# A Class of Second Order Difference Equations Inspired by Euler's Discretization Method

**H. Sedaghat [1] and M. Shojaei[2]**

[1]Department of Mathematics
Virginia Commonwealth University
Richmond, Virginia 23284-2014, USA
hsedagha@vcu.edu

[2]Department of Applied Mathematics
Amirkabir University of Technology
No. 424, Hafez Ave. Tehran 15914, Iran
m_shojaeiarani@aut.ac.ir

**Abstract**

In this paper we study a multiparameter, nonlinear second order difference equation that is motivated by the Euler discretization of derivatives in the autonomous, second order differential equation derived from Newton's second law in mechanics. Our objective is mainly to analyze qualitative properties of the second order difference equation such as convergence, periodicity and chaos. With proper restrictions, two different semiconjugate factorizations facilitate our work.

**Keywords**: Euler's forward method, difference equation, global stability, persistent oscillations, periodicity, chaos, semiconjugates

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# 1 Introduction

Euler's simple method of rendering derivatives discrete in time has, over the centuries led to interesting classes of difference equations that have inspired a significant amount of research. The bulk of this research has been done during the past 30 years when digital computing has been available and increasingly accessible.

We start with the differential equation

$$x'' = \phi(x, x') \tag{1}$$

of classical mechanics. Using Euler's forward difference method (1) may be transformed into a second order difference equation. The time axis is made discrete as $t_0, t_1, t_2, \ldots$ with a fixed step size $\tau$ so that for each $n = 0, 1, 2, \ldots$ we have $t_{n+1} - t_n = \tau$. Then we estimate the first and second derivatives of the function $x(t)$ using forward differences as

$$\begin{aligned} x'(t_n) &\approx \frac{x(t_n + \tau) - x(t_n)}{\tau} = \frac{x_{n+1} - x_n}{\tau}, \\ x''(t_n) &\approx \frac{x'(t_n + \tau) - x'(t_n)}{\tau} = \frac{1}{\tau}\left[\frac{x_{n+2} - x_{n+1}}{\tau} - \frac{x_{n+1} - x_n}{\tau}\right]. \end{aligned}$$

Inserting these into (1) yields

$$\frac{1}{\tau}\left[\frac{x_{n+2} - x_{n+1}}{\tau} - \frac{x_{n+1} - x_n}{\tau}\right] = \phi\left(x_n, \frac{x_{n+1} - x_n}{\tau}\right). \tag{2}$$

This is the Euler discretization of (1) with a fixed step size. For sufficiently small $\tau$ and a wide range of functions $\phi$ Eq.(2) gives good estimates of the solutions of (1) over a chosen time interval $[a, b]$, in which case $t_0 = a$ and $t_N = b$ where $N$ is the largest index that one would consider. For more details on Euler's and other methods for solving differential equations a standard numerical analysis text such as [4] may be consulted.

In this paper we consider a slightly more general form of (2) that is capable of producing a much richer variety of asymptotic behavior through parameter adjustments. Our discussion is focused on the asymptotics of that general difference equation rather than on estimating solutions of (1) using (2).

Relabling $x_{n+1}/\tau$ as $y_n$ and rearranging terms in Equation (2) gives

$$y_{n+1} = 2y_n - y_{n-1} + \tau\phi(\tau y_{n-1}, y_n - y_{n-1}). \tag{3}$$

This is a special case of the second order difference equation

$$x_{n+1} = ax_n + bx_{n-1} + f(x_{n-1}, x_n - cx_{n-1}) \tag{4}$$

where the parameters $a, b, c$ are given real numbers and $f : \mathbb{R}^2 \to \mathbb{R}$ is a given function. This equation may also be written succinctly as

$$x_{n+1} = F(x_n, x_{n-1})$$

which is reminicent of (1) but with

$$F(u, v) = au + bv + f(v, u - cv). \tag{5}$$

It may be mentioned in passing that difference equations may be used directly as equations of motion in mechanics. An interesting study of this approach is given in [7].

## 2 General Concepts and Results

Each fixed point or equilibrium $\bar{x}$ of Eq.(4) is given by the equation

$$\bar{x} = a\bar{x} + b\bar{x} + f(\bar{x}, (1-c)\bar{x})$$

or equivalently,

$$(1 - a - b)\bar{x} = f(\bar{x}, (1-c)\bar{x}). \tag{6}$$

For example, if $f$ is a homogeneous function of degree $k$, i.e. $f(tu, tv) = t^k f(u, v)$ then the origin $\bar{x} = 0$ is a fixed point (if the domain of $f$ contains it) and for $k \neq 1$ another isolated fixed point

$$\bar{x} = \left[\frac{1 - a - b}{f(1, 1 - c)}\right]^{1/(k-1)}$$

may exist provided that the various quantities are well defined. For $k = 1$ if $1-a-b = f(1, 1-c)$ then all points on the diagonal (and in the domain of $f$) are fixed and thus none are isolated; otherwise, origin is the unique isolated fixed point if it is in the domain of $f$. We refer the reader to texts such as [1], [6], [12], [13] and [17] for basic background material, including the definitions of stability, asymptotic stability and instability for fixed points and cycles of difference equations.

## 2.1 Global stability

Let $\bar{x}$ be an isolated fixed point of (4) and let $F$ be the function defined in (5). If $f$ is continuously differentiable, then so is $F$ and through linearization it may be shown that $\bar{x}$ is locally stable if

$$\left|\frac{\partial F}{\partial u}(\bar{x}, \bar{x})\right| < 1 + \frac{\partial F}{\partial v}(\bar{x}, \bar{x}) < 2.$$

If $\bar{x}$ is the only fixed point of (4) then we also have the following general result in which $f$ is only assumed to be continuous.

**Theorem 1**. *Assume that $f$ is continuous on $\mathbb{R}^2$ and define $g(u, v) = f(v, u - cv)$. If $\bar{x}$ is the only fixed point of (4) and there is $\delta \in (0, 1)$ such that*

$$|a| + |b| + \delta < 1$$

*and*

$$|g(u, v) - g(\bar{x}, \bar{x})| \leq \delta \max\{|u - \bar{x}|, |v - \bar{x}|\}, \quad (u, v) \in \mathbb{R}^2 \tag{7}$$

*then $\bar{x}$ is globally asymptotically stable.*

*Proof.* Note that

$$\begin{aligned}|F(u, v) - \bar{x}| &= |F(u, v) - F(\bar{x}, \bar{x})| \\ &\leq |a||u - \bar{x}| + |b||v - \bar{x}| + |f(v, u - cv) - f(\bar{x}, (1 - c)\bar{x})| \\ &\leq (|a| + |b| + \delta) \max\{|u - \bar{x}|, |v - \bar{x}|\}.\end{aligned}$$

Therefore, by Corollary 4.3.5 in [17] $\bar{x}$ is globally asymptotically stable.

**Example 1.** Consider the difference equation

$$x_{n+1} = 0.5x_{n-1} + 1 + \sqrt[3]{x_n + 1}. \tag{8}$$

which is a special case of (4) with $a = c = 0$, $b = 0.5$ and $f(v,u) = 1+\sqrt[3]{u+1}$. Equation (8) has a unique fixed point $\bar{x}$ with a value of approximately 5.8. Inequality (7) holds for (8) because

$$\begin{aligned}
|g(u,v) - g(\bar{x},\bar{x})| &= |f(v,u) - f(\bar{x},\bar{x})| \\
&= |\sqrt[3]{u+1} - \sqrt[3]{\bar{x}+1}| \\
&= \frac{|u-\bar{x}|}{\left|\sqrt[3]{(u+1)^2} + \sqrt[3]{u+1}\sqrt[3]{\bar{x}+1} + \sqrt[3]{(\bar{x}+1)^2}\right|}
\end{aligned}$$

and using the approximate value of $\bar{x}$ we find that the denominator of the fraction above exceeds $5/2$ for all real $u$, so we may set $\delta = 2/5 = 0.4$ in Theorem 1 and conclude that the unique fixed point $\bar{x}$ of (8) is globally asymptotically stable.

**Remark**. Condition (7) in particular holds if $f$ is a contraction on the plane, i.e.

$$|f(x,y) - f(s,t)| \le \gamma \max\{|x-s|, |y-t|\}, \quad \gamma < \frac{1}{1+|c|}. \tag{9}$$

If (9) holds then

$$\begin{aligned}
|g(u,v) - g(\bar{x},\bar{x})| &= |f(v, u-cv) - f(\bar{x}, (1-c)\bar{x})| \\
&\le \gamma \max\{|v-\bar{x}|, |u-cv-(1-c)\bar{x}|\} \\
&\le \gamma \max\{|v-\bar{x}|, |u-\bar{x}| + c|v-\bar{x}|\} \\
&\le \gamma(1+|c|) \max\{|u-\bar{x}|, |v-\bar{x}|\}
\end{aligned}$$

and (7) follows if $\gamma(1+|c|) < 1$.

Inequality (7) is essentially weaker than (9) because the latter inequality is assumed to hold globally whereas the former only requires sufficient flatness of the graph of $g$ near the point $(\bar{x},\bar{x})$. For instance, in Example 1 above the function $f(v,u)$ is not a contraction on the plane (in fact, $f$ is not differentiable when $u = -1$), but near the point $(\bar{x},\bar{x})$ the cylindrical surface $1 + \sqrt[3]{u+1}$ flattens out significantly. See [17, Sec.4.3] for further general remarks with regard to the geometric aspects of Theorem 1.

## 2.2 Persistent oscillations

Suppose that $\bar{x}$ is an isolated fixed point of $F$. Then the following inequalities imply that both eigenvalues of the linearization of (4) have modulus greater than 1 (see, e.g. [17, p.168]):

$$\left|\frac{\partial F}{\partial v}(\bar{x},\bar{x})\right| > 1, \quad \left|\frac{\partial F}{\partial v}(\bar{x},\bar{x}) - 1\right| > \left|\frac{\partial F}{\partial u}(\bar{x},\bar{x})\right|. \tag{10}$$

These inequalities and a basic result from [17, p.166] imply the next theorem. We say that a bounded solution of (4) will *oscillate persistently* if it has at least two distinct limit points.

**Theorem 2**. *Suppose that inequalites (10) hold at an isolated fixed point $\bar{x}$ of (4) and further, the equation*

$$a\bar{x} - bv + f(v, \bar{x} - cv) = \bar{x} \tag{11}$$

*has no real solution $v \neq \bar{x}$. Then all bounded, non-constant solutions of (4) oscillate persistently.*

**Example 2**. In Equation (4) let $c = 1$ and assume that $f(v, u - cv) = g(u - v) - bv$ for some real function $g$ so that (4) takes the form

$$x_{n+1} = ax_n + g(x_n - x_{n-1}). \tag{12}$$

We make the following additional assumptions:

(a) $0 \leq a < 1$;

(b) $g$ is continuous, nondecreasing and bounded below on $\mathbb{R}$;

(c) There is $\alpha \in (0, 1)$ and $t_0 > 0$ such that $g(t) \leq \alpha t$ for all $t > t_0$;

Then (12) has a unique fixed point $\bar{x} = g(0)/(1-a)$ and all of its solutions are bounded ([17, T4.1.1]). If we also assume that:

(d) $g$ is continuously differentiable at 0 with $g'(0) > 1$,

then every solution of (12) with at least one initial value different from $\bar{x}$ oscillates persistently.

To prove this last assertion, we note with regard to (10) that if $F(u, v) = au + g(u - v)$ then

$$\left|\frac{\partial F}{\partial v}(\bar{x},\bar{x})\right| = g'(0) > 1,$$

$$\left|\frac{\partial F}{\partial v}(\bar{x},\bar{x}) - 1\right| = |-g'(0) + 1| > |a + g'(0)| = \left|\frac{\partial F}{\partial u}(\bar{x},\bar{x})\right|.$$

Further, Eq.(11) takes the form

$$g(\bar{x} - v) = g(0)$$

whose only solution by assumptions (b) and (d) is $v = \bar{x}$. Thus by Theorem 2 all nontrivial solutions of (12) oscillate persistently. A specific example of $g$ that satisfies (a)-(d) above is $g(t) = \tan^{-1} \beta t$ with $\beta > 1$.

# 3 Semiconjugate factorizations

A second order equation such as (4) may be viewed as a mapping of the two dimensional space upon unfolding in vector form. Such an equation in principle admits factorizations into two mappings of the real line ([17]). When a difference equation is stated in scalar form as (4) is, we may obtain the semiconjugate factors through substitutions. We obtain our first (of two) such factorization by subtracting the term $cx_n$ from both sides of (4) and rearrange terms to obtain

$$x_{n+1} - cx_n = (a - c)x_n + bx_{n-1} + f(x_{n-1}, x_n - cx_{n-1}).$$

We now make two assumptions:

**(SC1)** *$f$ is linear in the first coordinate,* i.e., $f(u, v) = du + g(v)$ where $d$ is a real number (possibly 0) and $g$ is a function.

Then the terms on the right hand side of the preceding expression may be rearranged to give

$$x_{n+1} - cx_n = (a - c)x_n + (b + d)x_{n-1} + g(x_n - cx_{n-1}). \tag{13}$$

Our second assumption is as follows:

**(SC2)** *The constants $a, b, c, d$ satisfy*

$$b + d = c(c - a). \tag{14}$$

Note that the constant values in Eq.(3) namely, $a = 2$, $b = -1$, $c = 1$ satisfy condition (14) if we assume hypothesis (SC1) above with $d = 0$. The case $d = 0$ corresponds to the function $\phi$ in (1) being “space independent”.

Under assumptions (SC1) and (SC2), we substitute $t_n = x_n - cx_{n-1}$ into Eq.(13) and obtain the equivalent system of first order difference equations

$$t_{n+1} = (a-c)t_n + g(t_n) \tag{15a}$$

$$x_{n+1} = cx_n + t_{n+1} \tag{15b}$$

These two first order equations represent the first semiconjugate factorization of (4) that we discuss here. We may call this type of factorization *semiconjugacy by sums*. For reference, we note that under the assumptions (SC1) and (SC2) Eq.(4) takes the following form:

$$x_{n+1} = ax_n + c(c-a)x_{n-1} + g(x_n - cx_{n-1}). \tag{16}$$

The second type of semiconjugate factorization requires the following assumption:

**(SC3)** *$f$ is homogeneous of degree one, i.e. $f(tu, tv) = tf(u, v)$ for all real values of $t$ for which $f$ is defined.*

Examples of mappings that satisfy (SC3) include linear maps $f(u, v) = \alpha u + \beta v$ as well as the following:

$$|\alpha u + \beta v|, \quad \sqrt{\alpha u^2 + \beta uv + \gamma v^2}, \quad \frac{\alpha u^2 + \beta uv + \gamma v^2}{\delta u + \xi v}$$

under suitable domain restrictions where necessary. Under (SC3) we may divide both sides of (4) by $x_n$ to obtain

$$\begin{aligned}\frac{x_{n+1}}{x_n} &= a + b\frac{x_{n-1}}{x_n} + \frac{1}{x_n}f(x_{n-1}, x_n - cx_{n-1})\\ &= a + b\frac{x_{n-1}}{x_n} + \frac{x_{n-1}}{x_n}f(1, \frac{x_n}{x_{n-1}} - c).\end{aligned}$$

In the preceding expression we substitute

$$r_n = \frac{x_n}{x_{n-1}} \tag{17}$$

to obtain

$$r_{n+1} = a + \frac{b + f(1, r_n - c)}{r_n}.$$

Note that this is a first order difference equation that together with (17) gives the following factorization of (4) that we call *semiconjugacy by ratios*:

$$r_{n+1} = a + \frac{b + f(1, r_n - c)}{r_n} \tag{18a}$$

$$x_{n+1} = r_{n+1} x_n \tag{18b}$$

The essential or structural difference between (15) and (18) is in their second equations (15b) and (18b), respectively. These latter equations translate the dynamics of fibers given by equations (15a) and (18a) in different ways into behaviors for solutions of (4). For instance, even if in both (15a) and (18a) all solutions converge to a unique fixed point, the resulting behaviors for (4) will be quite different in the two cases because (15b) and (18b) give different outcomes.

We note that *these two semiconjugate types are essentially complementary, becasuse if both of the assumptions (SC1) and (SC3) hold then $f$ is linear, which reduces (4) to a linear equation*. It may also be mentioned that equations (15) and (18) are examples of "triangular" systems; these types of systems have been studied at a general level for their periodic structure; see, e.g. [2] and [11]. Since we are dealing with somewhat specific systems, we can obtain substantial information (some of which go beyond periodicity) without having to appeal to the more general results.

## 3.1 Semiconjugacy by sums

Throughout this section, we assume that (SC1) and (SC2) hold. We demonstrate that solutions of Eq.(16) exhibit a wide variety of dynamic behaviors ranging from periodic to chaotic. Solutions of Eq.(15a) are orbits, or sequences of iterates $\{t_n\} = \{h^n(t_0)\}$ where

$$h(t) = (a - c)t + g(t), \quad t_0 = x_0 - cx_{-1}.$$

For each given sequence of real numbers $\{t_n\}$, the general solution of (15b) is

$$x_n = c^n x_0 + \sum_{j=1}^{n} c^{n-j} t_j, \quad n \geq 1. \tag{19}$$

For nontriviality, we assume that $c \neq 0$ in the sequel. The sum in (19) is of convolution type but here the sequence $\{t_n\}$ is rarely given in explicit form.

Often we only know some of the qualitative features of $\{t_n\}$ as a solution of (15a), e.g. whether it is stable or periodic. We use (19) to translate those qualitative properties into properties of solutions of Eq.(16).

**Lemma 1**. *Assume that* $|c| \neq 1$.

(a) *Let* $\{t_n\}$ *be a periodic sequence of real numbers with period* $p$*, and let* $\{\tau_0, \ldots, \tau_{p-1}\}$ *be one cycle of* $\{t_n\}$*. If*

$$\xi_i = \frac{1}{1-c^p}\sum_{j=0}^{p-1} c^{p-j-1}\tau_{(i+j) \bmod p} \quad i = 0, 1, \ldots, p-1 \tag{20}$$

*then the solution* $\{x_n\}$ *of Eq.(15b) with* $x_0 = \xi_0$ *and* $t_1 = \tau_0$ *has period* $p$ *and* $\{\xi_0, \ldots, \xi_{p-1}\}$ *is a cycle of* $\{x_n\}$*.*

(b) *If for a given sequence* $\{t_n\}$ *of real numbers Eq.(15b) has a solution* $\{x_n\}$ *of period* $p$ *then* $\{t_n\}$ *is periodic with period* $p$*.*

*Proof.* (a) With $x_0 = \xi_0$ and $t_1 = \tau_0$ we find that

$$x_1 = cx_0 + t_1 = c\xi_0 + \tau_0$$

which upon using (20) for $\xi_0$ gives

$$x_1 = \frac{c}{1-c^p}\left(\sum_{j=0}^{p-1} c^{p-j-1}\tau_j\right) + \tau_0 = \frac{1}{1-c^p}\left(\sum_{j=0}^{p-2} c^{p-j-1}\tau_{j+1} + \tau_0\right) = \xi_1$$

Proceeding in an inductive fashion, we show in this way that $x_i = \xi_i$ for $i = 0, \ldots, p-1$. Next, we show that $x_p = x_0$. Using (19) we have

$$x_p = c^p\xi_0 + \sum_{j=0}^{p-1} c^{p-j-1}\tau_j = \frac{c^p}{1-c^p}\sum_{j=0}^{p-1} c^{p-j-1}\tau_j + \sum_{j=0}^{p-1} c^{p-j-1}\tau_j = \xi_0 = x_0.$$

Hence $\{x_n\}$ is a solution with period $p$, as claimed.

(b) Suppose that for a given sequence $\{t_n\}$ of real numbers, the corresponding solution of (15b) is periodic with period $p$. Let $t_1 = x_1 - cx_0$ and from (15b) obtain

$$t_{p+1} = x_{p+1} - cx_p = x_1 - cx_0 = t_1.$$

It follows that $\{t_n\}$ is periodic with period $p$.

**Theorem 2** (periodic solutions) *Assume that* $|c| \neq 1$ *and let* $\{t_n\}$ *be a periodic solution of the first order equation (15a) with prime period* $p$. *If* $\{\tau_0, \ldots, \tau_{p-1}\}$ *is one cycle of* $\{t_n\}$ *then (16) has a solution* $\{x_n\}$ *of prime period* $p$ *with a cycle* $\{\xi_0, \ldots, \xi_{p-1}\}$ *given by (20).*

*Proof.* By Lemma 1(a) the periodic sequence $\{t_n\}$ generates a periodic solution of (15b). By construction, this periodic solution is also a solution of (16) if $\{t_n\}$ is a solution of (15a). It remains to show that $p$ is the prime or minimal period. Let $q$ be the prime period of $\{x_n\}$ so that $q \leq p$. Then by Lemma 1(b) $\{t_n\}$ has period $q \geq p$ since $p$ is the prime period for $\{t_n\}$. Therefore, $q = p$.

The periodic orbits in iterates of a continuous one dimensional map of an interval satisfy the following ordering known as the Sharkovski ordering of cycles; see [5], [17], [20].

$$3 \rhd 5 \rhd 7 \rhd \cdots 2 \cdot 3 \rhd 2 \cdot 5 \rhd \cdots 2^2 \cdot 3 \rhd 2^2 \cdot 5 \rhd \cdots 2^k \rhd 2^{k-1} \rhd \cdots \rhd 2 \rhd 1.$$

In particular, if a continuous mapping has an orbit with period 3, then it has periodic orbits with all possible periods. The following is an immediate consequence of Theorem 2.

**Corollary 1**. (coexisting periods) *If Eq.(15a) has a solution of period 3 (e.g. satisfies the Li-Yorke conditions; see[14]) then Eq.(16) has periodic solutions with all possible periods that are arranged in the Sharkovski ordering.*

**Lemma 2** (boundedness) *Let* $|c| < 1$. *If* $\{t_n\}$ *is a bounded sequence with* $|t_n| \leq B$ *for some* $B > 0$, *then the corresponding solution* $\{x_n\}$ *for Eq.(15b) is also bounded and* $|x_n| < |x_0| + B/(1 - |c|)$ *for all* $n \geq 1$. *Further, there is a positive integer* $N$ *such that*

$$|x_n| \leq 1 + \frac{B}{1 - |c|} \quad \textit{for all } n \geq N. \tag{21}$$

*Proof.* From (19) we obtain

$$|x_n| \leq |c|^n |x_0| + \sum_{j=1}^{n} |c|^{n-j} B < |x_0| + B \sum_{k=0}^{\infty} |c|^k = |x_0| + \frac{B}{1 - |c|}.$$

From the preceding inequalities it also follows that if $n$ is large enough, then $|c|^n |x_0| \leq 1$ from which (21) follows.

In the literature, the term “chaotic” usually indicates non-periodic, oscillatory behavior that is sensitive to initial values. See [5], [14], [15] and [17] for some background on this concept. In particular, Theorem 3.3.3 in [17] implies the following:

**Theorem 3** (chaotic behavior) *Assume that* $|c| < 1$ *and that the first order equation (15a) is chaotic within an invariant closed interval* $[A, B]$ *on the line. Then the second order equation (16) is chaotic in the following invariant compact, convex set in the plane*

$$\{(u,v) : cu + A \leq v \leq cu + B\} \cap \left[-1 - \frac{\max\{|A|,|B|\}}{1-|c|}, 1 + \frac{\max\{|A|,|B|\}}{1-|c|}\right]^2 .$$

**Example 3**. Consider the one parameter family of rational second order equations

$$x_{n+1} = \frac{6x_n^2 - 5x_n x_{n-1} + x_{n-1}^2 - \alpha(2x_n - x_{n-1}) + 4}{4x_n - 2x_{n-1}}, \quad 0 < \alpha < 4 \qquad (22)$$

which is obtained from (16) by setting $a = 3/2$, $c = 1/2$ and $g(t) = 1/t - \alpha/2$. We may write the mapping $h$ in (15a) as

$$h(t) = t - \frac{\alpha}{2} + \frac{1}{t}.$$

$h$ has a positive fixed point $\bar{t} = 2/\alpha$ and a positive global minimum value of $h_{\min} = 2 - \alpha/2$ at $t = 1$, so $h(t) > 0$ for all $t > 0$ if $0 < \alpha < 4$. If $\alpha \leq 2\sqrt{2}$ then the fixed point $\bar{t}$ attracts all positive iterates of $h$. To see this, we note that if

$$h^2(t) = h(h(t)) = t + \frac{1}{t} - \alpha + \frac{t}{t^2 - (\alpha/2)t + 1}$$

then

$$h^2(t) - t = \frac{-\alpha[t - (2/\alpha)][t^2 - (\alpha/2)t + 1/2]}{t[t^2 - (\alpha/2)t + 1]}$$

Since both of the quadratic terms in the preceding expression are positive for all $t$ if $\alpha \leq 2\sqrt{2}$ it follows that $h^2(t) > t$ for $0 < t < 2/\alpha$ so by Theorem 2.1.2 of [17] $\bar{t} = 2/\alpha$ is a global attractor of all positive orbits of $h$. Thus when $\alpha \leq 2\sqrt{2}$, Eq.(22) has a fixed point $\bar{x} = \bar{t}/(1-c) = 4/\alpha$ (this can also be computed directly from (22)) which attracts all positive solutions of (22).

The attractivity of $\bar{x}$ can be established directly using (19) or by observing that the invariant fiber $v = u/2 + \bar{t}$ is attracting and all points on this fiber approach $\bar{x}$.

If $\alpha > 2\sqrt{2}$ then it is not hard to see that all iterates of $h$ will eventually enter and remain in the invariant interval $[2 - \alpha/2, h(2 - \alpha/2)]$ and with increasing value of $\alpha$, a sequence of bifurcations of periodic orbits ensues that progresses through the Sharkovski ordering to lead to a period 3 orbit at about $\alpha = 3.48$. Since for each point $(u, v)$ in the plane, $v - u/2 = t$, it follows that for $2 < \alpha < 4$, each solution of Eq.(22) eventually enters the invariant bounded set

$$\left\{(u,v) : \frac{u}{2} + 2 - \frac{\alpha}{2} \leq v \leq \frac{u}{2} + h\left(2 - \frac{\alpha}{2}\right)\right\} \cap (0, 1 + 2h(2 - \alpha/2)]^2 \quad (23)$$

For $\alpha$ up to 3.48, each period-$p$ orbit of $h$ uniquely generates a period-$p$ trajectory of (22) in the set (23) according to (20). For instance, solving the equation

$$t = h^2(t) = t + \frac{1}{t} - \alpha + \frac{2t}{2t^2 - \alpha t + 2}$$

for $\alpha > 2\sqrt{2}$ yields the period-2 orbit

$$\tau_0 = \frac{\alpha - \sqrt{\alpha^2 - 8}}{4}, \quad \tau_1 = \frac{\alpha + \sqrt{\alpha^2 - 8}}{4}.$$

Now using (20) we obtain a period-2 solution of (22) as

$$\xi_0 = \frac{c\tau_0 + \tau_1}{1 - c^2} = \frac{3\alpha + \sqrt{\alpha^2 - 8}}{6},$$
$$\xi_1 = \frac{c\tau_1 + \tau_0}{1 - c^2} = \frac{3\alpha - \sqrt{\alpha^2 - 8}}{6}.$$

On the other hand, if $\alpha$ is close enough to 4, e.g. $\alpha > 3.48$, then the trajectories of (22) will exhibit sensitivity to initial conditions and undergo nonperiodic oscillations within the set (23).

**Remark.** As the preceding results show, when $|c| < 1$ then Eq.(16) rather faithfully duplicates the qualitative behavior of solutions of Eq.(15a). When $|c| > 1$ then it is evident from (19) that solutions of (16) are typically unbounded and thus any bounded solutions (including periodic ones) of (16) that correspond to bounded behavior in (15a) must be unstable. Therefore, different qualitative behaviors will be exhibited by (16) and (15a) when $|c| > 1$.

The relationship between the solutions of (16) and (15a) in cases $c = \pm 1$ is also different from $|c| \neq 1$. It is worth noting that if $\phi$ is linear in its first coordinate, then (2) becomes a special case of Eq.(16) with $c = 1$ (upon re-scaling the mapping $\phi$). With $c = 1$ (19) changes into a sum (or discrete integral) so the nature of solutions of (16) will be qualitatively different from that of the solutions of (15a). In particular, a periodic solution of (15a) with a cycle $\{\tau_0, \ldots, \tau_{p-1}\}$ can translate into periodic solutions of (16) if and only if $\sum_{i=0}^{p-1} \tau_i = 0$. Thus there is a significant loss of periodicity in the second order equation. For more details on the case $c = 1$ in certain special cases of Eq.(16) see [16], [17] and [19].

## 3.2 Semiconjugacy by ratios

In this section we assume only that (SC3) holds, i.e., $f$ is homogeneous of degree 1. We do not put any further restrictions such as (14) on the coefficients $a, b, c$ in Eq.(4). The solutions of (4) under (SC3) exhibit a very different type of behavior than was the case with (SC1) and (SC2). In order to avoid singularities in (18), solutions of (4) that contain zero may be singled out and treated differently.

Since for each given solution $\{r_n\}$ of (18a) the corresponding solution of (4) is obtained from (18b) as

$$x_n = r_n r_{n-1} \cdots r_0 x_{-1} \tag{24}$$

the following result is easy to establish.

**Theorem 4**. *Let $x_0, x_{-1}$ be given initial values with $x_{-1} \neq 0$ and let $\{r_n\}$ be a solution of Eq.(18a) with $r_0 = x_0/x_{-1}$. Assume that $r_n$ is a real number for all $n \geq 0$ (e.g. $r_n$ is contained in an invariant set of (18a) which does not contain 0). Then the following is true:*

*(a) If there is $n_0 \geq 0$ such that $|r_n| < 1$ for all $n \geq n_0$ then the corresponding solution $\{x_n\}$ of (4) converges to 0.*

*(b) If there is $n_0 \geq 0$ such that $|r_n| > 1$ for all $n \geq n_0$ then the corresponding solution $\{x_n\}$ of (4) is unbounded.*

*(c) If $\{r_n\}$ converges to a cycle $\{\rho_1, \ldots, \rho_p\}$ with $\rho_1 \rho_2 \cdots \rho_p = 1$ then $\{x_n\}$ converges to a periodic solution of (4) with period $p$.*

*(d) If $\{x_n\}$ converges to a nonzero value, then the infinite product $\prod_{n=0}^{\infty} r_n$ is convergent; in particular, there are disjoint, infinite sets of positive integers $K_0$ and $K_1$ such that $|r_n| < 1$ (or $|x_n| < |x_{n-1}|$) for $n \in K_0$ and $|r_n| > 1$ (or $|x_n| > |x_{n-1}|$) for $n \in K_1$.*

The following corollary illustrates the various points made in Theorem 4 as well as the fact that semiconjugates can sometimes be useful in the derivation of solutions in quantitatively explicit form. If we set $\beta = a - \alpha c$ for arbitrary $a, c$ then the difference equation given in next corollary is a version of (4) with $b = 0$ and

$$f(v, u - cv) = \frac{\alpha(u - cv)^2}{v} + \alpha c(u - cv).$$

**Corollary 2**. *Consider the following rational difference equation*

$$x_{n+1} = \frac{\alpha x_n^2 + \beta x_n x_{n-1}}{x_{n-1}}, \quad x_{-1} \neq 0 \tag{25}$$

(a) *If for $n \geq 0$ we set $x_{n+1} = 0$ when $x_{n-1} = x_n = 0$ then the general solution of (25) is given in explicit form as*

$$x_n = x_0 \prod_{k=1}^{n} \left[ \left( \frac{x_0}{x_{-1}} - \frac{\beta}{1-\alpha} \right) \alpha^k + \frac{\beta}{1-\alpha} \right], \quad \alpha \neq 1 \tag{26a}$$

$$x_n = x_0 \prod_{k=1}^{n} \left( \frac{x_0}{x_{-1}} + \beta k \right), \quad \alpha = 1. \tag{26b}$$

(b) *If $|\alpha| < 1$ and $|\beta| < 1 - \alpha$ then every solutions of (25) converges to zero.*

(c) *If $|\alpha| > 1$ or $|\beta| > 1 - \alpha$ then almost all solutions of (25) are unbounded.*

(d) *If $|\alpha| = 1$ then certain solutions of (25) are periodic with period 2, hence bounded and not converging to zero. There are also both unbounded solutions and solutions converging to zero in this case.*

*Proof.* (a) Since the function on the right hand side of (25) is homogeneous of degree 1 in $x_n$ and $x_{n-1}$, we may divide by $x_n$ and use the $r_n$ notation to obtain

$$r_{n+1} = \alpha r_n + \beta. \tag{27}$$

The solution of this linear first order equation is easily obtained and then transformed into the appropriate form in (26) using (24) to complete the proof.

(b) If $|\alpha| < 1$ then every solution of (27) converges to the unique fixed point $\beta/(1-\alpha)$. This point has absolute value less than unity if $|\beta| < 1-\alpha$ so by Theorem 4(a) every solution of (25) converges to zero.

(c) If $|\alpha| > 1$ then almost every solution of (27) is unbounded exponentially, where as if $|\alpha| < 1$ but $|\beta| > 1-\alpha$ then every solution of (27) converges to the fixed point $\beta/(1-\alpha)$ with magnitude greater than unity. In either case, the conclusion follows upon an application of Theorem 4(b).

(d) In this case it is more efficient to use Eq.(26). If $\alpha = 1$ and $\beta \neq 0$ then using (26b) it is clear that every solution of (25) is unbounded whereas if $\beta = 0$ then solutions with initial values satisfying $|x_0| \leq |x_{-1}|$ are bounded (periodic if $x_0 = -x_{-1} \neq 0$).

If $\alpha = -1$ then consider solutions with initial values satisfying $x_0/x_{-1} = 1 + \beta/2$. For these solutions (26a) reduces to

$$x_n = x_0 \prod_{k=1}^{n} \left[(-1)^k + \frac{\beta}{2}\right] = K_n \left(\frac{\beta^2}{4} - 1\right)^{[n/2]}$$

where $[n/2]$ is the greatest integer less than or equal to $n/2$ and $K_n = x_0$ if $n$ is even and $K_n = x_0(\beta/2 - 1)$ if $n$ is odd. It follows that if $|\beta| \leq 2\sqrt{2}$ then $\{x_n\}$ is bounded whereas if $\beta > 2\sqrt{2}$ then $\{x_n\}$ is unbounded. In particular, if $\beta = 0, \pm 2\sqrt{2}$ then $\{x_n\}$ is periodic with period 2.

To give further applications of ratios, the next two results are quoted from the literature ([9], [10], [18]) concerning the following equation

$$x_{n+1} = |\alpha x_n - \beta x_{n-1}| \tag{28}$$

which is a special case of Eq.(4) where $a = b = 0$ and $f(v, u - cv) = \alpha|u - cv|$ if we define $c = \beta/\alpha$. In this case, Eq.(18a) takes the form

$$r_{n+1} = \left|\alpha - \frac{\beta}{r_n}\right| \tag{29}$$

**Theorem 5**. [18] *Let $\alpha = \beta = 1$ in (28) and let $\mathbb{Q}^+$ denote the set of all non-negative rational numbers.*

(a) *If* $x_0/x_{-1} \in \mathbb{Q}^+$ *or* $x_{-1} = 0$ *then the corresponding solution* $\{x_n\}$ *of* (28) *has period 3 eventually and for all large* $n$ *its cycles are* $\{0, \alpha, \alpha\}$ *where* $\alpha > 0$.

(b) *If* $x_0/x_{-1} \notin \mathbb{Q}^+$ *then the corresponding solution* $\{x_n\}$ *of* (28) *converges to zero.*

(c) *Equation* (29) *has a period-p solution or a p-cycle* $\{\rho_1, \ldots, \rho_p\}$ *for every* $p \neq 3$. *These p-cycles are given as*

$$\rho_1 = \frac{1+\sqrt{5}}{2}, \ \rho_2 = \frac{\sqrt{5}-1}{\sqrt{5}+1} \quad (p=2)$$

$$\rho_1 = \frac{1}{2}\left[y_{p-4} + \sqrt{y_{p-4}^2 + 4y_{p-4}y_{p-1}}\right], \ \rho_k = \frac{y_{k-4}\rho_1 - y_{k-2}}{y_{k-3} - y_{k-5}\rho_1}, \ 2 \leq k \leq p, \ (p \geq 4)$$

*where* $y_n$ *is the n-th Fibonacci number; i.e.,* $y_{n+1} = y_n + y_{n-1}$ *for* $n \geq -2$ *where we define*

$$y_{-3} = -1, \ y_{-2} = 1.$$

(d) *If* $\{\rho_1, \ldots, \rho_p\}$ *is a periodic solution of* (29) *then for the corresponding solution* $\{x_n\}$ *of* (28) *it is true that*

$$x_n = x_0 \rho^{n/p}, \quad \text{if } n/p \text{ is an integer}$$
$$x_n \leq x_0 \alpha \rho^{n/p}, \quad \text{otherwise}$$

*where*

$$\rho = \prod_{i=1}^{p} \rho_i < 1, \quad \alpha = \max\{r_1, \ldots, r_p\}\rho^{-(1-1/p)} > 1.$$

**Theorem 6**. [9] (a) *Eq.*(28) *has a positive period-2 solution if and only if*

$$\beta^2 - \alpha^2 = 1, \quad \alpha > 0.$$

*Further, these period-2 solutions are confined to the pair of lines* $y = r_1 x$ *and* $y = r_2 x$ *in phase space, where the slopes* $r_1, r_2$ *are given by*

$$r_1 = \frac{\beta - 1}{\alpha}, \quad r_2 = \frac{\beta + 1}{\alpha}.$$

*On the other hand, the only period-2 solutions of* (28) *that pass through the origin occur at* $\alpha = 0$ *where* $\beta = 1$.

(b) *Eq.*(28) *has a positive period-3 solution if and only if*

$$\alpha^3 + \alpha\beta - \beta^3 = 1, \quad \alpha > 1. \tag{30}$$

*Further, these period-3 solutions are confined to the three lines* $y = r_i x$ *in phase space where for* $i = 1, 2, 3$, *the slopes* $r_i$ *are given by*

$$r_1 = \frac{\alpha\beta + 1}{\alpha^2 + \beta}, \quad r_2 = \frac{\beta^2 - \alpha}{\alpha\beta + 1}, \quad r_3 = \frac{\beta + \alpha^2}{\beta^2 - \alpha}. \tag{31}$$

*On the other hand, the only period-3 solutions of* (28) *that pass through the origin occur at* $\alpha = 1$ *where* $\beta = 1$ *also (see Theorem... above).*

(c) *Let* $\beta = 1$. *Then there is a strictly increasing sequence of parameter values* $\{\alpha_p\}$, $p \geq 3$, *such that*

$$\alpha_3 = 1 \quad \text{and} \quad \lim_{p \to \infty} \alpha_p = 2$$

*and for each* $p = 3, 4, 5, \ldots$ *the particular solution* $\{x_n\}$ *of* (28) *with initial values* $x_{-1} = 1$, $x_0 = \alpha_p$ *is periodic with period* $p$.

It is a curious fact that the behaviors of solutions of (25) are considerably simpler than those of (28). This is not easy to understand through a direct comparison of the two second order difference equations which seem to have little in common except that they are both 2-parameter difference equations. However, the essential difference becomes apparent when we contrast the relatively simple dynamics of the linear mapping (27) with the much more complex dynamics of the first order mapping (29).

## References


[1] Agarwal, R.P., 2000, *Difference Equations and Inequalities*, 2nd ed., Dekker, New York.

[2] Alsedà, L. and Llibre, J. 1993, Periods for triangular maps, *Bull. Australian Math. Soc.*, **47**: 41-53.

[3] Berezansky, L., Braverman, E. and Liz, E., 2005, Sufficient conditions for the global stability of nonautonomous higher order difference equation, *J. Difference Eq. and Appl.*, **11**: 785-798.

[4] Burden, R.L. and Fairs, J.D., 1997, *Numerical Analysis,* 6th ed., Brooks/Cole, Pacific Grove.

[5] Collet, P. and Eckmann, J-P., 1980, *Iterated Maps on the Interval as Dynamical Systems,* Birkhäuser, Boston.
[6] Elaydi, S.N., 1999, *An Introduction to Difference Equations,* 2nd ed, Springer, New York.
[7] Greenspan, D., 1973, *Discrete Models*, #3 in the Applied Math. and Computation Series, Addison-Wesley, Reading.
[8] Kelley, W. and Petersen, A.C., 2001, *Difference Equations: An Introduction with Applications*, 2nd. ed., Academic Press, San Diego.
[9] Kent, C.M. and Sedaghat, H., 2004, Convergence, periodicity and bifurcations for the two-parameter, absolute difference equation, *J. Difference Eq. and Appl.*, **10**: 817-841.
[10] Kent, C.M. and Sedaghat, H., 2005, Difference equations with absolute values, *J. Difference Eq. and Appl.*, **11**: 677-685.
[11] Kloeden, P.E., 1979, On Sharkovsky's cycle coexistence ordering, *Bull. Australian Math. Soc.*, **20**: 171-177.
[12] Kocic, V.L. and Ladas, G., 1993, *Global Behavior of Nonlinear Difference Equations of Higher Order with Applications*, Kluwer, Dordrecht.
[13] Kulenovic, M.R.S. and Ladas, G., 2001, *Dynamics of Second Order Rational Difference Equations with Open Problems and Conjectures*, Chapman and Hall/CRC Boca Raton.
[14] Li, T-Y. and Yorke, J.A., 1975, Period 3 implies chaos, *Amer. Math. Monthly,* **82**: 985-992.
[15] Marotto, F.R., 1978, Snap-back repellers imply chaos in $\mathbb{R}^n$, *J. Math. Anal. and Appl.*, **63**: 199-223.
[16] Puu, T., 1993, *Nonlinear Economic Dynamics*, 3rd. ed., Springer, New York.
[17] Sedaghat, H., 2003, *Nonlinear Difference Equations: Theory with Applications to Social Science Models*, Kluwer, Dordrecht.
[18] Sedaghat, H., 2004, Periodicity and convergence for $x_{n+1} = |x_n - x_{n-1}|$, *J. Math. Anal. and Appl.*, **291**: 31-39.
[19] Sedaghat, H., 2005, Global attractivity, oscillations and chaos in a class of nonlinear second order difference equations, *CUBO Math. J.*, **7**: 89-110.
[20] Sharkovsky, A.N., 1964, Co-existence of cycles of a continuous mapping of the line into itself, *Ukrain. Math. Zh.* **16**: 61-71 (Russian).