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Global Attractivity and Semiconjugacy

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Part I
Global Attractivity:
A General Result

1 Basic concepts at a glance

Let $\mathbb{X}$ be a (real or complex) Banach space and consider the difference equation

$$x_{n+1} = f_n(x_n, x_{n-1}, \ldots, x_{n-k}), \quad n = 0, 1, 2, \ldots \tag{1}$$

with a prescribed sequence of functions $f_n : D \to \mathbb{X}$ where $D \subset \mathbb{X}^{k+1}$.

- This equation represents the most general difference equation of *order* $k+1$ that is of *recursive type* – i.e., each state x_{n+1} is explicitly and uniquely determined by the preceding $k+1$ states $x_n, x_{n-1}, \ldots, x_{n-k}$.

- If the functions $f_n = f$ are all equal then we obtain the *autonomous* difference equation

$$x_{n+1} = f(x_n, x_{n-1}, \ldots, x_{n-k}).$$

- For each n, the vector function $F_n : D \to \mathbb{X}^{k+1}$ that is defined as

$$F_n(\xi_0, \xi_1, \ldots, \xi_k) = (f_n(\xi_0, \xi_1, \ldots, \xi_k), \xi_0, \xi_1, \ldots, \xi_{k-1})$$

  represents an *unfolding* of f_n to a mapping of $\mathbb{X}^{k+1}$. The sequence $\{F_n\}$ for all $n \geq 0$ may be said to unfold the difference equation (1).

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- Due to the recursive nature of (1), from a set of $k+1$ initial values

$$x_0, x_{-1}, \ldots, x_{-k} \in \mathbb{X} \text{ such that } (x_0, x_{-1}, \ldots, x_{-k}) \in D$$

  a unique sequence $\{x_n\}$ of points in $\mathbb{X}$ is generated upon iteration: $n = 0, 1, 2, \ldots$

$$\begin{aligned} x_1 &= f_0(x_0, x_{-1}, \ldots, x_{-k}), \quad n = 0, \\ x_2 &= f_1(x_1, x_0, \ldots, x_{-k+1}), \quad n = 1, \\ x_3 &= f_2(x_2, x_1, \ldots, x_{-k+2}), \quad n = 2, \\ &\vdots \end{aligned}$$

  A corresponding sequence of points $\{X_n\} = \{(x_n, x_{n-1}, \ldots, x_{n-k})\}$ is obtained in $\mathbb{X}^{k+1}$

$$X_{n+1} = F_n(X_n) = (F_n \circ F_{n-1} \circ \cdots \circ F_0)(X_0)$$

- In the autonomous case, $F_n = F$ for all n so

$$X_{n+1} = F^n(X_0).$$

- If $X_0 \in D$ and $\{X_n\}$ stays in D for all $n \geq 1$ then $\{X_n\}$ is an *orbit* of the unfolding of (1). The sequence of points $\{x_n\}$ is then a *solution* of (1).

- If $F_n(D) \subset D$ for every n then D is an *invariant set* of (1). In this case, the *existence* of a solution for (1) is guaranteed starting from any initial point in D. This is always true if $D = \mathbb{X}^{k+1}$.

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Theorem 1 *Let $\mathbb{X}$ be a Banach space and assume that for some real $\alpha \in (0,1)$ the functions f_n satisfy the norm inequality*

$$\|f_n(\xi_0, \xi_1, \ldots, \xi_k)\| \leq \alpha \max\{\|\xi_0\|, \ldots, \|\xi_k\|\} \tag{2}$$

for every n and all $(\xi_0, \ldots, \xi_k) \in \mathbb{X}^{k+1}$. Then every solution $\{x_n\}$ of (1) with given initial values $x_0, x_{-1}, \ldots, x_{-k} \in \mathbb{X}$ satisfies

$$\|x_n\| \leq \alpha^{n/(k+1)} \max\{\|x_0\|, \|x_{-1}\|, \ldots, \|x_{-k}\|\}.$$

That is, the origin is globally exponentially stable.

Proof. Let $\mu = \max\{\|x_0\|, \|x_{-1}\|, \ldots, \|x_{-k}\|\}$.

If $\{x_n\}$ is the solution of (1) with the given initial values then we first claim that $\|x_n\| \leq \alpha\mu$ for all $n \geq 1$.

By (2)

$$\|x_1\| = \|f_0(x_0, x_{-1}, \ldots, x_{-k})\| \leq \alpha \max\{\|x_0\|, \|x_{-1}\|, \ldots, \|x_{-k}\|\} = \alpha\mu$$

and if for any $j \geq 1$ it is true that $\|x_n\| \leq \alpha\mu$ for $n = 1, 2, \ldots, j$ then

$$\begin{aligned}\|x_{j+1}\| &= \|f_j(x_j, x_{j-1}, \ldots, x_{j-k})\| \leq \alpha \max\{\|x_j\|, \|x_{j-1}\|, \ldots, \|x_{j-k}\|\} \\ &\leq \alpha \max\{\mu, \alpha\mu\} = \alpha\mu.\end{aligned}$$

Therefore, our claim is true by induction.

In particular, since $0 < \alpha < 1$ we have shown that $\|x_n\| \leq \alpha^{n/(k+1)}\mu$ for $n = 1, 2, \ldots, k+1$. Now suppose that $\|x_n\| \leq \alpha^{n/(k+1)}\mu$ is true for $n \leq m$ where $m \geq k+1$. Then

$$\begin{aligned}\|x_{m+1}\| &= \|f_m(x_m, x_{m-1}, \ldots, x_{m-k})\| \leq \alpha \max\{\|x_m\|, \|x_{m-1}\|, \ldots, \|x_{m-k}\|\} \\ &\leq \alpha\mu \max\{\alpha^{m/(k+1)}, \alpha^{(m-1)/(k+1)}, \ldots, \alpha^{(m-k)/(k+1)}\} \\ &= \alpha^{(m+1)/(k+1)}\mu\end{aligned}$$

and the proof is complete by induction. ■

History/background of last theorem...

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Part II
Semiconjugate Factorization on Banach Algebras

2 Banach algebras at a glance

A *Banach algebra* with an *identity* 1 is a Banach space together with a *multiplication operation xy* that is *associative*, *distributes over addition* and satisfies

$$||xy|| \leq ||x||\,||y||, \quad ||1|| = 1 \tag{3}$$

The multiplication by *scalars* (real or complex numbers) that is inherited from the vector space structure of $\mathbb{X}$ is made consistent with the main multiplication by assuming that the following equalities hold for all scalars α:

$$\alpha(xy) = (\alpha x)y = x(\alpha y).$$

- Elements of type $\alpha 1$ where α is a scalar are the *constants* in $\mathbb{X}$.

- The set $\mathbb{R}$ ($\mathbb{C}$) is a real (complex) commutative Banach algebra with identity over itself with respect to the ordinary addition and multiplication of complex numbers and the absolute value as norm.

- The set $C[0,1]$ of all continuous real-valued functions on the interval [0,1] forms a commutative real Banach algebra relative to the sup (or max) norm. The identity element is the constant function $x(r) = 1$ for all $r \in [0,1]$. The other constants in $C[0,1]$ are just the constant functions on [0,1].

- An element $x \in \mathbb{X}$ is *invertible*, or a *unit*, if there is $x^{-1} \in \mathbb{X}$ (the inverse of x) such that $x^{-1}x = 1$. The collection of all invertible elements of $\mathbb{X}$ forms a group $\mathcal{G}$ (the *group of units*) that contains all nonzero constants. It can be shown that $\mathcal{G}$ is open relative to the metric topology of $\mathbb{X}$.

- Since the zero element is not invertible, $\mathcal{G} \neq \mathbb{X}$. If $\mathbb{X}$ is either $\mathbb{R}$ or $\mathbb{C}$ then $\mathcal{G} = \mathbb{X}\backslash\{0\}$. In the algebra $C[0,1]$ units are functions that do not assume the (scalar) value 0.

3 The difference equation of interest

Consider the difference equation

$$x_{n+1} = \sum_{i=0}^{k} a_i x_{n-i} + g_n \left(\sum_{i=0}^{k} b_i x_{n-i} \right) \tag{4}$$

- For each n, the function $g_n : \mathbb{X} \to \mathbb{X}$ is defined on a real or complex Banach algebra $\mathbb{X}$ with identity.
- The parameters a_i, b_i are constants in $\mathbb{X}$ such that

$$a_k \neq 0 \text{ or } b_k \neq 0.$$

Several special cases of (4) in $\mathbb{X} = \mathbb{R}$ have been studied in the literature – for example:

- For the second-order case ($k = 1$)

$$x_{n+1} = cx_n + g(x_n - x_{n-1}) \tag{5}$$

  - Sedaghat (2003 and earlier): Economic models of the business cycle
  - C.M. Kent and H. Sedaghat (2004): boundedness and global asymptotic stability of (5)
  - S. Li and W. Zhang (2008): bifurcations of solutions of (5), including the Neimark-Sacker bifurcation (discrete analog of Hopf)
  - El-Morshedy (2011): improves some results of Kent and Sedaghat and gives necessary and sufficient conditions for the occurrence of oscillations

- The global attractivity and stability of equilibrium for the following delay version of (5) is studied by B. Dai and N. Zhang (2005)

$$x_{n+1} = cx_n + g(x_n - x_{n-k})$$

- For the following generalization of (5)

$$x_{n+1} = ax_n + bx_{n-1} + g_n(x_n - cx_{n-1}) \tag{6}$$

  - Sedaghat (2009): sufficient conditions for the occurrence of periodic solutions, limit cycles and chaotic behavior in (6) are obtained using reduction of order and semiconjugate factorization
  - Dehghan, Kent, Mazrooei, Ortiz, Sedaghat (2008): sufficient conditions for occurrence of limit cycles and chaos in certain rational difference equations of the following type

$$x_{n+1} = \frac{ax_n^2 + bx_{n-1}^2 + cx_n x_{n-1} + dx_n + ex_{n-1} + f}{\alpha x_n + \beta x_{n-1} + \gamma}$$

    that are special cases of (6)

- Hamaya (2007): obtains sufficient conditions for the global attractivity of the origin for the following equation

$$x_{n+1} = \alpha x_n + a \tanh\left(x_n - \sum_{i=1}^{k} b_i x_{n-i}\right)$$

  with $0 \leq \alpha < 1$, $a > 0$ and $b_i \geq 0$.

A *direct application* of Theorem 1 to Equation (4) gives the following result.

Corollary 2 *Let $g_n : \mathbb{X} \to \mathbb{X}$ be a sequence of functions on a real or complex Banach algebra $\mathbb{X}$ with identity. Assume that there is a real number $\sigma > 0$ such that*

$$\|g_n(\xi)\| \leq \sigma||\xi||, \quad \xi \in \mathbb{X} \tag{7}$$

for all n and further, for coefficients a_i, b_i (real or complex) we assume that the inequality

$$\sum_{i=0}^{k}(||a_i|| + \sigma||b_i||) < 1 \tag{8}$$

holds. Then every solution $\{x_n\}$ of Equation (4)

$$x_{n+1} = \sum_{i=0}^{k} a_i x_{n-i} + g_n\left(\sum_{i=0}^{k} b_i x_{n-i}\right)$$

with initial values $x_0, x_{-1}, \ldots, x_{-k} \in \mathbb{X}$ satisfies

$$||x_n|| \leq \alpha^{n/(k+1)} \max\{||x_0||, ||x_{-1}||, \ldots, ||x_{-k}||\}, \quad \alpha = \sum_{i=0}^{k}(||a_i|| + \sigma||b_i||).$$

Proof. If $(\xi_0, \xi_1, \ldots, \xi_k) \in \mathbb{X}^{k+1}$ then by the triangle inequality, (3) and (7)

$$\begin{aligned}\left\|\sum_{i=0}^{k} a_i\xi_i + g_n\left(\sum_{i=0}^{k} b_i\xi_i\right)\right\| &\leq \sum_{i=0}^{k}(||a_i|| + \sigma||b_i||)||\xi_i|| \\ &\leq \left[\sum_{i=0}^{k}(||a_i|| + \sigma||b_i||)\right] \max\{||\xi_0||, \ldots, ||\xi_k||\}\end{aligned}$$

Therefore, given (8), by Theorem 1 the origin is globally asymptotically stable. ∎

4 Semiconjugate factorization

In previous studies of special cases of Equation (4):

$$x_{n+1} = \sum_{i=0}^{k} a_i x_{n-i} + g_n \left(\sum_{i=0}^{k} b_i x_{n-i} \right)$$

it is often the case that the restriction

$$\sum_{i=0}^{k} (||a_i|| + \sigma ||b_i||) < 1$$

on coefficients can be relaxed.

General sufficient conditions for improving the ranges of parameters a_i, b_i can be obtained through an *indirect application* of Theorem 1.

- We first split the above difference equation into two equations of lower orders;
- One equation will be similar to the above but with order reduced by 1; Theorem 1 is then applied to this lower order equation.
- The second equation will be linear non-homogeneous of order 1 and its solution is easy to analyze.
- The system of two equations will be *triangular* since the first equation is independent of the second.
- The triangular system constitutes a *semiconjugate factorization* of Equation (4) above.

The next result sets things up by supplying the crucial semiconjugate factorization:

Lemma 3 *Let $g_n : \mathbb{X} \to \mathbb{X}$ be a sequence of functions on an algebra $\mathbb{X}$ with identity (not necessarily normed) over a field $\mathcal{F}$. If for $a_i, b_i \in \mathbb{X}$ the polynomials*

$$P(\xi) = \xi^{k+1} - \sum_{i=0}^{k} a_i \xi^{k-i}, \quad Q(\xi) = \sum_{i=0}^{k} b_i \xi^{k-i}$$

have a common root $\rho \in \mathcal{G}$, the group of units of $\mathbb{X}$, then each solution $\{x_n\}$ of (4) in $\mathbb{X}$ satisfies

$$x_{n+1} = \rho x_n + t_{n+1} \tag{9}$$

where the sequence $\{t_n\}$ is the unique solution of the equation:

$$t_{n+1} = -\sum_{i=0}^{k-1} p_i t_{n-i} + g_n \left(\sum_{i=0}^{k-1} q_i t_{n-i} \right) \tag{10}$$

in $\mathbb{X}$ with initial values $t_{-i} = x_{-i} - \rho x_{-i-1} \in \mathbb{X}$ for $i = 0, 1, \ldots, k-1$ and coefficients

$$p_i = \rho^{i+1} - a_0 \rho^i - \cdots - a_i \quad \text{and} \quad q_i = b_0 \rho^i + b_1 \rho^{i-1} + \cdots + b_i$$

in $\mathbb{X}$. Conversely, if $\{t_n\}$ is a solution of (10) with initial values $t_{-i} \in \mathbb{X}$ then the sequence $\{x_n\}$ that it generates in $\mathbb{X}$ via (9) is a solution of (4).

The pair of equations (9) and (10) that are equivalent to (4) represents a *semiconjugate factorization* of (4) relative to the linear form symmetry.

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Part III
Global Attractivity plus Semiconjugate Factorization

5 Global attractivity revisited

Theorem 4 *Let* $g_n : \mathbb{X} \to \mathbb{X}$ *be a sequence of functions that satisfy (7):* $||g_n(\xi)|| \leq \sigma||\xi||$*, for each* n*. Then every solution* $\{x_n\}$ *of (4):*

$$x_{n+1} = \sum_{i=0}^{k} a_i x_{n-i} + g_n\left(\sum_{i=0}^{k} b_i x_{n-i}\right)$$

converges to zero if either (a) or (b) below is true:

(a) Inequality (8) holds:

$$\sum_{i=0}^{k}(||a_i|| + \sigma||b_i||) < 1.$$

(b) The polynomials P, Q *in Lemma 3 have a common root* $\rho \in \mathcal{G}$ *such that* $||\rho|| < 1$ *and*

$$\sum_{i=0}^{k-1}(||p_i|| + \sigma||q_i||) < 1 \tag{11}$$

with the coefficients p_i, q_i

$$p_i = \rho^{i+1} - a_0\rho^i - \cdots - a_i \quad \text{and} \quad q_i = b_0\rho^i + b_1\rho^{i-1} + \cdots + b_i$$

in the factor equation (10), $i = 0, 1, \ldots, k-1$*.*

Proof. (a) Corollary 2.

(b) Observe that the solution of Equation (9) $x_{n+1} = \rho x_n + t_{n+1}$ in terms of t_n is given by

$$x_n = \rho^n x_0 + \sum_{j=1}^{n} \rho^{n-j} t_j \tag{12}$$

Now let Equation (10) which has order one less than (4) be as in Lemma 3. Then, given $||g_n(\xi)|| \leq \sigma ||\xi||$, apply Corollary 2 to the lower order equation (10) to conclude that

$$||t_n|| \leq \alpha^{n/(k+1)} \mu$$

where $\mu = \max\{||t_0||, ||t_{-1}||, \ldots, ||t_{-k+1}||\}$ with $t_{-i} = x_{-i} - \rho x_{-i-1}$ for $i = 0, 1, \ldots, k-1$ and

$$\alpha = \sum_{i=0}^{k-1} (||p_i|| + \sigma ||q_i||).$$

Since in $\mathbb{X}$, $||\rho^j|| \leq ||\rho||^j$ for each j, taking norms in (12) yields

$$||x_n|| \leq \|\rho\|^n ||x_0|| + \sum_{j=1}^{n} \|\rho\|^{n-j} ||t_j|| \leq \|\rho\|^n ||x_0|| + \mu \|\rho\|^n \sum_{j=0}^{n-1} \left(\frac{\alpha^{1/(k+1)}}{\|\rho\|} \right)^j . \tag{13}$$

If $\alpha^{1/(k+1)} \neq \|\rho\|$ then

$$\begin{aligned} ||x_n|| &\leq \|\rho\|^n ||x_0|| + \mu \alpha^{1/(k+1)} \|\rho\|^{n-1} \frac{[\alpha^{1/(k+1)} / \|\rho\|]^n - 1}{[\alpha^{1/(k+1)} / \|\rho\|] - 1} \\ &= \|\rho\|^n ||x_0|| + \mu \alpha^{1/(k+1)} \frac{\alpha^{n/(k+1)} - \|\rho\|^n}{\alpha^{1/(k+1)} - \|\rho\|} \end{aligned}$$

Since $\alpha, \|\rho\| < 1$ it follows that $\{x_n\}$ converges to zero. If $\alpha^{1/(k+1)} = \|\rho\|$ then (13) reduces to

$$||x_n|| \leq \|\rho\|^n ||x_0|| + \mu \|\rho\|^n n$$

and by L'Hospital's rule $\{x_n\}$ again converges to zero. ■

Corollary 5 *Let g_n be functions on $\mathbb{X}$ satisfying (7) for all $n \geq 0$. Every solution of the difference equation*

$$x_{n+1} = ax_n + g_n\left(b_0 x_n + b_1 x_{n-1} + \cdots + b_k x_{n-k}\right), \qquad (14)$$
$$a \in \mathcal{G}, b_i \in \mathbb{X},\ \ b_k \neq 0$$

converges to zero if $||a|| < 1$ and the following conditions hold:

$$b_0 a^k + b_1 a^{k-1} + b_2 a^{k-2} + \cdots + b_k = 0, \qquad (15)$$
$$\sum_{i=0}^{k-1} ||b_0 a^i + b_1 a^{i-1} + \cdots + b_i|| < \frac{1}{\sigma}. \qquad (16)$$

Proof. For equation (14) the polynomials P, Q are

$$P(\xi) = \xi^{k+1} - a\xi^k, \quad Q(\xi) = b_0\xi^k + b_1\xi^{k-1} + \cdots + b_k.$$

Thus $\rho = a$ is their common root in $\mathcal{G}$ if (15) holds. The numbers p_i, q_i that define the factor equation (10) in this case are

$$p_i = \rho^{i+1} - a\rho^i = 0, \quad q_i = b_0 a^i + b_1 a^{i-1} + \cdots + b_i$$

Thus, if $||a|| < 1$ then by Theorem 4 every solution of (14) converges to zero. ■

The parameter range determined by the inequality $\sum_{i=0}^{k-1}(||p_i||+\sigma||q_i||) < 1$ is generally distinct from that given by $\sum_{i=0}^{k}(||a_i|| + \sigma||b_i||) < 1$.

Example 1. To illustrate the difference between the two inequalities above, consider the following equation on the set of real numbers:

$$x_{n+1} = ax_n + \alpha_n \tanh(x_n - bx_{n-k}) \tag{17}$$

Suppose that the sequence $\{\alpha_n\}$ of real numbers is bounded by $\sigma > 0$ and it is otherwise arbitrary. Then

$$|\alpha_n \tanh t| = |\alpha_n||\tanh t| \leq |\alpha_n||t| \leq \sigma|t|$$

for all n. If $0 < |a| < 1$ and $b = a^k$ then by Corollary 5 every solution of (17) converges to zero if

$$\frac{1}{\sigma} > \sum_{i=0}^{k-1} |a|^i = \frac{1-|a|^k}{1-|a|} \Leftrightarrow \sigma < \frac{1-|a|}{1-|a|^k}. \tag{18}$$

On the other hand, by Corollary 2 the origin is globally attracting for (17) with $b = a^k$ if

$$|a| + \sigma(1 + |a|^k) < 1 \Rightarrow \sigma < \frac{1-|a|}{1+|a|^k}$$

which is clearly more limited than (18).

Corollary 6 *Let g_n be functions on $\mathbb{X}$ satisfying (7) for all $n \geq 0$. For the difference equation*

$$x_{n+1} = a_0 x_n + a_1 x_{n-1} + \cdots + a_k x_{n-k} + g_n\left(x_n - b x_{n-1}\right), \qquad (19)$$
$$a_i \in \mathbb{X}, b \in \mathcal{G},\ a_k \neq 0$$

assume that $||b|| < 1$ and the following conditions hold:

$$a_0 b^k + a_1 b^{k-1} + \cdots + a_k = b^{k+1}, \qquad (20)$$

$$\sum_{i=0}^{k-1} ||b^{i+1} - a_0 b^i - \cdots - a_i|| < 1 - \sigma. \qquad (21)$$

Then every solution of (19) converges to zero.

Proof. The polynomials P, Q in this case are

$$P(\xi) = \xi^{k+1} - a_0 \xi^k - \cdots - a_k, \quad Q(\xi) = \xi^k - b\xi^{k-1}.$$

Clearly, $Q(b) = 0$ and if Equality (20) holds then $P(b) = 0$ too, so Theorem 4 applies. We calculate the coefficients of the factor equation (10) as $q_0 = 1$, $q_i = 0$ if $i \neq 0$ and

$$p_i = b^{i+1} - a_0 b^i - \cdots - a_i.$$

Now, inequality (11) yields (21) via a straightforward calculation:

$$\begin{aligned} 1 &> \sum_{i=0}^{k-1} ||b^{i+1} - a_0 b^i - \cdots - a_i|| + \sigma \\ &= \sum_{i=0}^{k-1} ||b^{i+1} - a_0 b^i - \cdots - a_i|| + \sigma \end{aligned}$$

Thus, if $||b|| < 1$ then by Theorem 4 every solution of (19) converges to zero. ∎

Example 2. As an application of the preceding corollary, consider the case $k = 1$, i.e., the second-order equation

$$x_{n+1} = a_0 x_n + a_1 x_{n-1} + g_n(x_n - b x_{n-1}) \tag{22}$$

which is essentially Equation (6) on a Banach algebra $\mathbb{X}$. By Corollary 6, every solution of (22) converges to zero if the functions g_n satisfy (7) and

$$b \in \mathcal{G}, \quad ||b|| < 1, \quad a_0 b + a_1 = b^2, \quad ||a_0 - b|| + \sigma < 1. \tag{23}$$

On the other hand, according to Corollary 2, every solution of (22) converges to zero if the functions g_n satisfy (7) and

$$||a_0|| + ||a_1|| + \sigma(1 + ||b||) < 1. \tag{24}$$

Parameter values that do not satisfy (24) may satisfy (23). For comparison, if $a_1 = b^2 - a_0 b$ then (24) may be solved for σ to obtain

$$\sigma < \frac{1 - ||a_0|| - ||b||\,||a_0 - b||}{1 + ||b||}.$$

This is a stronger constraint on σ than $\sigma < 1 - ||a_0 - b||$ from (23), especially if b is not near 0.

Example 3. Consider the following difference equation on the real Banach algebra $C[0,1]$:

$$x_{n+1} = \frac{\alpha r}{r+1} x_n + \frac{\beta(\beta - \alpha r)}{(r+1)^2} x_{n-1} + \int_0^r \phi_n \left(x_n - \frac{\beta}{r+1} x_{n-1} \right) dr \tag{25}$$

where the functions $\phi_n : \mathbb{R} \to \mathbb{R}$ are integrable and for each n they satisfy the absolute value inequality

$$|\phi_n(r)| \leq \sigma |r|, \quad r \in \mathbb{R}$$

for some $\sigma > 0$. Assume that the following inequalities hold

$$0 < \beta < 1, \quad 3\beta \leq \alpha < 2 + \beta, \quad \sigma < \frac{2 + \beta - \alpha}{2}$$

and define the coefficient functions

$$a_0(r) = \frac{\alpha r}{r+1}, \quad a_1(r) = \frac{\beta(\beta - \alpha r)}{(r+1)^2}, \quad b(r) = \frac{\beta}{r+1}.$$

Then clearly $b \in \mathcal{G}$ and

$$b^2(r) - a_0(r) b(r) = \frac{\beta(\beta - \alpha r)}{(r+1)^2} = a_1(r)$$

Further, the following are true about the norms:

$$||a_0|| = \sup_{0 \leq r \leq 1} \frac{\alpha r}{r+1} = \frac{\alpha}{2}, \quad ||b|| = \sup_{0 \leq r \leq 1} \frac{\beta}{r+1} = \beta < 1$$

and

$$||a_0 - b|| = \sup_{0 \leq r \leq 1} \left| \frac{\alpha r - \beta}{r+1} \right| = \max \left\{ \frac{\alpha - \beta}{2}, \beta \right\} = \frac{\alpha - \beta}{2} < 1 - \sigma.$$

All the required conditions are met. Next, since the functions or operators $g_n : C[0,1] \to C[0,1]$ in (22) are defined as

$$g_n(x)(r) = \int_0^r (\phi_n \circ x)(r)dr$$

for $r \in [0,1]$ and all n, their norms satisfy

$$||g_n(x)|| \leq \sup_{0\leq r\leq 1} \int_0^r |\phi_n(x(r))|dr \leq \sup_{0\leq r\leq 1} \int_0^r \sigma|x(r)|dr \leq ||x||\sigma \sup_{0\leq r\leq 1} r = \sigma||x||.$$

Therefore, by Corollary 6 for every pair of initial functions $x_0(r), x_{-1}(r) \in C[0,1]$ the sequence of functions $x_n = x_n(r)$ that satisfy (25) in $C[0,1]$ converges uniformly to the zero function.

It is worth observing that $a_0 \notin \mathcal{G}$ and $||a_0|| \geq 1$ if $\alpha \geq 2$, in which Corollary 2 does not apply.

- The preceding corollaries and Theorem 4 are broad applications of the reduction of order method to very general equations that improve the range of parameters compared to Lemma 2.
- They show that different patterns of delays may be translated into algebraic problems about the polynomials P and Q and their root structures.
- In special cases a more efficient application of Lemma 3 may yields a greater amount of information about the behavior of solutions than Theorem 4.

The next result represents a deeper use of order reduction in that sense.

Theorem 7 *In Equation (22) assume that $b \in \mathcal{G}$ with $|b| < 1$ and $a_0, a_1 \in \mathbb{X}$ such that $a_0 b + a_1 = b^2$. If x_0, x_{-1} are given initial values for (22) for which the solution of the first order equation*

$$t_{n+1} = (a_0 - b)t_n + g_n(t_n) \tag{26}$$

converges to zero with the initial value $t_0 = x_0 - bx_{-1}$ then the corresponding solution of (22) converges to zero. In particular, if the origin is a global attractor of the solutions of the first order Equation (26) then it is also a global attractor of the solutions of (22).

Proof. In this case $Q(\xi) = \xi - b$ so there is only one root b. Now Lemma 3 gives Equation (26) if $P(b) = 0$, i.e., if $a_0 b + a_1 = b^2$. Finally, we complete the proof by arguing similarly to the proof of Theorem 4(b), using (12). ■

Example 4. Consider the following autonomous equation on the real numbers

$$x_{n+1} = ax_n + b(b-a)x_{n-1} + \sigma \tanh(x_n - bx_{n-1}) \tag{27}$$

where $\sigma > 0$, $0 < b < 1$ and $a < b$. Equation (26) in this case is

$$t_{n+1} = h(t_n), \quad h(\xi) = (a-b)\xi + \sigma \tanh \xi. \tag{28}$$

- The function h has a fixed point at the origin since $h(0) = 0$.
- The origin is the unique fixed point of h if $|h(\xi)| < |\xi|$ for $\xi \neq 0$.
- Since h is an odd function, it is enough to consider $\xi > 0$. In this case, $h(\xi) < \xi$ if and only if $\sigma \tanh \xi < (1 - a + b)\xi$. Since $\tanh \xi < \xi$ for $\xi > 0$ we may conclude that

$$\sigma < 1 - a + b. \tag{29}$$

- Given that $a < b$, it is possible to choose $1 \leq \sigma < 1 - a + b$ and extend the range of σ beyond what is possible with previous corollaries which require that $\sigma < 1$. In particular, the function $\sigma \tanh \xi$ is not a contraction near the origin in this discussion.

Routine analysis of the properties of $h(\xi) = (a-b)\xi + \sigma \tanh \xi$ leads to the following bifurcation scenario:

1. Suppose that $b-1 \leq a < b < 1$ and (29) holds, i.e., $\sigma < 1-a+b$. Then $b-a \leq 1$ and all solutions of $t_{n+1} = h(t_n)$, hence, also all solutions of the second order equation (27) converge to zero.

2. Now fix b, σ and reduce the value of a so that $a < b-1 < 0$. Then the function $h \circ h$ crosses the diagonal at two points $\tau > 0$ and $-\tau$ and a 2-cycle $\{-\tau, \tau\}$ emerges for $t_{n+1} = h(t_n)$. Note that (29) still holds when a is further reduced, but the origin is no longer globally attracting.

3. The cycle $\{-\tau, \tau\}$ itself is repelling and generates a repelling 2-cycle for (27). The emergence of this cycle implies that $\{t_n\}$ is unbounded if $|t_0| > \tau$ and it converges to 0 if $|t_0| < \tau$. Therefore, the corresponding solution $\{x_n\}$ of (27) also converges to 0 if

$$|x_0 - bx_{-1}| = |t_0| < \tau;$$

   i.e., if the initial point (x_{-1}, x_0) is between the two parallel lines $y = b\xi + \tau$ and $y = b\xi - \tau$ in the (ξ, y) plane.

4. Suppose that a continues to decrease. Then the value of τ also decreases and reaches zero when

$$a = b - \sigma - 1$$

   i.e., when the slope of h at the origin is -1. Now, the cycle $\{-\tau, \tau\}$ collapses into the origin and turns it into a repelling fixed point. In this case, *all* nonzero solutions of (28) and (27) are unbounded.

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