -ct_n + \alpha t_n^q$ converge to its unique fixed point

$$\bar{t} = \left(\frac{\alpha}{1+c}\right)^{1/(1-q)} = \left(\frac{\alpha}{1+c}\right)^{\frac{2k+1}{2(k-j)+1}}.$$

Hence the fixed point $\bar{x} = \bar{t}/(1-c)$ of (8) is globally attracting.

Now we consider conditions that imply chaotic behavior. For difference equations a chaotic solution is typically a non-periodic, oscillatory solution that is sensitive to initial values. See [2], [5], [6] and [7] for some background on this concept. For first order difference equations a more refined definition of chaotic solutions was given in [3]. We first give conditions for solutions of (1) to be uniformly bounded.

Lemma 2. (boundedness) *Let $|c| < 1$. If $\{t_n\}$ is a bounded sequence with $|t_n| \leq B$ for some $B > 0$, then the corresponding solution $\{x_n\}$ for Eq.(4b) is also bounded and there is a positive integer N such that*

$$|x_n| < |c| + \frac{B}{1-|c|} \quad \text{for all } n \geq N. \tag{9}$$

Proof. From (5) we obtain

$$|x_n| \leq |c|^n|x_0| + \sum_{j=1}^{n} |c|^{n-j}B < |c|^n|x_0| + B\sum_{k=0}^{\infty}|c|^k = |c|^n|x_0| + \frac{B}{1-|c|}.$$

Now if n is large enough, then $|c|^n|x_0| \leq |c|$ from which (9) follows.

In light of Lemma 2, the following result is easy to prove using Theorem 3.3.3 in [7].

Theorem 2. (chaotic behavior) *Assume that* $|c| < 1$ *and let* $f_n = f$ *for all* n *where* f *is continuous on an invariant closed interval* $[\mu, \nu]$ *on the line. If the first order equation (4a) is chaotic within* $[\mu, \nu]$ *then the second order equation (1) is chaotic in the following invariant compact, convex set in the plane:*

$$\{(x, y) : cx + \mu \leq y \leq cx + \nu\} \cap \left[-|c| - \frac{\max\{|\mu|, |\nu|\}}{1 - |c|}, |c| + \frac{\max\{|\mu|, |\nu|\}}{1 - |c|}\right]^2.$$

A straightforward example for illustration is the autonomous equation

$$x_{n+1} = cx_n + \alpha(x_n - cx_{n-1})(1 - x_n + cx_{n-1}) \tag{10}$$

where we have picked $a = c \in (-1, 1)$ and $f(t) = \alpha t(1 - t)$ in (1). As α varies in the interval [0,4] the familiar behavior of $f(t)$ on the interval [0,1] is translated via Theorems 1 and 2 into the analogous behavior for the solutions of (10) in the compact invariant region

$$\{(x, y) : cx \leq y \leq cx + 1\} \cap \left[-|c| - \frac{1}{1 - |c|}, |c| + \frac{1}{1 - |c|}\right]^2$$

in the plane. Similar observations apply to

$$x_{n+1} = c^2 x_{n-1} + (x_n - cx_{n-1})(\alpha - x_n + cx_{n-1}), \; 0 < |c| < 1 \leq \alpha \leq 4.$$

A less routine example is the one parameter family of autonomous rational equations

$$x_{n+1} = \frac{6x_n^2 - 5x_n x_{n-1} + x_{n-1}^2 + 4}{4x_n - 2x_{n-1}} - \alpha, \quad 0 < \alpha < 2 \tag{11}$$

obtained from (1) by setting $a = 3/2$, $c = 1/2$, $f(t) = 1/t - \alpha$ and $g(t) = t + f(t)$. For $\alpha > \sqrt{2}$ it can be shown that all iterates of g will eventually enter and remain in the invariant interval $[2 - \alpha, g(2 - \alpha)]$. With increasing value of α a sequence of bifurcations of periodic orbits ensues that progresses through the Sharkovski ordering. This behavior can then be translated into the analogous behavior for (11) using Theorems 1 and 2.

2 Extensions and future directions.

The results of the previous section can be readily extended to the equation

$$x_{n+1} = a_n x_n + b_n x_{n-1} + f_n(x_n - cx_{n-1}) \tag{12}$$

where the sequences $\{a_n\}$ and $\{b_n\}$ satisfy

$$c \neq 0, \; b_n + c(a_n - c) = 0, \text{ for all } n. \tag{13}$$

This would lead to essentially the same type of triangular system as (4) but with a slightly greater range of possibilities. Theorems and Lemmas 1 and 2 apply to Equation (12) essentially as they are presented above. A more distant generalization is to higher order equations of the following type:

$$x_{n+1} = c_n x_{n-k+1} + f_n(x_n - c_{n-1} x_{n-k}) \tag{14}$$

where k is a fixed positive integer, $\{c_n\}$ is a given sequence of real numbers and $\{f_n\}$ is a sequence of real valued functions all defined on a given interval I. Equation (14) is equivalent to the triangular system

$$t_{n+1} = f_n(t_n) \tag{15a}$$

$$x_{n+1} = t_{n+1} + c_n x_{n-k+1} \tag{15b}$$

Equation (15b) produces different results from those seen in Theorems 1 and 2 above, especially if c_n does not converge to a limit.

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