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Solving Linear Difference Equations in Rings Using Reduction of Order

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Consider the linear (nonhomogeneous) difference equation

$$x_{n+1} = a_{0,n}x_n + a_{1,n}x_{n-1} + \cdots + a_{k,n}x_{n-k} + b_n$$

where $\{b_n\}$ and the (variable) coefficients $\{a_{j,n}\}$ are given sequences in a nontrivial ring R for $j = 0, 1, \ldots, k$.

- The multiplicative subgroup G of R, i.e., the "unit group" is nonempty if R has an identity 1; e.g., if R is the ring of all $m \times m$ matrices of real numbers then G consists of all matrices in R that have nonzero determinants. If R is the ring $C(S)$ of all real-valued, continuous functions on a nonempty set S then G consists of all functions f that are either always positive or always negative.
- If R is a field then G consists of all nonzero elements of R –familiar examples: the real numbers $\mathbb{R}$, the finite field $\mathbb{Z}_p$ of integers modulo a prime p.
- Every element of G is a "unit" and each sequence in G is "unitary".

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- The "characteristic equation" of (the homogeneous part of) the linear equation

$$x_{n+1} = a_{0,n}x_n + a_{1,n}x_{n-1} + \cdots + a_{k,n}x_{n-k} + b_n \tag{1}$$

is the (nonlinear) equation

$$r_{n+1}r_n \ldots r_{n-k+1} - a_{0,n}(r_n r_{n-1} \ldots r_{n-k+1}) - a_{1,n}(r_{n-1} \ldots r_{n-k+1}) - \cdots - a_{k-1,n}r_{n-k+1} - a_{k,n} = 0 \tag{2}$$

- If coefficients $a_{j,n} = a_j$ are constants and $b_n = 0$ i.e., the linear equation is autonomous:

$$x_{n+1} = a_0x_n + a_1x_{n-1} + \cdots + a_kx_{n-k} \tag{3}$$

then a *fixed point* of its characteristic equation

$$r_{n+1}r_n \ldots r_{n-k+1} - a_0(r_n r_{n-1} \ldots r_{n-k+1}) - a_1(r_{n-1} \ldots r_{n-k+1}) - \cdots - a_{k-1}r_{n-k+1} - a_k = 0$$

is a root of the polynomial

$$r^{k+1} - a_0r^k - a_1r^{k-1} - \cdots a_k.$$

This is the familiar "characteristic polynomial" of (3) and so its roots are the "eigenvalues" of the (autonomous) linear difference equation.

- More generally, an "eigensequence" of (1) is any sequence that satisfies the characteristic equation (2).

- A "unitary eigensequence" is any solution of (2) that is contained in the unit group G.

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Theorem 1 *If the linear difference equation*

$$x_{n+1} = a_{0,n}x_n + \cdots + a_{k,n}x_{n-k} + b_n \tag{4}$$

has a unitary eigensequence $\{s_n\}$ *in a ring* R *with identity then the linear difference equation is equivalent to the following system of lower order linear difference equations*

$$t_{n+1} = a'_{0,n}t_n + a'_{1,n}t_{n-1} + \cdots + a'_{k-1,n}t_{n-k+1} + b_n \tag{5}$$

$$x_{n+1} = s_{n+1}x_n + t_{n+1} \tag{6}$$

where for $m = 0, \ldots, k-1$, $t_{m+1} = x_{m+1} - s_{m+1}x_m$ *and*

$$a'_{m,n} = -\sum_{i=m+1}^{k} a_{i,n} \left(\prod_{j=m+1}^{i} s_{n-j+1} \right)^{-1}.$$

- The system of difference equations (5) and (6) is a "semiconjugate factorization" or "sc-factorization" of the linear difference equation (4). This is a semi-coupled "triangular system" since the first equation is independent of the second.

- Equation (5), namely, the "factor equation" of (4) has order k which is one lower than the order of (4).

- Equation (6) of order 1 is the "cofactor equation".

- If the factor equation (5) has a unitary eigensequence in R then it has a sc-factorization into a triangular system with a factor of order $k-1$ and a cofactor of order 1. In this way the above theorem can be applied repeatedly as long as their factors have unitary eigensequences in R.

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Unitary eigensequences can be obtained from “unitary solutions” of the homogeneous part

$$x_{n+1} = a_{0,n}x_n + \cdots + a_{k,n}x_{n-k} \tag{7}$$

i.e., a solution that is contained in the unit group G.

Theorem 2 *Let R be a ring with identity. A (unitary) sequence $\{r_n\}$ in R is an eigensequence of (7) if and only if $r_n = u_n u_{n-1}^{-1}$ where $\{u_n\}$ is a unitary solution of (7).*

An example: With initial values $x_0 = 1$ and $x_1 = 1$ the “Fibonacci recurrence”

$$x_{n+1} = x_n + x_{n-1} \tag{8}$$

has a familiar solution in the ring $\mathbb{Z}$ of integers, namely the “Fibonacci sequence” $\{F_n\}$: 1,1,2,3,5,8,...

The sequence $\{F_n\}$ is not unitary in $\mathbb{Z}$ whose unit group is $\{-1, 1\}$. But $\{F_n\}$ is a unitary solution of (8) in the field $\mathbb{Q}$ of rational numbers since $F_n \neq 0$

The sequence $F_{n+1}F_n^{-1} = F_{n+1}/F_n$ is an eigensequence of (8) in $\mathbb{Q}$ because it satisfies the characteristic equation

$$r_{n+1}r_n - r_n - 1 = 0$$

Indeed,

$$\frac{F_{n+2}}{F_{n+1}}\frac{F_{n+1}}{F_n} - \frac{F_{n+1}}{F_n} - 1 = \frac{F_{n+1} + F_n}{F_n} - \frac{F_{n+1}}{F_n} - 1 = 0.$$

The above eigensequence yields the following sc-factorization of (8) in $\mathbb{Q}$

$$\begin{aligned} t_{n+1} &= -\frac{F_n}{F_{n+1}}t_n, \quad t_1 = x_1 - \frac{F_{n+1}}{F_n}x_0 \\ x_{n+1} &= \frac{F_{n+2}}{F_{n+1}}x_n + t_{n+1}. \end{aligned}$$

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To minimize notational clutter, in the rest of this talk we focus on the second-order homogeneous linear equation

$$x_{n+1} = \alpha_n x_n + \beta_n x_{n-1} \tag{9}$$

where α_n, β_n are given sequences in a ring R.

- The characteristic equation of (9) is

$$r_{n+1} r_n - \alpha_n r_n - \beta_n = 0 \tag{10}$$

  A solution $\{s_n\}$ of this equation in R is a unitary eigensequence if $s_n \in G$ for all n.

- If (10) has unitary solutions then they can be obtained by iterating the following recurrence starting with a unit $s_1 \in G$

$$r_{n+1} = \alpha_n + \beta_n r_n^{-1} \tag{11}$$

- As implied by the preceding theorem, this recurrence can be derived from a unitary solution $\{u_n\}$ of (9)

$$u_{n+1} = \alpha_n u_n + \beta_n u_{n-1} \Rightarrow u_{n+1} u_n^{-1} = \alpha_n + \beta_n u_{n-1} u_n^{-1} = \alpha_n + \beta_n (u_n u_{n-1}^{-1})^{-1}$$
$$s_n = u_n u_{n-1}^{-1} \Rightarrow r_{n+1} = \alpha_n + \beta_n r_n^{-1}$$

- The sc-factorization of (9) with a unitary eigensequence $\{s_n\}$ is

$$t_{n+1} = -a_{0,n} t_n = -\beta_n s_n^{-1} t_n, \quad t_1 = x_1 - s_1 x_0$$
$$x_{n+1} = s_{n+1} x_n + t_{n+1}.$$

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An example: For the difference equation

$$x_{n+1} = 3(-1)^n x_n + 2x_{n-1} \tag{12}$$

with coefficients in $\mathbb{Z}$ the recurrence (11) is

$$r_{n+1} = 3(-1)^n + \frac{2}{r_n}$$

With $s_1 = 1$ we calculate $s_2 = -1$, $s_3 = 1$, $s_4 = -1$, etc. This is an eigensequence $\{(-1)^{n+1}\}$ of period 2 which is unitary in $\mathbb{Z}$. A sc-factorization with integer coefficients is:

$$\begin{aligned} t_{n+1} &= -\frac{2}{(-1)^{n+1}} t_n = 2(-1)^n t_1, \quad t_1 = x_1 - x_0 \\ x_{n+1} &= (-1)^n x_n + t_{n+1}. \end{aligned}$$

Iteration of the factor equation gives

$$t_n = (-1)^{n(n-1)/2} 2^n (x_1 - x_0)$$

which yields the following formula for the solution for (12)

$$x_n = (-1)^{n(n-1)/2}[x_0 + (2^n - 1)(x_1 - x_0)].$$

In particular, for initial values x_0, x_1 such that $x_0 \neq x_1$, the ratio $x_n/2^n$ converges to the 4-cycle $(-1)^{n(n-1)/2}(x_1 - x_0)$. If $x_0 = x_1$ then the solution x_n itself is the 4-cycle $(-1)^{n(n-1)/2} x_0$.

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The difference equation (12) $x_{n+1} = 3(-1)^n x_n + 2x_{n-1}$ and its semiconjugate factorization are also valid in the finite rings $\mathbb{Z}_m$ for $m \geq 4$. For instance, in $\mathbb{Z}_6$

$$2^n = \begin{cases} 2, & n \text{ odd} \\ 4, & n \text{ even} \end{cases}$$

and using the above formula

$$x_n = (-1)^{n(n-1)/2}[x_0 + (2^n - 1)(x_1 - x_0)]$$

we obtain periodic solutions (typically of period 4) from the above formulas. In $\mathbb{Z}_7$ the distinct powers of 2 are

$$2^n = \begin{cases} 2, & n = 3j+1 \\ 4, & n = 3j+2 \\ 1, & n = 3j+3 \end{cases}, \quad j = 0, 1, 2, \ldots$$

which give solutions of a different period (12 if $x_0 \neq x_1$) using the same formulas.

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Let $R = \mathbb{R}$ the field of real numbers. A "Poincaré difference equation" (of order 2) is a difference equation

$$x_{n+1} = \alpha_n x_n + \beta_n x_{n-1} \tag{13}$$

in which $\alpha_n \to \alpha$ and $\beta_n \to \beta$ as $n \to \infty$ where α, β are real numbers. The autonomous difference equation

$$y_{n+1} = \alpha y_n + \beta y_{n-1} \tag{14}$$

is the "limiting equation" of the Poincaré equation.

- The classical theorem of Poincaré and Perron can be stated succinctly in terms of eigensequences as follows:

  "Each eigenvalue of (14) is a limit of a unitary eigensequence of (13)."

- Not every unitary eigensequence of (13) may converge to an eigenvalue, or to any other number. For instance, the autonomous difference equation

  $$x_{n+1} = 2x_n - 4x_{n-1}$$

  is a Poincaré equation and its own limiting equation with characteristic polynomial $r^2 - 2r + 4$ whose roots $1 \pm i\sqrt{3}$ are complex. It also has an eigensequence $\{1, -2, 4, 1, -2, 4, \ldots\}$ of period 3 which is unitary in $\mathbb{R}$.

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An example: Consider the following Poincaré equation in the field $\mathbb{R}$ of real numbers

$$x_{n+1} = \frac{1}{n}x_n + x_{n-1} \tag{15}$$

The limiting autonomous equation for this is $y_{n+1} = y_{n-1}$ whose eigenvalues are ± 1, i.e., the roots of $r^2 - 1 = 0$. The characteristic equation of (15) is

$$r_{n+1} = \frac{1}{n} + \frac{1}{r_n}. \tag{16}$$

It is readily verified by induction that the solution of (16) with $s_1 = 1$ may be expressed as

$$s_{2n-1} = 1, \quad s_{2n} = \frac{2n}{2n-1}.$$

Thus $\lim_{n\to\infty} s_n = 1$, as expected. The eigensequence $\{s_n\}$ also yields a sc-factorization of (15)

$$t_{n+1} = -\frac{1}{s_n}t_n, \quad t_1 = x_1 - x_0$$
$$x_{n+1} = s_{n+1}x_n + t_{n+1}$$

that can be used to obtain a formula for the general solution of (15). The factor equation can be written as

$$t_{2n} = -t_{2n-1}, \quad t_{2n+1} = -\frac{2n-1}{2n}t_{2n}$$

which yields by straightforward iteration

$$t_{2n+1} = \frac{(2n)!}{4^n (n!)^2}, \quad t_{2n+2} = -t_{2n+1}.$$

Finally, using the cofactor equation $x_{n+1} = r_{n+1}x_n + t_{n+1}$ a formula for the solution of (15) may be obtained.

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Regarding rings of real valued functions under pointwise addition and multiplication of functions we have the following corollary.

Corollary 3 *Let $R(S)$ be a ring of real-valued functions on a nonempty set S. Assume that $a_{j,n}(s) \geq 0$ for all $s \in S$, $j = 0, 1, \ldots, k$ and all n. If*

$$\sum_{j=0}^{k} a_{j,n}(s) > 0 \tag{17}$$

for all $s \in S$ and all n then the linear difference equation

$$x_{n+1} = a_{0,n}x_n + a_{1,n}x_{n-1} + \cdots + a_{k,n}x_{n-k} \tag{18}$$

has positive (hence, unitary) solutions and a sc-factorization in $R(S)$.

Proof. Let $a_{j,n}(s) \geq 0$ for all $s \in S$ and all n. Choose constant initial values $u_j = 1$ for $j = 0, 1, \ldots, k$ in (18). By (17), $u_{k+1}(s) = \sum_{j=0}^{k} a_{j,n}(s) > 0$ for all $s \in S$. Thus $u_{k+1}(s)$ is a unit in $R(S)$ and

$$u_{k+2}(s) = \sum_{j=0}^{k} a_{j,k+1}(s)u_{k+1-j}(s) = a_{0,k+1}(s)u_{k+1}(s) + \sum_{j=1}^{k} a_{j,k+1}(s)$$

for all $s \in S$. If $\sum_{j=1}^{k} a_{j,k+1}(s) = 0$ for some s then by (17) $a_{0,k+1}(s) \neq 0$. It follows that u_{k+2} is also positive on S, hence a unit in $R(S)$. Proceeding in this fashion, it follows that $u_n(s) > 0$ for all $s \in S$ and all n. Thus $\{u_n(s)\}$ is a unitary solution of (18). Hence, the ratios sequence $\{u_n(s)/u_{n-1}(s)\}$ is a unitary eigensequence in $R(S)$ and this yields a sc-factorization for (18). ■

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As an example, consider the second order, linear difference equation

$$x_{n+1}(s) = \frac{2n}{s} x_n(s) + x_{n-1}(s), \quad s \in (0, \infty). \tag{19}$$

This is the recurrence relation for the "modified Bessel functions $K_n(s)$ of the second kind", so-called because they are solutions of the second-order linear differential equation known as Bessel' s modified differential equation. In fact, the sequence of functions $\{K_n(s)\}$ is a particular solution of (19) from specified initial values $K_0(s), K_1(s)$. According to the preceding Corollary a unitary solution $\{u_n(s)\}$ of (19) is generated by any pair of positive functions; e.g., $u_0(s) = u_1(s) = 1$. The first few terms are

$$u_2(s) = \frac{2}{s} + 1,\ u_3(s) = \frac{8}{s^2} + \frac{4}{s} + 1,\ u_4(s) = \frac{48}{s^3} + \frac{24}{s^2} + \frac{2}{s} + 1$$

Now the ratios $u_n(s)/u_{n-1}(s)$ define an eigensequence for (19) and yield the sc-factorization

$$t_{n+1}(s) = -\frac{u_{n-1}(s)}{u_n(s)} t_n(s) \quad x_{n+1}(s) = \frac{u_{n+1}(s)}{u_n(s)} x_n(s) + t_{n+1}(s)$$

with $t_1(s) = x_1(s) - [u_1(s)/u_0(s)]x_0(s) = x_1(s) - x_0(s)$. Iteration of the factor equation yields $t_n(s) = (-1)^{n-1} t_1(s)/u_{n-1}(s)$. If we insert this into the cofactor then summation yields a formula for the general solution of (19) in terms of the unitary solution $\{u_n(s)\}$ as follows:

$$\begin{aligned} x_n(s) &= u_n(s) x_1(s) + \sum_{i=2}^{n-1} \frac{u_n(s)}{u_i(s)} t_i(s) \\ &= u_n(s) \left[x_0(s) + t_1(s) \sum_{i=1}^{n-1} \frac{(-1)^{i-1}}{u_i(s) u_{i-1}(s)} \right]. \end{aligned}$$

Different values of positive functions $u_0(s), u_1(s)$ yield different formulas but of course, the same quantity $x_n(s)$.