Difference Equations as Discrete Dynamical Systems

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1 Introduction.

Interest in difference equations goes a long way back to the times before the discovery of differential and integral calculus. For instance, the famous sequence 0,1,1,2,3,5,8,... that appeared in the work of Fibonacci (circa 1202) is the solution of a difference equation, namely $x_n = x_{n-1} + x_{n-2}$ with given initial values $x_{-1} = 0$, $x_0 = 1$. This difference equation also generates the well-known Lucas numbers if $x_{-1} = 1$, $x_0 = 3$. Even after the invention of the concept of derivative until around the middle of the twentieth century, difference equations found numerous applications in numerical analysis where they were used in the solution of algebraic and differential equations. Indeed, the celebrated Newton's method for finding roots of scalar equations is an example of a difference equation, as is the equally famous Euler's method for estimating solutions of differential equations through estimation of the derivative by a finite difference [6]. These are just two among many other and more refined difference methods for dealing with complex problems in calculus and differential equations. By the mid-twentieth century, the theory of linear difference equations had been developed in sufficient detail to rival, indeed parallel its differential analog. This theory had already been put to use in the 1930's and 40's by economists ([23, 45]) in their analyses of discrete-time models of the business cycle.

Interest of a different sort began to emerge in the 1960's and 70's with important discoveries such as the Mandelbrot and Julia sets, the Sharkovsky ordering of cycles and the Li-Yorke "chaos theorem". A substantial amount of work by numerous researchers since then led to the creation of a qualitative theory of difference equations that no longer paralleled similar discoveries in differential equations. Although many analogs can be found between the two disciplines, there are also significant differences; for example, the Poincare-Bendixon theorem [24] establishes dimension 3 as the minimum needed for the occurrence of deterministic chaos in differential equations, whereas such behavior can appear even in simple, one-dimensional difference equations. The work in the latter part of the 20th century inspired a further development of the qualitative theory of difference equations which included the study of conditions for asymptotic stability of equilibria and cycles and other significant aspects of nonlinear difference equations.

Solutions of difference equations are sequences and their existence is often not a significant problem, in contrast to differential equations. Further, it is unnecessary to estimate solutions of difference equations. Because of their recursive nature, it is easy to generate actual solutions on a digital computer starting from given initial values. Therefore, modelers are quickly rewarded with insights about both the transient and the asymptotic behavior of their equation of interest. A deeper understanding can then be had from the qualitative theory of nonlinear difference equations which has now been developed sufficiently to make it applicable to a wide variety of modeling problems in the biological and social sciences. In studying nonlinear difference equations, qualitative methods are not simply things to use in the absense of quantitative exactitude. In the relatively rare cases where the general solution to a nonlinear difference equation can be found analytically, it is often the case that such a solution has a complicated form that is more difficult to use and analyze than the comparatively simple equation that gave rise to it (see, e.g. Example 9 below). Thus even with an exact solution at hand, it may not be easy to answer basic questions such as whether an equilibrium exists, or if it is stable, or if there are periodic or non-periodic solutions.

In this article we present some of the fundamental aspects of the modern qualitative theory of nonlinear difference equations of order one or greater. The primary purpose is to acquaint the reader with the outlines of the standard theory. This includes some of the most important results in the field as well as a few of the latest findings so as to impart a sense that a coherent area of mathematics exists in the discrete settings that is independent of the continuous theory. Indeed, there are no continuous analogs for many of the results that we discuss below. As it is not possible to cover so broad an area in a limited number of pages, we leave out all proofs. The committed reader may pursue the matters further through the extensive list of references provided. Entire topics, such as bifurcation theory, fractals and complex dynamics and measure theoretic or stochastic dynamics had to be left out; indeed, each of these topics is quite extensive and it would be impossible to meaningfully include more than one of these within the confines of a single chapter.

2 Basic Concepts.

A *discrete dynamical system* (autonomous, finite dimensional) basically consists of a mapping $F : D \to D$ on a nonempty set $D \subseteq \mathbb{R}^m$. We usually assume that F is continuous on D. We abbreviate the composition $F \circ F$ by F^2, and refer to the latter as an *iterate* of F. The meaning of F^n for $n = 3, 4, \ldots$ is inductively clear; for convenience, we also define F^0 to be the identity mapping. For each $\mathbf{x}_0 \in D$, the sequence $\{F^n(\mathbf{x}_0)\}$ of iterates of F is called a *trajectory* or *orbit* of F through $\mathbf{x}_0$ (more specifically, a *forward orbit* through $\mathbf{x}_0$). Sometimes, $\mathbf{x}_0$ is called the *initial point* of the trajectory. In analogy with differential equations, we sometimes refer to the system domain as the *phase space,* and call the plot of a trajectory in D a *phase plot.* Also, the plot of a scalar component of $F^n(\mathbf{x}_0)$ versus n is often called a *time series.*

Associated with the mapping F is the recursion

$$\mathbf{x}_n = F(\mathbf{x}_{n-1}) \tag{1}$$

which is an example of a *first order, autonomous vector difference equation.* The vector equation (1) is equivalent to a *system* of scalar difference equations, analogously to systems of differential equations which are composed of a finite number of ordinary differential equations. If the map $F(\mathbf{x}) = A\mathbf{x}$ is linear, where $\mathbf{x} \in \mathbb{R}^m$ and A is an $m \times m$ matrix of real numbers, then (1) is called a *linear difference equation*. Otherwise, (1) is *nonlinear* (usually this excludes cases like the linear-affine map $F(\mathbf{x}) = A\mathbf{x}+B$, where B is an $m\times m$ matrix, since such cases are easy to convert to linear ones by a translation). Each trajectory $\{F^n(\mathbf{x}_0)\}$ is a *solution* of (1) with initial point $\mathbf{x}_0$ and may be abbreviated $\{\mathbf{x}_n\}$. Unlike a first order nonlinear differential equation, it is clear that (1) always has solutions as long as F is defined on D, and that each solution of (1) may be recursively generated from some point of D by iterating F. A (*scalar, autonomous*) *difference equation of order* m is defined as

$$x_n = f(x_{n-1}, x_{n-2}, \ldots, x_{n-m}) \tag{2}$$

where the scalar map $f : \mathbb{R}^m \rightarrow \mathbb{R}$ is continuous on some domain $D \subset \mathbb{R}^m$. Given any set of m initial values $x_0, x_{-1}, \ldots, x_{-m+1} \in \mathbb{R}$, equation (2) recursively generates a solution $\{x_n\}$, $n \geq 1$.Equation (2) may be expressed in terms of vector equations as defined previously. Associated with f or with (2) is a mapping

$$V_f(u_1, \ldots, u_m) \doteq [f(u_1, \ldots, u_m), u_1, u_2, \ldots, u_{m-1}]$$

of $\mathbb{R}^m$ which we call the *standard vectorization* or "unfolding" of f. Note that if we define

$$\mathbf{x}_n \doteq [x_n, \ldots, x_{n-m+1}], \quad n \geq 0$$

then

$$\begin{aligned} V_f(\mathbf{x}_{n-1}) &= [f(x_{n-1}, \ldots, x_{n-m}), x_n, \ldots, x_{n-m+1}] \\ &= [x_n, x_{n-1}, \ldots, x_{n-m+1}] \\ &= \mathbf{x}_n. \end{aligned}$$

Hence, the solutions of (2) are known if and only if the solutions of the vector equation $\mathbf{x}_n = V_f(\mathbf{x}_{n-1})$ are known. The latter equation is the standard vectorization of Eq.(2) and a common way of expressing (2) as a system of difference equations.

A *fixed point* of F is a point $\bar{\mathbf{x}}$ such that $F(\bar{\mathbf{x}}) = \bar{\mathbf{x}}$. Clearly, iterations of F do not affect $\bar{\mathbf{x}}$, so $\bar{\mathbf{x}}$ is a *stationary point* or *equilibrium* of (1). For Eq.(2), $\bar{x}$ is a fixed point if and only if $\bar{\mathbf{x}} = (\bar{x}, \ldots, \bar{x})$ is a fixed point of V_f. A fixed point of F^k for some fixed integer $k \geq 1$ is called a *k-periodic point* of F. The orbit of a k-periodic point $\mathbf{p}$ of F is a finite set $\{\mathbf{p}, F(\mathbf{p}), \ldots, F^{k-1}(\mathbf{p})\}$ which is called a *cycle* of F *of length* k, or a *k-cycle* of F. A point $\mathbf{q}$ is *eventually periodic* if there is $l \geq 1$ such that $\mathbf{p} = F^l(\mathbf{q})$ is periodic. These definitions imply that a k-cycle of Eq.(2) is a finite set $\{p_1, \ldots, p_k\}$ of real numbers such that $p_{kn+i} = p_i$ for $i = 1, \ldots, k$ and $n \geq 1$. Evidently, the point $\mathbf{p} = (p_1, \ldots, p_k)$ is a period-k point of V_f.

A fixed point $\bar{\mathbf{x}}$ of F is said to be *stable* if for each $\varepsilon > 0$, there is $\delta > 0$ such that $x_0 \in B_\delta(\bar{\mathbf{x}})$ implies that $F^n(x_0) \in B_\varepsilon(\bar{\mathbf{x}})$ for all $n \geq 1$. Here, $B_r(\bar{\mathbf{x}})$ is the open ball of radius $r > 0$ with center $\bar{\mathbf{x}}$, which consists of all points in $\mathbb{R}^m$ that are within a distance r from $\bar{\mathbf{x}}$. Thus $\bar{\mathbf{x}}$ is stable if trajectories starting near $\bar{\mathbf{x}}$ stay close to $\bar{\mathbf{x}}$. If a fixed point is not stable, then it is called *unstable.* If there is $r > 0$ such that for all $x_0 \in B_r(\bar{\mathbf{x}})$ the trajectory $\{F^n(x_0)\}$ converges to $\bar{\mathbf{x}}$, then $\bar{\mathbf{x}}$ is *attracting.* If $\bar{\mathbf{x}}$ is both stable and attracting, then $\bar{\mathbf{x}}$ is said to be *asymptotically stable.*

In the case of the scalar difference equation (2), linearization is more easily done than for (1) because the characteristic polynomial of the derivative is easily determined. The Jacobian of the vectorization V_f is given by the $m \times m$ matrix

$$\begin{bmatrix} \partial f/\partial x_1 & \partial f/\partial x_2 & \cdots & \partial f/\partial x_{m-1} & \partial f/\partial x_m \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \tag{3}$$

where the partial derivatives are evaluated at an equilibrium point $(\bar{x}, \ldots, \bar{x})$.

The characteristic polynomial of this matrix (and hence, also of the linearization of the scalar difference equation at the equilibrium) is computed easily as:

$$P(\lambda) = \lambda^m - \sum_{i=1}^{m} \frac{\partial f}{\partial r_i}(\bar{x}, \ldots, \bar{x})\lambda^{m-i}. \tag{4}$$

The roots of this polynomial give the eigenvalues of the linearization of (2) at the fixed point $\bar{x}$. If all roots of $P(\lambda)$ have modulus less than unity (i.e., if all roots lie within the interior of the unit disk in the complex plane) then the fixed point $\bar{x}$ is *locally* asymptotically stable, i.e. it is stable and attracts trajectories starting in some usually small neighborhood of $\bar{x}$. If any root lies in the exterior of the unit circle then $\bar{x}$ is unstable as some trajectories are repelled away from $\bar{x}$ because of the eigenvalues outside the unit disk.

If the mappings F or f above depend on the index n, then we obtain the more general *non-autonomous* versions of equations (1) and (2) as

$$\mathbf{x}_n = F(n, \mathbf{x}_{n-1}) \tag{5}$$

and

$$x_n = f(n, x_{n-1}, x_{n-2}, \ldots, x_{n-m}) \tag{6}$$

respectively. Equations (5) and (6) are useful in modeling such things as periodic changes in the environment in the case of modelling discrete-time population growth or to account for non-homogeneous media in the case of discrete spaces as in infarcted cardiac tissue.

For additional reading on fundamentals of difference equations, see [1, 2, 4, 12, 14-16, 18, 19, 26-28, 31, 37, 46, 51, 62].

3 First Order Difference Equations.

Equations (1) and (2) are the same when $m = 1$, i.e., the dimension of the phase space is 1, the same as the order of the equation. In this section we present conditions for the asymptotic stability or instability of equilibria and cycles, the ordering of cycles for continuous mappings and conditions for the occurrence of chaotic behavior for first order equations. Equations (5) and (6) are also the same when $m = 1$, but they are not first order equations, since the variable n in F or f adds an additional dimension (see, e.g. [18])

3.1 Asymptotic stability: Necessary and sufficient conditions.

Let $f : I \to I$ be a mapping of an interval I of real numbers. Here the *invariant interval* I may be any interval, bounded or unbounded as long as it is not empty or a singleton. Let $\bar{x}$ be a fixed point of f in I that is *isolated,* meaning that there is an open interval J such that $x \in J \subset I$ and J contains no other fixed points of f. The one-dimensional version of (3) is the following:

$$|f'(\bar{x})| < 1. \tag{7}$$

In (7) it is assumed that f has a continuous derivative at the fixed point $\bar{x}$. If (7) holds, then $\bar{x}$ is stable and the orbit of each point x_0 will converge to $\bar{x}$ provided that x_0 is sufficiently near $\bar{x}$. Furthermore, the rate of convergence is *exponential*, i.e. $|f^n(x_0) - \bar{x}|$ is proportional to the quantity e^{-an} where $a > 0$ for all $n \geq 1$. On the other hand, if $|f'(\bar{x})| > 1$ then nearby points x_0 will be repelled by the unstable fixed point $\bar{x}$, also at an exponential rate.

Example 1. The equation

$$x_n = ax_{n-1}(1 - x_{n-1}) \tag{8}$$

is called the one-dimensional *logistic equation* (with the underlying "logistic map" $f(x) = ax(1-x)$.) For $a > 1$ this equation has a unique positive fixed point $\bar{x} = (a-1)/a$. Further, if $a \leq 4$ then f has an invariant interval [0,1]. Since

$$f'(\bar{x}) = a - 2a\bar{x} = 2 - a$$

it follows that $\bar{x}$ is (i) asymptotically stable if $1 < a < 3$ or (ii) repelling if $a > 3$. The origin 0 is the only other fixed point of the Logistic equation and it is unstable because $|f'(0)| = a > 1$.

Despite its ease of use, Condition (7) which is based only on a linear approximation of f has some limitations: (i) It does not specify *how near* $\bar{x}$ the initial point x_0 needs to be (it may have to be quite near in some cases); (ii) it requires that f be smooth; (iii) it conveys no information when $|f'(\bar{x})| = 1$; (iv) it is *not a necessary condition*.

We may remove one or more of these deficiencies by using more detailed information about the function f than its tangent line approximation can provide. In particular, *necessary and sufficient* conditions for asymptotic stability are given next.

Theorem 1. (Asymptotic Stability) *Let $f : I \to I$ be a continuous mapping and let $\bar{x}$ be an isolated fixed point of f in I. The following statements are equivalent*:

(a) *$\bar{x}$ is asymptotically stable*;

(b) *There is a neigborhood U of $\bar{x}$ such that for all $x \in U \subset I$ the following inequalities hold*:

$$f^2(x) > x \text{ if } x < \bar{x}, \quad f^2(x) < x \text{ if } x > \bar{x}. \tag{9}$$

(c) *There is a neighborhood U of $\bar{x}$ such that for all $x \in U \subset I$ the following inequalities hold*:

$$f(x) > x \text{ if } x < \bar{x}, \quad f(x) < x \text{ if } x > \bar{x}. \tag{10}$$

and the graph of f_r^{-1} (the inverse image of the part of f to the right of $\bar{x}$) lies above the graph of f.

Reversing the inequalities in Theorem 1 gives conditions that are necessary and sufficient for $\bar{x}$ to be repelling. This follows from the next result that characterizes all possible types of behavior at an isolated equilibrium for a mapping of the real number line.

Theorem 2. *Let $f : I \to I$ be a continuous mapping and assume that f has an isolated fixed point $\bar{x} \in I$. Then precisely one of the following is true*:

(i) $\bar{x}$ *is asymptotically stable*;

(ii) $\bar{x}$ *is unstable and repelling*;

(iii) $\bar{x}$ *is semistable (attracting from one side of $\bar{x}$ and repelling from the other side).*

(iv) *There is a sequence of period-2 points of f converging to $\bar{x}$.*

Example 2. Let us examine the Logistic Equation (8) in cases $a = 1, 3$. If

$a = 1$, then it is easy to verify that

$$f(x) = x(1 - x) < x \quad \text{if} \quad x \neq 0.$$

It follows that 0 is a semistable fixed point. Now consider $a = 3$. This value of the parameter a represents the boundary between stability and instability. Linearization fails in this case so we apply Theorem 1. Note that $\bar{x} = 2/3$ when $a = 3$ and

$$f^{\,2}(x) = 3[3x(1 - x)][1 - 3x(1 - x)] = 9x(1 - x)(1 - 3x + 3x^2).$$

The function $9(1 - x)(1 - 3x + 3x^2)$ is decreasing on $(-\infty, \infty)$ and has the value 1 at $x = 2/3 = \bar{x}$. Thus (9) is satisfied and the fixed point is asymptotically stable.

The situation in Theorem 2(iv) does not occur for the Logistic map, but it does occur for maps of the line; a trivial example is $f(x) = -x$ which has a unique fixed point at the origin. See [51, p.32] for a more interesting example.

3.2 Cycles and Limit Cylces.

One of the most striking features of continuous mappings of the line is the way in which their periodic points can coexist. The following result establishes the peculiar manner and ordering in which the cycles of a continuous map of an interval appear.

Theorem 3. (Coexisting cycles) *Suppose that a continuous map $f : I \to I$ of the interval I has a cycle of length m, and consider the following total ordering relation $\rhd$ of the positive integers:*

$$\begin{array}{l}
3 \rhd 5 \rhd 7 \rhd 9 \rhd 11 \rhd \cdots \\
2 \times 3 \rhd 2 \times 5 \rhd 2 \times 7 \rhd 2 \times 9 \rhd \cdots \\
\vdots \\
2^i \times 3 \rhd 2^i \times 5 \rhd 2^i \times 7 \rhd 2^i \times 9 \rhd \cdots \\
\vdots \\
\cdots \rhd 2^n \rhd \cdots \rhd 2^3 \rhd 2^2 \rhd 2 \rhd 1.
\end{array}$$

Then for every positive integer k such that $m \rhd k$, there is a cycle of length k for f. In particular, if f has a cycle of length 3, then it has cycles of all possible lengths.

It may be shown that a continuous mapping f has a 3-cycle in the interval I if and only if there is $\alpha \in I$ such that

$$f^3(\alpha) \leq \alpha < f(\alpha) < f^2(\alpha), \text{ or } f^3(\alpha) \geq \alpha > f(\alpha) > f^2(\alpha). \tag{11}$$

These conditions have come to be known as the *Li-Yorke Conditions* [41]. The Logistic map of Example 1 has a 3-cycle when $a \approx 3.38$.

It must be emphasized that even if a 3–cycle exists for a mapping f, most or even all of the cycles may be unstable. Therefore, in numerical simulations one will not see all of the above cycles. The question as to which cycles, if any, can be stable is the subject of the next theorem.

Theorem 4. (Limit cycles) *Let $f \in C^3(I)$ where $I = [a, b]$ is any closed and bounded interval, with Schwarzian*

$$Sf = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 < 0$$

on I. Here $Sf = -\infty$ is a permitted negative value.

(a) *If p is an asymptotically stable periodic point of f, then either a critical point of f or an endpoint of I converges to the orbit of p.*

(b) *If f has $N \geq 0$ critical points in I, then f has at most $N + 2$ limit cycles (including any asymptotically stable fixed points).*

Example 3. Consider the Logistic map $f(x) = ax(1 - x)$, $x \in [0, 1]$, $1 < a \leq 4$. We know that $Sf < 0$ on $[0, 1]$ since f is a quadratic polynomial and $f'''(x) = 0$ for all x. Since $f'(0) = a > 1$, it follows that 0 is unstable so it cannot attract the orbit of the unique critical point $c = 1/2$, unless

$c \in f^{-k}(0)$ for some integer $k \geq 1$. If $a < 4$, then the maximum value of f on [0,1] is $f(c) = a/4 < 1$, so that $f^{-k}(0) = \{0, 1\}$. Thus when $a < 4$, Theorem 4 implies that there can be at most one limit cycle, namely the one whose orbit attracts 1/2. In fact, because Theorem 4 holds for non-hyperbolic periodic points also [51, p.47], it follows that for the logistic map all cycles, except possibly one, must be repelling! If a limit cycle exists for some value of a then certainly, we may compute $f^{n}(1/2)$ for sufficiently large n to estimate that limit cycle. If $a = 4$, then $f^{2}(1/2) = 0$, the unstable fixed point. Hence, by Theorem 4 there are no limit cycles in this case, even though a 3-cycle exists for $a = 4$ and therefore, by Theorem 3 there are (unstable) cycles of all lengths.

3.3 Chaos.

Recall from the Introduction that by the Poincare-Bendixon theorem chaotic behavior does not occur in *differential* equations in dimensions 1 and 2 (i.e first and second order differential equations), so dimension 3 as the minimum needed for the occurrence of deterministic chaos in the continuous case. However, even a first order *difference* equation can exhibit complex, aperiodic solutions that are stable, in the sense that "most" solutions display such unpredictable behavior. The essential characteristic of these solutions is that they exhibit *sensitive dependence on the initial value* x_0; i.e., a trajectory starting from a piont arbitrarily close to x_0 rapidly (at an exponential rate) diverges from the trajectory that starts from x_0. This unpredictability exists even though the underlying map f is completely known; therefore, this phenomenon has been termed "deterministic chaos." Since sensitive dependence on initial values exists near any repelling fixed point, usually deterministic chaos involves certain other features also, such as complicated orbits that may be dense in some subspace if not the whole space. The following famous result shows that "period 3 implies chaos".

Theorem 5. (Chaos: Period 3) *Let I be a bounded, closed interval and let f be a continuous function on I satisfying one of the inequalities in* (11). *Then the following are true*:

(a) *For each positive integer k, f has a k-cycle.*

(b) *There is an uncountable set $S \subset I$ such that S contains no periodic points of f and satifies the following conditions*:

(b1) *For every* $p,q \in S$ *with* $p \neq q$,

$$\limsup_{n\to\infty} |f^n(p) - f^n(q)| > 0 \tag{12}$$

$$\liminf_{n\to\infty} |f^n(p) - f^n(q)| = 0. \tag{13}$$

(b2) *For every* $p \in S$ *and periodic* $q \in I$,

$$\limsup_{n\to\infty} |f^n(p) - f^n(q)| > 0. \tag{14}$$

The set S is called the *scrambled set* of f. Its existence characterizes chaotic behavior on the line in the sense of Li and Yorke. By (12), the orbits of two arbitrarily close points in S will be pulled a finite distance away under iterations of f so there is sensitivity to initial conditions. By (13) the orbits of any two points in S can get arbitrarily close to each other, and by (14) no orbit starting from within S can converge to a periodic solution. The Logistic map with $3.83 < a \leq 4$ has a 3-cycle in $I = [0, 1]$ and thus satisfies the conditions of Theorem 5. If a is close to 3.83, then the 3-cycle is stable and it follows that S is a proper subset of I. In many cases, S is a "large" subset of I (e.g., when a is large enough that the 3-cycle becomes unstable) and for $a = 4$, in fact S is dense in I. On the other hand, S could be a fractal set of Lebesgue measure zero in some cases.

The Logistic map displays erratic behavior for smaller values of a also and this suggests that the existence of period 3 may not be necessary for the occurrence of chaotic behavior. Indeed, for difference equations in dimensions 2 or greater this is often the case. In particular, we can define a mapping of the unit disk that exhibits sensitive dependence on initial values and its trajectories are dense in the disk, but which has no k-cycles for $k > 1$ [51, p.127] There is also a result which applies to higher dimensional maps and establishes a Li-Yorke type chaos in which Condition (a) in Theorem 5 is relaxed; under conditions of Theorem 12 below, chaotic behavior occurs in the Logistic map for $a \geq 3.7$.

Chaotic behavior may occur even on unbounded intervals. The following example illustrates this feature.

Example 4. Consider the continuous, piecewise smooth mapping

$$\rho(r) = \left|1 - \frac{1}{r}\right|, \quad r > 0.$$

This mapping does not leave the interval $(0, \infty)$ invariant since it maps 1 to 0, but it does have a scrambled set in $(0, \infty)$. We show this through an indirect application of Theorem 5 because ρ itself does not have a period-3 point. Remarkably, ρ does have period-p points for all poisitive integers $p \neq 3$ and in [49] these points were explicitly determined using the Fibonacci numbers. We can complete the list of periodic solutions for ρ by adding a "3-cycle that passes through ∞" as follows:

Let $[0, \infty]$ be the one-point compactification of $[0, \infty)$ and define ρ^* on $[0, \infty]$ as

$$\rho^*(0) = \infty, \quad \rho^*(\infty) = 1, \quad \rho^*(r) = \rho(r), \; 0 < r < \infty.$$

Note that ρ^* extends ρ continuously to $[0, \infty]$ and furthermore, ρ^* has a 3-cycle $\{1, 0, \infty\}$. The interval $[0, \infty]$ is homeomorphic to [0,1] so ρ^* is chaotic on $[0, \infty]$ in the sense of Theorem 5. To show that ρ is chaotic on $(0, \infty)$, we show that it has a scrambled set. Let S^* be a scrambled set for ρ^* and take out ∞ and the set of all backward iterates of 0, namely, $\cup_{n=0}^{\infty} \rho^{-n}(0)$ from S^*. The subset S that remains is contained in $(0, \infty)$. Further, S is uncountable because S^* is uncountable and each inverse image $\rho^{-n}(0)$ is countable for all $n = 0, 1, 2, \ldots$ (in fact $\cup_{n=0}^{\infty} \rho^{-n}(0)$ is the set of all non-negative rational numbers; see [49]). Further, since $\rho(S) \subset S$ and $\rho|_S = \rho^*|_S$ it follows that S is a scrambled set for ρ that has the properties stated in Theorem 5.

3.4 Notes.

A proof of the equivalence of (a) and (b) in Theorem 1 first appeared in [61]; also see [62]. A different proof of this fact which also established the equivalence of Part (c) was first given in [54]. Theorem 1 is true only for maps on an invariant subset of the real number line. It is not true for invariant subsets of the Euclidean plane; see [56]. For complete proofs of Theorem 1 and Theorem 2, including other equivalent conditions not stated here and some further comments, see [51].

Theorem 3 was first proved in [60]; also see [10, 16, 51]. Like Theorems 1 and 2, this result is peculiar to the maps of the interval and is not shared by the continuous mappings of a closed, one dimensional manifold like the circle, or by continuous mappings of higher dimensional Euclidean spaces. See e.g. [49] for an example of a second order difference equation all of whose solutions are either periodic of period 3 or else they converge to zero.

Theorem 4 was proved almost simultaneously in [3] and [63]. Also see [10, 16, 51]. Theorem 5 was first proved in [41]; also see [51]. Example 4 is extracted from [48].

4 Higher Order Difference Equations.

Equations of type (2) or (6) with $m \geq 2$ are higher order difference equations. As noted above, these can be converted to first order vector equations.

Unfortunately, the theory of the preceding section largely fails for the higher order equations or for vector equations. Higher order equations provide a considerably greater amount of flexibility for modeling applications, but results that are comparable to those of the previous section in power and generality are lacking. Nevertheless, there do exist results that apply to general classes of higher order difference equations and we shall be concerned with those results in this section. Because of space limitations we will not discuss results that are applicable to narrowly defined types of difference equations. Although many interesting results of profound depth have been discovered for such classes of difference equations, these types of equations do not offer the flexibility needed for scientific modeling.

4.1 Asymptotic stability: Weak contractions.

A general sufficient condition for the asymptotic stability (non-local) of a fixed point is given next.

Theorem 6. (Asymptotic Stability) *Let f be continuous with an isolated fixed point $\bar{x}$, and let M be an invariant closed set containing $\bar{\mathbf{x}} = (\bar{x}, \ldots, \bar{x})$. If A is the containing $\bar{\mathbf{x}}$ and all $\mathbf{u} = (u_1, \ldots, u_m) \in M$ such that*

$$|f(\mathbf{u}) - \bar{\mathbf{x}}| < \max\{|u_1 - \bar{x}|, \ldots, |u_m - \bar{x}|\}$$

then $\bar{\mathbf{x}}$ *is asymptotically stable relative to each invariant subset* S *of* A *that is closed in* M; *in particular,* $\bar{\mathbf{x}}$ *attracts every trajectory with a vector of initial values* $(x_{1-m}, \ldots, x_0) \in S$.

Further, if A *is open and contains* $\bar{\mathbf{x}}$ *in its interior, then* $\bar{x}$ *is asymptotically stable relative to* $(\bar{x}-r, \bar{x}+r)$, *where* $r > 0$ *is the largest real number such that* $B_r(\bar{\mathbf{x}}) \subset A$. *In particular, if* $A = \mathbb{R}^m$, *then* $\bar{x}$ *is globally asymptotically stable.*

When f satisfies the inequality in Theorem 6 we may say that f is a *weak contraction* at $\bar{x}$ ("weak" because the vectorization V_f is not strictly a contraction in the usual sense).

Example 5. Consider the third order equation

$$x_n = ax_{n-1} + bx_{n-3} \exp\left(-cx_{n-1} - dx_{n-3}\right), \qquad (15)$$

$$a, b, c, d \geq 0, \;\; c + d > 0$$

This equation was derived from a model for the study of observed variations in the flour beetle population; see [33]. For an entertaining account of the flour beetle experiments see [8]. We show that the origin is asymptotically stable (so that the beetles go extinct) if

$$a + b \leq 1, \quad b > 0. \qquad (16)$$

We note that the linearization of (15) at the origin, i.e., the characteristic polynomial (4), has a unit eigenvalue $\lambda = 1$ when $a + b = 1$. Therefore, linear stability analysis is not applicable in this particular case.

Now, observe that if (16) holds, then for every $(x, y, z) \in [0, \infty)^3$,

$$\begin{aligned} ax + bz\exp(-cx - dz) &\leq [a + b\exp(-cx - dz)]\max\{x, z\} \\ &< (a + b)\max\{x, y, z\} \\ &\leq \max\{x, y, z\} \end{aligned}$$

so by Theorem 6 the origin is stable and attracts all non-negative solutions of (15).

A generalization of Example 5 is the first part of the next corollary of Theorem 6, which in particular provides a simple tool for establishing the global stability of the zero equilibrium. The simple proof (showing that the map is a weak contraction) is omitted.

Corollary 1. (a) *Let* $f_i \in C([0, \infty)^m, [0, 1))$ *for* $i = 1, \ldots, k$ *and* $k \geq 2$. *If* $\sum_{i=1}^{k} f_i(u_1, \ldots, u_m) < 1$ *for all* $(u_1, \ldots, u_m) \in [0, \infty)^m$ *then the origin is the unique, globally asymptotically stable fixed point of the following equation*:

$$x_n = \sum_{i=1}^{k} f_i(x_{n-1}, \ldots, x_{n-m})x_{n-i}.$$

(b) *The origin is a globally asymptotically stable fixed point of the equation*

$$x_n = x_{n-k} g(x_{n-1}, \ldots, x_{n-m}), \quad 1 \leq k \leq m$$

where $g \in C(\mathbb{R}^m, \mathbb{R})$, *if* $|g(\mathbf{x})| < 1$ for all $\mathbf{x} \neq (0, \ldots, 0)$.

The following result gives an interesting and useful version of Theorem 6 for the non-autonomous equation (6).

Theorem 7. *Let* f *be the function in Eq.(6) and assume that there is a sequence* $a_n \geq 0$ *such that for all* $u \in \mathbb{R}^m$ *and all* $n \geq 1$,

$$|f(n, \mathbf{u})| \leq a_n \max\{|u_1|, \ldots, |u_m|\}.$$

(a) *If* $\limsup_{n \to \infty} a_n = a < 1$ *then the origin is the globally exponentially stable fixed point of (6).*

(b) *If* $b_k = \max\{a_{mk}, a_{mk+1}, \ldots, a_{mk+k}\}$ *and* $\prod_{k=0}^{\infty} b_k = 0$, *then the origin is the globally asymptotically stable fixed point of (6).*

4.2 Asymptotic stability: Coordinate-wise monotonicity.

In case the function f in (2) or (6) is either non-decreasing or non-increasing in all of its arguments, it is possible to obtain general conditions for asymptotic stability of a fixed point. There are several results of this type and we discuss some of them in this section along with their applications. The first result is the most general of its kind on attractivity of a fixed point within a given interval.

Theorem 8. *Assume that* $f : [a,b]^m \to [a,b]$ *in (2) is continuous and satisfies the following conditions:*

(i) *For each* $i \in \{1., \ldots, m\}$ *the function* $f(u_1, \ldots, u_m)$ *is monotone in the coordinate* u_i *(with all other coordinates fixed);*

(ii) *If* (μ, ν) *is a solution of the system*

$$f(\mu_1, \mu_2, \ldots, \mu_m) = \mu$$

$$f(\nu_1, \nu_2, \ldots, \nu_m) = \nu$$

then $\mu = \nu$, *where for* $i \in \{1., \ldots, m\}$ *we define*

$$\mu_i = \begin{cases} \mu & \text{if } f \text{ is non-decreasing in } u_i \\ \nu & \text{if } f \text{ is non-increasing in } u_i \end{cases}$$

and

$$\nu_i = \begin{cases} \nu & \text{if } f \text{ is non-decreasing in } u_i \\ \mu & \text{if } f \text{ is non-increasing in } u_i \end{cases}.$$

Then there is a unique fixed point $\bar{x} \in [a,b]$ *for (2) that attracts every solution of (2) with initial values in* $[a,b]$.

The following variant from [7] has less flexibility in the manner in which f depends on variations in coordinates, but it does not involve a bounded interval and adds stability to the properties of $\bar{x}$.

Theorem 9. *Let r_0, s_0 be extended real numbers where $-\infty \leq r_0 < s_0 \leq \infty$ and consider the following hypotheses:*

(H1) $f(u_1, \ldots, u_m)$ is non-increasing in each $u_1, \ldots, u_m \in I_0$ where $I_0 = (r_0, s_0]$ if $s_0 < \infty$ and $I_0 = (r_0, \infty)$ otherwise;

(H2) $g(u) = f(u, \ldots, u)$ is continuous and decreasing for $u \in I_0$;

(H3) There is $r \in [r_0, s_0)$ such that $r < g(r) \leq s_0$. If $r_0 = -\infty$ or $\lim_{t \to r_0^+} g(t) = \infty$ then we assume that $r \in (r_0, s_0)$.

(H4) There is $s \in [r, x^*)$ such that $g^2(s) \geq s$, where $g^2(s) = g(g(s))$.

(H5) There is $s \in [r, x^*)$ such that $g^2(u) > u$ for all $u \in (s, x^*)$

Then the following is true:

(a) *If (H2) and (H3) hold then Equation (2) has a unique fixed point x^* in the open interval $(r, g(r))$.*

(b) *Let $I = [s, g(s)]$. If (H1)-(H4) hold then I is an invariant interval for (2) and $x^* \in I$.*

(c) *If (H1)-(H3) and (H5) hold then x^* is stable and attracts all solutions of (2) with initial values in $(s, g(s))$.*

(d) *If (H1)-(H3) hold then* x^* *is an asymptotically stable fixed point of (2) if it is an asymptotically stable fixed point of the mapping* g; *e.g., if* g *is continuously differentiable with* $g'(x^*) > -1$.

It may be emphasized that the conditions of Theorems 8 and 9 imply asymptotic stability over an interval and as such, they impart considerably greater information than linear stability results about the ranges on which convergence occurs. They also have the added advantage that if the extent of the interval I is not an issue, then we may reduce the amount of calculations considerably by using Theorem 9(d) instead of examining the roots of the characteristic polynomial (4).

Example 6. Consider the difference equation

$$x_n = \frac{\alpha - \sum_{i=1}^{m} a_i x_{n-i}}{\beta + \sum_{i=1}^{m} b_i x_{n-i}}, \quad \alpha, \beta > 0, \tag{17}$$

$$a_i, b_i \geq 0, \ 0 < a = \sum_{i=1}^{m} a_i < \beta, b = \sum_{i=1}^{m} b_i > 0$$

The function f for this equation is

$$f(u_1, \ldots, u_m) = \frac{\alpha - \sum_{i=1}^{m} a_i u_i}{\beta + \sum_{i=1}^{m} b_i u_i}$$

which is non-increasing in each of its m coordinates if $u_i < \alpha/a$ for all $i = 1, \ldots, m$. Eq.(17) has one positive fixed point $\bar{x}$ which is a solution of the equation $g(t) = t$ where

$$g(t) = \frac{\alpha - at}{\beta + bt}.$$

Eq.(17) satisfies (H1)-(H3) in Theorem 11 with $s = g^{-1}(\alpha/a)$ and

$$g'(\bar{x}) = \frac{-a\beta - \alpha b}{(\beta + b\bar{x})^2}.$$

It is easy to verify that $\bar{x} \in [s, \alpha/a]$ with $|g'(\bar{x})| < 1$ so $\bar{x}$ is asymptotically stable according to Theorem 9(d).

We finally mention the following result for the non-autonomous equation

$$x_n = f\left(\sum_{i=1}^{m} [a_{n-i} g\left(x_{n-i}\right) + g_i\left(x_{n-i}\right)]\right), \quad n = 1, 2, 3, \ldots \tag{18}$$

which is of type (6) whose *autonomous* version (the coefficients a_{n-i} all have the same value) is a special case of Theorem 9. In Eq.(18) we assume that:

$$a_{mn+i} = a_i \geq 0,\ \ i = 1, \ldots, m,\ \ n = 1, 2, 3, \ldots,\ \ a = \sum_{i=1}^{m} a_i. \tag{19}$$

The functions f, g, g_i are all continuous on some interval (t_0, ∞) of real numbers $\mathbb{R}$ and monotonic (non-increasing or non-decreasing). It is assumed for non-triviality that all a_i and all g_i are not simultaneously zero. Define the function h as

$$h(t) = f\left(\sum_{i=0}^{m} g_i\,(t)\right), \quad g_0(t) = ag(t).$$

and assume that h satisfies the condition

$$h(t) \text{ is decreasing for } t > t_0 \geq -\infty. \tag{20}$$

Theorem 10. *Suppose that (19) and (20) hold and consider the following assumptions:*

(A1) *For some* $r > t_0 \geq -\infty$, $h(r) > r$;

(A2) *For some* $s \in (r, \bar{x})$, $h^2(s) \geq s$;

(A3) *For some* $s \in (r, \bar{x})$, $h^2(t) > t$ for all $t \in (s, \bar{x})$.

Then:

(a) *If (A1) holds, then Eq.(18) has a unique fixed point* $\bar{x} > r$.

(b) *If (A1) and (A2) hold, then the interval* $(s, h(s))$ *is invariant for (18) and* $\bar{x} \in (s, h(s))$.

(c) *If (A1) and (A3) hold, then the fixed point* $\bar{x}$ *of Eq.(18) is stable and attracts every point in the interval* $(s, h(s))$.

Example 7. The propagation of an action potential pulse in a ring of excitable media (e.g. cardiac tissue) can be modeled by the equation

$$x_n = \sum_{i=1}^{m} a_{n-i} C\left(x_{n-i}\right) - A\left(x_{n-m}\right) \tag{21}$$

if certain threshold and memory effects are ignored and if all the cells in the ring have the conduction properties. Models of this type can aid in gaining a better understanding of causes of cardiac arrhythmia. The ring is composed of m units (e.g. cardiac cell aggregates) and the functions A and C in (21) represent the restitutions of action potential duration and the conduction time, respectively, as functions of the diastolic interval x_n. The numbers a_n represent lengths of the ring's excitable units which are not generally constant, but the sequence $\{a_n\}$ is m-periodic since cell aggregate $m+1$ is the same as cell aggregate 1 and another cycle through the same cell aggregates in the ring starts (the reentry process).

The following conditions are generally assumed:

(C1) There is $r_A \geq 0$ such that the APD restitution function A is continuous and increasing on the interval $[r_A, \infty)$ with $A(r_A) \geq 0$.

(C2) There is $r_C \geq 0$ such that the CT restitution function C is continuous and nonincreasing on the interval $[r_C, \infty)$ with $\inf_{x \geq r_C} C(x) \geq 0$.

(C3) There is $r \geq \max\{r_A, r_C\}$ such that $mC(r) > A(r) + r$.

Define the function $F = mC - A$ and note that by (C1)-(C3), F is continuous and decreasing on the interval $[r, \infty)$ and satisfies

$$F(r) > r. \tag{22}$$

Thus (A1) holds and there is a unique fixed point $\bar{x}$ for (21). Now assume that the following condition is also ture:

(C4) There is $s \in [r, \bar{x})$ such that $F^2(x) > x$ for all $x \in (s, \bar{x})$.

Then (A1) and (A3) in Theorem 10 are satisfied with $h = F$, $f(t) = t$, $g = C$ and $g_i = -A/m$ for all i and the existence of an asymptotically stable fixed point for (21) is established. In the context of cardiac arrhythmia, this means that there is an equilibrium heart beat period of $A(\bar{x}) + \bar{x}$ that is usually shorter than the normal beat period by a factor of 2 or 3.

4.3 Persistent oscillations and chaos.

It is possible to establish the existence of oscillatory solutions for Eq.(2) if the linearization of this equation exhibits such behavior; i.e., if some of the roots of the characteristic polynomial (4) are either complex or negative.

Such linear oscillations occur both in stable cases where all roots of (4) have modulus less than one and in unstable cases where some roots have modulus greater than one. Further, linear oscillations take place about the equilibrium (i.e., going past the fixed point repeatedly infinitely often) and by their very nature, if linear oscillations are bounded and non-decaying, then they are not structurally stable or robust.

Our focus here is on a different and less familiar type of oscillation. This type of nonlinear oscillation is bounded and persistent (non-decaying), and unlike linear oscillations, it is structurally stable. Further, it need not take place about the equilibrium, although it is caused by the instability of the equilibrium.

We define a *presistently oscillating* solution of (2) simply as one that is bounded and has two or more (finite) limit points.

Theorem 11. (Persistent Oscillations) *Assume that f in Eq.*(2) *has an isolated fixed point $\bar{x}$ and satisfies the following conditions*:

(a) *For $i = 1, \ldots, m$, the partial derivatives $\partial f / \partial x_i$ exist continuously at $\bar{\mathbf{x}} = (\bar{x}, \ldots, \bar{x})$, and every root of the characteristic polynomial* (4) *has modulus greater than 1*;

(b) $f(\bar{x}, \ldots, \bar{x}, x) \neq \bar{x}$ *if* $x \neq \bar{x}$.

Then all bounded solutions of (2) *except the trivial solution* $\bar{x}$ *oscillate persistently. If only* (a) *and* (b) *hold, then all bounded solutions that do not converge to some* $\bar{x}$ *in a finite number of steps oscillate persistently.*

Example 8. The second order difference equation

$$x_{n+1} = cx_n + g(x_n - x_{n-1}) \tag{23}$$

has been used in the classical theories of the business cycle where g is often assumed to be non-decreasing also; see [20, 23, 44, 45, 51, 52, 57]. If $0 \leq c < 0$ and g is a bounded function, then it is easy to see that all solutions of (23) are bounded and confined to a closed interval I. If additionally g is continuously differentiable at the origin with $g'(0) > 1$ then by Theorem 11 for all initial values x_0, x_{-1} that are not both equal to the fixed point $\bar{x} = g(0)/(1-c)$, the corresponding solution of (23) oscillates persistently, eventually in the absorbing interval I.

We also mention that if $tg(t) \geq 0$ for all t in the interval I that contains the fixed point 0 of (23), then every eventually non-negative and every eventually non-positive solution of (23) is eventually monotonic [50, 52]. This is true in particular if $g(t)$ is linear or an odd function. But in general it is possible for (23) to have oscillatory solutions that are eventually non-negative (or non-positive). For example, if

$$g(t) = \min\{1, |t|\}, \quad c = 0$$

then (23) has a period-3 solution $\{0, 1, 1\}$ which is clearly non-negative and oscillatory. This type of behavior tends to occur when g has a global minimum (not necessarily unique) at the origin, including even functions; see [52] for more details.

With regard to chaotic behavior, it may be noted that persistently oscillating solutions need not be erratic. Indeed, they could be periodic as in the preceding example. The conditions stated in the next theorem are more restrictive than those in Theorem 11, enough to ensure that erratic behavior does occur. The essential concept for this result is defined next.

Let $F : D \to D$ be continuously differentiable where $D \subset \mathbb{R}^m$, and let the closed ball $\bar{B}_r(\bar{\mathbf{x}}) \subset D$ where $\bar{\mathbf{x}} \in D$ is a fixed point of F and $r > 0$. If for every $\mathbf{x} \in \bar{B}_r(\bar{\mathbf{x}})$, all the eigenvalues of the Jacobian $DF(\mathbf{x})$ have magnitudes greater than 1, then $\bar{\mathbf{x}}$ is an *expanding fixed point.* If in addition, there is $\mathbf{x}_0 \in \bar{B}_r(\bar{\mathbf{x}})$ such that: (a) $\mathbf{x}_0 \neq \bar{\mathbf{x}}$; (b) there is a positive integer k_0 such that $F^{k_0}(\mathbf{x}_0) = \bar{\mathbf{x}}$; (c) $\det[DF^{k_0}(\mathbf{x}_0)] \neq 0$, then $\bar{\mathbf{x}}$ is a *snap-back repeller.* If $F = V_f$ is a vectorization so that $\bar{\mathbf{x}} = (\bar{x}, \ldots, \bar{x})$ then we refer to the fixed point $\bar{x}$ as a snap-back repeller for Eq.(2). Note that because of its expanding nature, a snap-back repeller satisfies the main Condition (a) of Theorem 11.

Theorem 12. (Chaos: Snap-back repellers) *Let* $D \subset \mathbb{R}^m$ *and assume that a continuously differentiable mapping* F *has a snap-back repeller. Then the following are true*:

(a) *There is a positive integer* N *such that* F *has a point of period* n *for every integer* $n \geq N$.

(b) *There is an uncountable set* S *satisfying the following properties*:

(i) $F(S) \subset S$ *and there are no periodic points of* F *in* S;

(ii) *For every* x,$y \in S$ *with* $x \neq y$,

$$\limsup_{n\to\infty} \|F^n(x) - F^n(y)\| > 0;$$

$$\limsup_{n\to\infty} \|F^n(x) - F^n(y)\| > 0.$$

(c) *There is an uncountable subset* S_0 *of* S *such that for every* $x, y \in S_0$:

$$\liminf_{n\to\infty} \|F^n(x) - F^n(y)\| = 0.$$

The norm $\|\cdot\|$ in Parts (b) and (c) above may be assumed to be the Euclidean norm in $\mathbb{R}^m$. The set S in (b) is analogous to the similar set in Theorem 5, so we may call it a “scrambled set” too. Unlike Theorem 5, Theorem 12 can be applied to models in any dimension; see [17, 51] for applications to social science models.

4.4 Semiconjugacy: First order equations revisited.

Given that there is a comparatively better-established theory for the first order difference equations than for equations of order 2 or greater, it is of interest that certain classes of higher order difference equations can be related to suitable first order ones. For these classes of equations, it is possible to discuss stability, convergence, periodicity, bifurcations and chaos using results from the first-order theory. Due to limitations of space, in this section we do not present the basic theory, but use specific examples to indicate how the main concepts can be used to study higher order difference equations. For the basic theory and additional examples, see [49, 51].

We say that the higher order difference equation (2) is *semiconjugate* to a first order equation if there are functions h and φ such that

$$h(V_f(u_1,\ldots,u_m)) = \varphi(h(u_1,\ldots,u_m)).$$

The function h is called a *link* and the function φ is called a *factor* for the semiconjugacy. The first order equation

$$t_n = \varphi(t_{n-1}), \quad t_0 = h(x_0,\ldots,x_{-m+1}) \tag{24}$$

is the equation to which Eq.(2) is semiconjugate. A relatively straightforward theory exists that relates the dynamics of (2) to that of (24) and bears remarkable similarity to the theory of Liapunov functions [38, 39, 51]. The link function h plays a crucial role also, and in some cases, two or more different factor and link maps can be found that shed light on different aspects of the higher order difference equation.

Example 9. Consider the second order scalar difference equation

$$x_n = \frac{a}{x_{n-1}} + bx_{n-2} \qquad a, b, x_0, x_{-1} > 0. \tag{25}$$

It can be readily shown that this equation (see [42]) is semiconjugate to the first order equation

$$t_n = a + bt_{n-1} \tag{26}$$

with the link $H(x, y) = xy$ over the positive quadrant $D = (0, \infty)^2$. Indeed, multiplying both sides of (25) by x_{n-1} gives

$$x_n x_{n-1} = a + bx_{n-1}x_{n-2}$$

which has the same form as the first order equation if $t_n = x_n x_{n-1}$. This semiconjugacy can be used to establish unboundedness of solutions for Eq.(25). Note that if $b \geq 1$ then all solutions of the first order equation (26) diverge to infinity, so that the product

$$x_n x_{n-1} = t_n \tag{27}$$

is unbounded. It follows from this observation that every positive solution of (25) is unbounded if $b \geq 1$.

If $b < 1$, then every solution of the first order equation converges to the positive fixed point $\bar{t} = a/(1-b)$. It is evident that the curve $xy = \bar{t}$ is an invariant set for the solutions of (25) in the sense that if $x_0 x_{-1} = \bar{t}$ then $x_n x_{n-1} = \bar{t}$ for all $n \geq 1$. Hence every solution of (25) converges to this invariant curve. Now since all solutions of the first order equation $x_n x_{n-1} = \bar{t}$ are periodic with period 2, we conclude that if $b < 1$ then all non-constant, positive solutions of (25) converge to period 2 solutions.

In this example we saw how semiconjugacy allows a factorization of the second order equation into two simple first order ones. In this particular example, we can actually solve Eq.(25) exactly, by first solving (26) to get

$$t_n = \alpha + \beta b^n, \quad \alpha = \frac{a}{1-b}, \ \beta = t_0 - \alpha, \ t_0 = x_0 x_{-1}$$

(for $b \neq 1$) and then using this solution in (27) to find an explicit solution for (25) as

$$x_n = x_0 \delta_n \prod_{k=1}^{n/2} \frac{1 + cb^{n-2k+2}}{1 + cb^{n-2k+1}} + x_{-1}(1 - \delta_n) \prod_{k=1}^{(n+1)/2} \frac{1 + cb^{n-2k+2}}{1 + cb^{n-2k+1}}$$

where $c = \beta/\alpha$ and $\delta_n = [1 + (-1)^n]/2$.

Example 10. Consider again the second order difference equation from

Example 8 above

$$x_n = cx_{n-1} + g(x_{n-1} - x_{n-2}), \qquad x_0, x_{-1} \in \mathbb{R}. \tag{28}$$

where g is continuous on $\mathbb{R}$ and $0 \leq c \leq 1$. In particular, in [44] the equation

$$y_n = (1-s)y_{n-1} + sy_{n-2} + Q(y_{n-1} - y_{n-2})$$

is considered where $0 \leq s \leq 1$ and Q is the "investment function" (taken as a cubic polynomial in [44]). We may re-write this equation as

$$y_n = y_{n-1} - s(y_{n-1} - y_{n-2}) + Q(y_{n-1} - y_{n-2})$$

which has the form (28) with $c = 1$ and $g(t) = Q(t) - st$. In the case $c = 1$ Eq.(28) is semiconjugate with link function $h(x, y) = x - y$ and factor $g(t)$. In this case, since $x_n - x_{n-1} = t_n$, the solutions of (28) are just sums of the solutions of the first order equation

$$t_n = g(t_{n-1}), \quad t_0 = x_0 - x_{-1}$$

i.e., $x_n = \sum_{k=1}^{n} t_k$. If g is a chaotic map (e.g., if it has a snap-back repeller or a 3-cycle) then the solutions of (28) exhibit highly complex and unpredictable behavior. See [44, 51] for further details.

Example 11. Let a_n, b_n, d_n be given sequences of real numbers with $a_n \geq 0$ and $b_{n+1} + d_n \geq 0$ for all $n \geq 0$. The difference equation

$$x_{n+1} = a_n|x_n - cx_{n-1} + b_n| + cx_n + d_n, \quad c \neq 0 \tag{29}$$

is non-autonomous of type (6). Using a semiconjugate factorization we show that its general solution is given by

$$x_n = x_0 c^n + \sum_{k=1}^{n} c^{n-k}\left(d_{k-1} + |t_0|\prod_{j=0}^{k-1} a_j + \sum_{i=1}^{k-1}(b_i + d_{i-1})\prod_{j=i}^{k-1} a_j\right). \tag{30}$$

where $t_0 = x_0 - cx_{-1} + b_0$. To see this, note that Eq.(29) has a semiconjugate factorization as

$$x_n - cx_{n-1} + b_n = t_n, \quad t_n = a_{n-1}|t_{n-1}| + b_n + d_{n-1}. \tag{31}$$

The second equation in (31) may be solved recursively to get

$$t_n = |t_0|\prod_{j=0}^{n-1} a_j + b_n + d_{n-1} + \sum_{i=1}^{n-1}(b_i + d_{i-1})\prod_{j=i}^{n-1} a_j. \tag{32}$$

Substituting (32) in the first equation in (31) and using another recursive argument yields (30). For additional details on the various interesting features of this and related equations see [29].

Example 12. As our final example we consider the difference equation

$$x_{n+1} = |ax_n - bx_{n-1}|, \qquad a, b \geq 0, \ n = 0, 1, 2, \ldots \tag{33}$$

An equation such as this may appear implicitly in smooth difference equations (or difference relations) that are in the form of e.g., quadratic polynomials. We may assume that the initial values x_{-1}, x_0 in Eq.(33) are non-negative and for non-triviality, at least one is positive. Dividing both sides of (33) by x_n we obtain a ratios equation

$$\frac{x_{n+1}}{x_n} = \left| a - \frac{bx_{n-1}}{x_n} \right|$$

which can be written as

$$r_{n+1} = \left| a - \frac{b}{r_n} \right|, \quad n = 0, 1, 2, \ldots \tag{34}$$

where we define $r_n = x_n/x_{n-1}$ for every $n \geq 0$. Thus we have a semiconjugacy in which Eq.(34) is the factor equation with mapping

$$\phi(r) = \left| a - \frac{b}{r} \right|, \quad r > 0$$

and the link function is $h(x, y) = x/y$. Note that a general solution of (33) is obtained by computing the solution r_n of (34) and then using these values of r_n in the non-autonomous linear equation $x_n = r_n x_{n-1}$ to obtain the solution $x_n = x_0 \prod_{k=1}^{n} r_k$. However, closed forms are not known for the nontrivial solutions of (34). In fact, when $a = b = 1$ we saw in Example 4 above that Eq.(34) has chaotic solutions. For a detailed analysis of the soltuions of equations (33) and (34) see [30, 49].

4.5 Notes.

Theorem 6 and related results were established in [55]; see also [51].For some results concerning the rates of convergence of solutions in Theorem 6, see [64]. Theorem 6 has also been shown to hold in any complete metric space not just $\mathbb{R}$; [65]. For a proof of Theorem 7 and related results see [5].

Theorem 8 appeared without a proof in [21]; it is based on similar results for second-order equations that are proved in [35]. This theorem in particular generalizes the main result in [22] where f is assumed to be non-decreasing in every coordinate. The requirement that the invariant interval $[a, b]$ be bounded is essential for the validity of Theorem 8, though it may be restrictive in some applications. Theorem 9 was motivated by the study of Eq.(21) in Example 7 and was established in different cases in [58] and [47]. The more complete version on which Theorem 9 is based is proved in [7]. Example 6 is based on results from [13]. Theorem 10 is proved in [47] and Example 7 is based on results from [56]. For additional background material behind the model in Example 7 also see [11, 26].

Theorem 11 is based on results in [56] where some classical economic models are also studied. Theorem 12 was first proved in [43]; see [51] for various applications of this result to social science models.

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