Fields Institute Communications
Volume **00**, 0000

On the Equation $x_{n+1} = cx_n + f(x_n - x_{n-1})$

Hassan Sedaghat
Department of Mathematics
Virginia Commonwealth University
Richmond, VA 23284-2014, USA
hsedagha@vcu.edu

Abstract. In this note we present two results about the equation in the title. The first concerns oscillatory behavior and the non-existence of period-2 solutions. The second gives a condition for the existence of monotonic solutions.

In this paper, we consider the second order difference equation

$$x_{n+1} = cx_n + f(x_n - x_{n-1}), \quad c \in [0, 1) \tag{0.1}$$

with real initial values x_0, x_{-1}. We assume that f is continuous on the real line. Variants of this equation have appeared in macroeconomic models of the business cycle for over half a century. One of the earliest cases was Samuelson's linear model in [4] in which the function f was linear-affine of type $at + b$. Nonlinear models appeared subsequently when it was noticed that persistent oscillatory behavior (bounded, non-convergent) in linear models is not structurally stable. The model by Hicks in [2] used a piecewise linear form for f that incorporated known economic mechanisms. Goodwin proposed (in a continuous time model in [1]) that the "investment function" f must be a non-decreasing sigmoid, similar to the inverse tangent. More recently, we have seen variations of this sigmoid in the work of Puu [3]. For more details see [5], [6] and [8].

In the preceding models, conditions leading to persistent oscillatory behavior were of particular interest, since data always indicated the existence of such a behavior in the actual business cycle; see [5]- [8]. Stochastic terms could then be added to a viable nonlinear equation for more realistic modeling. Strange and unusual behavior, such as the existence of an unstable, yet globally attracting fixed point is seen to occur for (0.1); see [6] and [8]. Further, in [8] Puu's variation is discussed in detail and it is shown how chaotic behavior may arise from a version of (0.1) in the case where savings are fully consumed in each period. In this note two additional new results about solutions of (0.1) are presented. The first is about the oscillatory behavior of solutions and the second gives a condition that implies the monotonicity of all solutions.

2000 *Mathematics Subject Classification.* Primary 39A10, 39A11.

1 The Main Results

In general, oscillatory solutions of (0.1) need not oscillate about the equilibrium. For example, if

$$f(t) = \min\{1, |t|\}, \quad c = 0$$

then $f(0) = 0$ so that the origin is the unique fixed point or equilibrium of (0.1). However, (0.1) has a period-3 solution $\{0, 1, 1\}$ which is clearly non-negative and oscillatory. The condition in the following lemma forces all oscillations to occur about the origin.

Lemma 1.1. *If $tf(t) \geq 0$ for all t, then every eventually non-negative and every eventually non-positive solution of* (0.1) *is eventually monotonic.*

Proof Suppose that $\{x_n\}$ is a solution of (0.1) that is eventually non-negative, i.e., there is $k > 0$ such that $x_n \geq 0$ for all $n \geq k$. Either $x_n \geq x_{n-1}$ for all $n > k$ in which case $\{x_n\}$ is eventually monotonic, or there is $n > k$ such that $x_n \leq x_{n-1}$. In the latter case,

$$x_{n+1} = cx_n + f(x_n - x_{n-1}) \leq cx_n \leq x_n$$

so that by induction, $\{x_n\}$ is eventually non-increasing, hence monotonic. The argument for an eventually non-positive solution is similar and omitted. □

Theorem 1.1. *If $tf(t) \geq 0$ for all t, then* (0.1) *has no solutions that are eventually of period two.*

Proof Let $\{x_n\}$ be a solution of (0.1). We claim that if $c > 0$, then for all $k \geq 1$

$$x_k > 0 > x_{k+1} \Rightarrow x_{k+2} < 0 \quad \text{and} \quad x_k < 0 < x_{k+1} \Rightarrow x_{k+2} > 0.$$

For suppose that $x_k > 0 > x_{k+1}$ for some $k \geq 1$. Then

$$x_{k+2} = cx_{k+1} + f(x_{k+1} - x_k) \leq cx_{k+1} < 0.$$

The argument for the other case is similar and omitted. Now by Lemma 1, if a solution $\{x_n\}$ eventually has period 2, then for all sufficiently large n, there is $x_n > 0$, $x_{n+1} < 0$ and $x_{n+2} = x_n > 0$. If $c > 0$, then this contradicts the above claim. If $c = 0$ then

$$0 < x_n = x_{n+2} = f(x_{n+1} - x_n) \leq 0$$

which is again a contradiction. Hence, no solution of (0.1) can eventually have period two. □

Theorem 1.2. *Assume that $tf(t) \geq 0$ and there is $a \in (0, 1)$ such that $|f(t)| \leq a|t|$ for all t. If*

$$c \geq 2\sqrt{a} - a \tag{1.1}$$

then every solution of (0.1) *is eventually monotonic and converges to zero.*

Proof Inequality (1.1) is equivalent to $(a + c)^2 \geq 4a$ so that

$$p = \frac{a + c - \sqrt{(a + c)^2 - 4a}}{2}$$

is a real number (the significance of p becomes clear below). First, assume that

$$0 \leq px_{-1} \leq x_0 \leq x_{-1} \tag{1.2}$$

and let $\{x_n\}$ be the solution generated by the initial values x_{-1}, x_0. Suppose that there is an integer $m \geq 1$ such that $x_m \leq 0$ but $x_{m-1} > 0$. Then $x_{-1} \geq x_0 > \cdots > x_m$ and

$$0 \geq x_m = cx_{m-1} + f(x_{m-1} - x_{m-2}) \geq cx_{m-1} + a(x_{m-1} - x_{m-2})$$

so that

$$x_{m-1} \leq \frac{a}{a+c} x_{m-2}. \tag{1.3}$$

Further,

$$\frac{a}{a+c} x_{m-2} \geq x_{m-1} = cx_{m-2} + f(x_{m-2} - x_{m-3}) \geq cx_{m-2} + a(x_{m-2} - x_{m-3})$$

which yields

$$x_{m-2} \leq \frac{a}{a+c-a/(a+c)} x_{m-3}. \tag{1.4}$$

The development of coefficients in inequalities (1.3) and (1.4) follows a pattern that is described next. Define the mapping

$$\phi(t) = \frac{a}{a+c-t}, \qquad 0 \leq t < a+c.$$

In terms of ϕ, inequalities (1.3) and (1.4) may be written as

$$x_{m-1} \leq \phi(0) x_{m-2}$$

and

$$x_{m-2} \leq \phi^2(0) x_{m-3}$$

respectively. This pattern continues inductively, since if for any $n = 2, \ldots, m$ we have

$$x_{m-n+1} \leq \phi^{n-1}(0) x_{m-n}$$

then

$$\phi^{n-1}(0) x_{m-n} \geq cx_{m-n} + f(x_{m-n} - x_{m-n-1}) \geq cx_{m-n} + a(x_{m-n} - x_{m-n-1})$$

which yields

$$x_{m-n} \leq \frac{a}{a+c-\phi^{n-1}(0)} x_{m-n-1} = \phi^n(0) x_{m-n-1}.$$

In particular, for $n = m$, we obtain

$$x_0 \leq \phi^m(0) x_{-1}.$$

But, upon iterating the mapping ϕ above, it is easy to see that the real sequence $\{\phi^n(0)\}$ is strictly increasing towards p, which is a fixed point of ϕ. Therefore, $\phi^m(0) < p$ and we obtain

$$x_0 \leq \phi^m(0) x_{-1} < p x_{-1}$$

which contradicts (1.2). It follows that $x_n > 0$ for all n if (1.2) holds so by Lemma 1 $\{x_n\}$ is monotonic, and in fact, non-increasing. Since the origin is the only fixed point of (0.1), it follows that $\{x_n\}$ converges to zero.

A similar argument shows that if $x_{-1} \leq x_0 \leq px_{-1} \leq 0$, then $x_n < 0$ for all $n \geq 1$ and thus again the solution is non-decreasing and converges to zero.

In the general case, the solution starting from an arbitrary pair of initial values x_{-1}, x_0 if not monotonic, will have a term x_k which is either a positive maximum or a negative minimum, i.e., $x_k \geq 0$ and $x_{k-1}, x_{k+1} \leq x_k$ or $x_k \leq 0$ and $x_{k-1}, x_{k+1} \geq x_k$. Consider the positive maximum case. Then

$$x_k \geq x_{k+1} = cx_k + f(x_k - x_{k-1}) \geq cx_k. \tag{1.5}$$

Now, since

$$(c-a)^2 = c^2 - 2ac + a^2 > c^2 - 2a + a^2 = (c+a)^2 - 4a$$

it follows that

$$c - p = \frac{c - a - \sqrt{(a+c)^2 - 4a}}{2} > 0.$$

From (1.5) it follows that $x_k \geq x_{k+1} > px_k \geq 0$ at which point the argument for case (1.2) holds and establishes that the solution $\{x_n\}$ is monotonically decreasing for $n > k$. A similar argument for the negative minimum case completes the proof. □

An interesting conjecture that might naturally follow Theorem 2 is that all solutions of (0.1) converge to zero in Theorem 2 *even if* (1.1) *does not hold*. Condition (1.1) may be re-written equivalently as

$$a \leq \left(1 - \sqrt{1-c}\right)^2 = b. \tag{1.6}$$

It may be conjectured that *if* $b < a < 1$ *then all solutions converge to zero in an oscillatory fashion*. This claim is true for the linear version of (0.1) where $f(t) = at$. In the linear case, the number b is the demarkation line where solutions change from monotonic to oscillatory as a exceeds b. Indeed, if (1.6) holds, then the linear version of (0.1) has two positive eigenvalues the smaller of which is the number p in the proof of Theorem 2. If (1.6) does not hold, then there are two complex eigenvalues whose common modulus is a.

References

[1] Goodwin, R.M. *The Nonlinear Accelerator and The Persistence of Business Cycles*, Econometrica, **19** (1951) 1–17

[2] Hicks, J.R. *A Contribution to the Theory of the Trade Cycle*, (2nd ed.) Clarendon Press, Oxford, 1965 [First edition: Oxford University Press, Oxford, 1950]

[3] Puu, T. *Nonlinear Economic Dynamics*, (3rd ed.) Springer-Verlag, 1993

[4] Samuelson, P.A. *Interaction between the Multiplier Analysis and the Principle of Acceleration*, The Review of Economic Statistics, **21** (1939) 75–78

[5] Sedaghat, H. *A Class of Nonlinear Second Order Difference Equations from Macroeconomics*, Nonlinear Analysis, **29** (1997) 593–603

[6] Sedaghat, H. *Bounded Oscillations in the Hicks Business Cycle Model and Other Delay Equations*, J. Difference Equations and Applications, **4** (1998) 325–341

[7] Sedaghat, H. *Geometric Stability Conditions for Higher Order Difference Equations*, J. Mathematical Analysis and Applications, **224** (1998) 255–272

[8] Sedaghat, H. *Nonlinear Difference Equations: Theory with Applications to Social Science Models*, Kluwer Academic Publishers (forthcoming)