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GEOMETRIC PROPERTIES OF FACTORABLE PLANAR SYSTEMS OF DIFFERENTIAL EQUATIONS*

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Abstract. Systems of two ordinary differential equations whose expressions factor in rectangular coordinates into separate functions of each of the two variables are called factorable planar systems. Special geometrical properties of the solutions and the equilibria of these systems are discussed and, in particular, it is shown that such systems do not have limit cycles.

Key words. factorable, limit cycle, periodic solution, equilibrium, second-order equations

AMS subject classifications. 34C05, 34C35

1. Introduction.

By a *factorable planar system* we mean the following autonomous system of two differential equations:

$$\dot{x} = f(x)h(y), \qquad \dot{y} = k(x)g(y), \tag{1}$$

where the dots represent time derivatives and the functions f, g, h, k are continuously differentiable on $(-\infty, \infty)$ or a suitable open subset of it. Special cases of system (1) appear frequently in lectures and textbooks on continuous dynamical systems. We mention, for instance, the well-known Volterra–Lotka predator–prey model and the standard phase plane representation of the Newtonian equation $\ddot{x} + \varphi(x) = 0$, among others (see, e.g., [4], [2]). Other special occurrences of (1) include the classical laws of Lanchester's combat theory [1], [3]. The flow of (1) displays a variety of possible behaviors, as the aforementioned applications indicate, but there are also some important omissions, notably the absence of foci and limit cycles. The purpose of this note is to discuss some of the basic qualitative features of factorable planar systems (in Cartesian coordinates) that may be of use or interest to those working or just interested in the field of differentiable dynamical systems.

2. The first integral and periodic solutions.

A significant property of (1) is that its phase equation is separable:

$$\frac{dy}{dx} = \frac{g(y)}{h(y)}\frac{k(x)}{f(x)}.$$

Integrating the phase equation, we obtain the first integral of (1) as

$$H(x, y) \equiv F(x) - G(y) = C, \tag{2}$$

where C is a constant of integration and the functions F and G are defined as follows:

$$F(x) = \int_a^x k(u)/f(u)du, \qquad G(y) = \int_b^y h(v)/g(v)dv$$

with a suitable choice of $(a, b) \in \mathbf{R}^2$. We now consider an interesting consequence of the separated form of the function H in (2).

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PROPOSITION 1. *The factorable planar system* (1) *has no limit cycles.*

Proof. Suppose, on the contrary, that there is a limit cycle Γ for (1). Then there is an open set U in $\mathbf{R}^2$ such that Γ is contained in the closure of U and the first integral H is constant on U [2, §11.5]. Since

$$\frac{k(x)}{f(x)} = F'(x) = \frac{\partial H}{\partial x} = 0, \qquad \frac{h(y)}{g(y)} = G'(y) = \frac{-\partial H}{\partial y} = 0$$

for all $(x, y) \in U$, we conclude that the Cartesian product $k^{-1}(0) \times h^{-1}(0)$ of inverse images contains the set U. Since $k^{-1}(0), h^{-1}(0)$ are both closed by continuity, the closure of U (hence also Γ) must be contained in the closed set $k^{-1}(0) \times h^{-1}(0)$. But this is impossible since $\dot{x} = \dot{y} = 0$ on $k^{-1}(0) \times h^{-1}(0)$. Thus there are no limit cycles. □

It is evident from Proposition 1 and the Poincaré–Bendixson theorem that factorability implies a significant simplification of the geometric behavior of planar systems. While not prone to limit cycles, factorable systems do have periodic solutions, as the positive solutions of the Volterra–Lotka equations demonstrate. General conditions for the existence of such orbitally (but not asymptotically) stable cycles are obtained next. Before stating the next result, we point out that (1) has the same phase equation as the Hamiltonian system

$$\text{(3)} \qquad \dot{x} = \frac{-h(y)}{g(y)}, \qquad \dot{y} = \frac{-k(x)}{f(x)}$$

whenever the right-hand side of (3) is well defined. Note that because (3) itself is of the form $\dot{x} = M(y)$, $\dot{y} = N(x)$, it is a special case of (1). The Hamiltonian function of (3) is, of course, the first integral H defined by (2).

PROPOSITION 2. *Assume that* $k(a) = h(b) = 0$. *If* $f(a)g(b)k'(a)h'(b) < 0$, *then the equilibrium* (a, b) *is a center, and the nearby solutions of* (1) *form closed orbits around* (a, b).

Proof. Since the Jacobian matrix of the vector field $(f(x)h(y), k(x)g(y))$ evaluated at (a, b) has the form

$$J(a, b) = \begin{bmatrix} 0 & f(a)h'(b) \\ k'(a)g(b) & 0 \end{bmatrix},$$

it follows that the eigenvalues of $J(a, b)$ are imaginary and given by $\pm i\sqrt{D}$, where $D = -f(a)g(b)k'(a)h'(b)$ is the determinant. To establish that the nonhyperbolic equilibrium (a, b) is indeed a center, we need only show that the surface $H(x, y)$ has a strict local minimum at the point (a, b) [4, p. 154]. We note first that the equilibrium is isolated by the negativity condition (a is not a zero of f and is only a simple zero of k, and similarly for b, g, and h). Observe next that (a, b) is a critical point of H and that

$$H_{xx}(a, b)H_{yy}(a, b) - H_{xy}(a, b)^2 = \frac{-k'(a)h'(b)}{f(a)g(b)} = \frac{D}{[f(a)g(b)]^2} > 0.$$

Now if $H_{xx}(a, b) = k'(a)/f(a) > 0$, then by the second-derivative test (a, b) is a strict local minimum of H. Otherwise, we use $-H$ as a first integral with H defined by (2), so that $(-H)_{xx}(a, b) > 0$, and (after the obvious minor modification of the above proof) the conclusion remains the same. □

As an immediate example, identify the Volterra–Lotka equations as a special case of (1) with $f(x) = x$, $h(y) = \alpha - \beta y$, $k(x) = \gamma x - \delta$, $g(y) = y$, where $\alpha, \beta, \gamma, \delta > 0$, to find

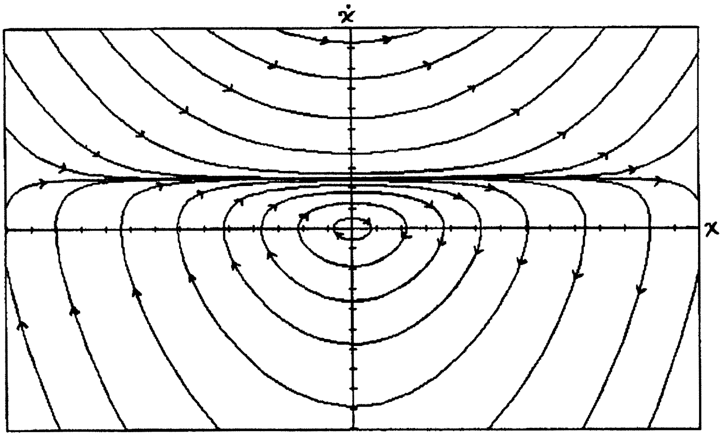

FIG. 1.

that $a = \delta/\gamma$, $b = \alpha/\beta$, so that $-D = (\delta/\gamma)(\alpha/\beta)(-\beta)\gamma < 0$. Therefore, the equilibrium $(\delta/\gamma, \alpha/\beta)$ is a center. Furthermore, by Proposition 1 there are no limit cycles.

Next, we recall that the standard representation of a second-order differential equation $\ddot{x} = \psi(x, \dot{x})$ in the plane has the form

$$\dot{x} = y,$$
$$\dot{y} = \psi(x, y),$$

so that the equilibria are always located on the x-axis. In case the function ψ is factorable, we have the following immediate consequence of the previous two propositions.

COROLLARY 1. *Let g, k be as in* (1) *and assume that a is an isolated zero of k such that $g(0)k'(a) < 0$. Then the second-order equation $\ddot{x} - g(\dot{x})k(x) = 0$ has periodic solutions that oscillate about a and which are not limit cycles.*

For example, with $g(\dot{x}) = \alpha\dot{x} - \beta$, where $\alpha, \beta > 0$, the Liénard-type equation $\ddot{x} - \alpha x\dot{x} + \beta x = 0$ has nonasymptotically stable periodic solutions that oscillate about the origin as long as $\dot{x} < \beta/\alpha$ (an "escape velocity"). A phase portrait for this equation in the phase plane is given in Figure 1.

Using the criteria of Bendixson or Dulac (which are simple applications of the familiar Green theorem; see, e.g., [4, p. 245]), any number of sufficient conditions may be obtained which imply that (1) has no periodic solutions whatsoever. The following result is a sample, and an example of its use appears at the end of this note.

PROPOSITION 3. *Assume in* (1) *that for all $u \in (-\infty, \infty)$, $f'(u)g'(u) \geq 0$, $f'(u) \neq 0$ (or $g'(u) \neq 0$) and also $h(u)k(u) \geq 0$, $h(u) \neq 0$ (or, respectively, $k(u) \neq 0$). Then* (1) *has no periodic solutions.*

Proof. Let $f'(u)h(u) \neq 0$ for all $u \in (-\infty, \infty)$. By continuity, each of f' and h is always positive or always negative. Hypotheses imply that where nonzero, k has the same sign as h, and similarly for f' and g'. Hence, the divergence of the vector field of (1), namely, $f'(x)h(y) + k(x)g'(y)$, is nonzero and does not change sign anywhere in the plane. Therefore, the Bendixson criterion implies that there are no periodic solutions for (1). □

3. Equilibria and their properties. A point (a, b) is an equilibrium point of (1) if and only if at least one of the pairs

$$(f(a), k(a)), \qquad (f(a), g(b)), \qquad (h(b), k(a)), \qquad (h(b), g(b))$$

is (0, 0), and an isolated equilibrium only if one of the middle two pairs is zero. Given the Jacobian matrix

$$J(a,b) = \begin{bmatrix} f'(a)h(b) & f(a)h'(b) \\ k'(a)g(b) & k(a)g'(b) \end{bmatrix},$$

it is clear from the characteristic equation

$$[\lambda - f'(a)h(b)][\lambda - k(a)g'(b)] - f(a)g(b)k'(a)h'(b) = 0$$

that the linearization of (1) near an arbitrary equilibrium (a, b) cannot have a focus. We can say more if (a, b) is *hyperbolic*, i.e., if the eigenvalues λ_1, λ_2 of $J(a, b)$ both have nonzero real parts. The next proposition completely determines the nature of a hyperbolic equilibrium of (1) and, in particular, it implies that no such equilibrium point can be a focus of the nonlinear system.

PROPOSITION 4. *Every hyperbolic equilibrium (a, b) of (1) is either a saddle point or a node. It is a node if $f(a) = g(b) = 0$ and $f'(a)h(b)$ has the same sign as $k(a)g'(b)$. Otherwise, it is a saddle point.*

Proof. Due to hyperbolicity, $f(a)$ and $k(a)$ cannot both be zero, and similarly for $g(b)$, $h(b)$ (this is also true for isolated but not necessarily hyperbolic equilibria). Now if $k(a) = h(b) = 0$, then the eigenvalues are $\pm\sqrt{f(a)g(b)k'(a)h'(b)}$, which are real and nonzero because of the hyperbolicity. Therefore, (a, b) is a saddle point in this case. If, on the other hand, $f(a) = g(b) = 0$, then $\lambda_1 = f'(a)h(b)$ and $\lambda_2 = k(a)g'(b)$, both of which are nonzero. By definition, if λ_1, λ_2 have the same signs then (a, b) is a node; otherwise, it is a saddle point. □

COROLLARY 2. *Every hyperbolic equilibrium $(a, 0)$ of $\ddot{x} - g(\dot{x})k(x) = 0$ (i.e., $k(a) = 0$, $g(0)k'(a) > 0$) is a saddle point.*

Corollaries 1 and 2 show that as long as zero eigenvalues do not arise at an equilibrium point $(a, 0)$ of the second-order ODE (i.e., $g(0)k'(a) \neq 0$), the equilibrium is either a center or a saddle point. If either $g(0) = 0$ or $k'(a) = 0$, then both eigenvalues are zero and other types of behavior are possible. See Concluding remarks (I) below.

COROLLARY 3. *Let (a, b) be a hyperbolic equilibrium of (1) with $f(a) = g(b) = 0$. Then (a, b) is a stable node (or a sink) if the eigenvalues $f'(a)h(b)$ and $k(a)g'(b)$ are both negative, and an unstable node (or a source) if both eigenvalues are positive.*

Concluding remarks. We close by mentioning a few points which we will not discuss in detail here.

(I) If a (or b) is a nonsimple zero of f or k (respectively, g or h) or, more generally, if at least one eigenvalue of $J(a, b)$ is zero, then more complicated and varied behavior may occur. For instance, the origin is a "cusp" for the Newtonian equation $\ddot{x} - x^2 = 0$ (Figure 2). More generally, consider the special class of factorable systems with zero eigenvalues given by the ODE $\ddot{x} = \alpha x^m(\dot{x})^n$, $\alpha \neq 0$, $n, m \geq 0$, $m + n > 1$. In the phase plane form

$$\dot{x} = y, \qquad \dot{y} = \alpha x^m y^n.$$

In this case, the origin is the unique equilibrium for $n = 0$ (the system is "Newtonian"), while for $n \geq 1$, the x-axis forms a continuum of nonisolated equilibria. All trajectories lie on the graphs of algebraic equations of the following type $(m + 1)y^{2-n} + (n - 2)\alpha x^{m+1} = C$, where C is a constant (if, exceptionally, $n = 2$, then the solution may be expressed as $(m + 1)\ln y^2 - 2\alpha x^{m+1} = C$ or, equivalently, as the family $y^2 = C_0 \exp[2\alpha x^{m+1}/(m + 1)]$ of exponential curves where $C_0 > 0$). When $n = 0$ and $m = 2$, the origin is a cusp for all

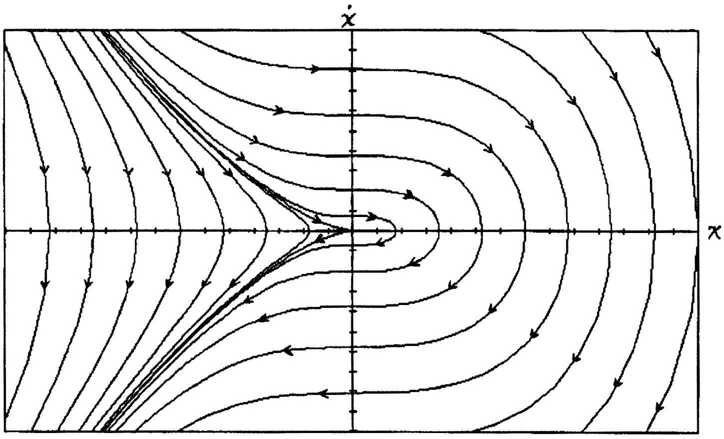

FIG. 2.

α (in the case where $n = 0$, $m = 1$, the eigenvalues of $J(0, 0)$ are nonzero and we obtain ellipses with $\alpha < 0$ and hyperbolas with $\alpha > 0$). When $n \geq 1$, no trajectory crosses the x-axis so that periodic solutions are not possible. If $n = 1$, we obtain a family of curves $y = x^{m+1}/(m+1) + C$ for each m. In this case, the origin and all other points on the x-axis are examples of equilibria which cannot be characterized as nodes, saddle points, centers, cusps, or foci. More general results on planar nonhyperbolic equilibria that are not limited to factorable systems may be found in [4, §2.11].

(II) Of course, the *stability* of a nonhyperbolic equilibrium can sometimes be determined by means of the Liapunov "direct method" [2], [4]. For example, the origin is the unique nonhyperbolic equilibrium of the system

$$\dot{x} = \frac{(\sin x - x)}{(y^2 + 1)},$$
$$\dot{y} = \frac{-y}{(x^2 + 1)}.$$

Defining a Liapunov function $V(x, y) = x^2 + y^2$, we can easily deduce that along a trajectory of the above system,

$$\frac{dV}{dt} = \frac{-2x(x - \sin x)}{y^2 + 1} - \frac{2y^2}{x^2 + 1},$$

which is negative definite for all $(x, y) \neq (0, 0)$. Hence $(0, 0)$ is globally asymptotically stable (*all* trajectories approach the origin). Also, Proposition 3 applies and implies that this system has no periodic solutions. See Figure 3 for a phase portrait.

(III) Factorability is preserved under certain coordinate transformations such as translations or reflections. However, more general transformations of the plane can create or destroy factorability. In particular, *polar* coordinate transformations do not preserve factorability. A factorable system in polar coordinates is generally not factorable when transformed into the Cartesian or rectangular coordinates. Thus a system may factor in polar coordinates but also possess foci and limit cycles, as seen in the system

$$\dot{r} = r(1 - r), \qquad \dot{\theta} = 1,$$

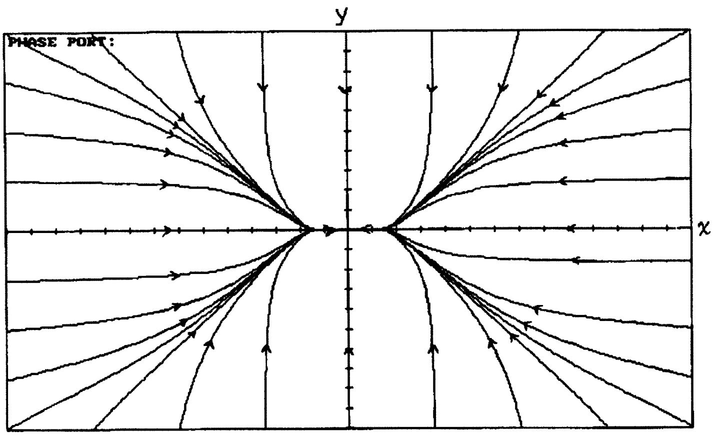

FIG. 3.

which has an unstable focus at the origin and a stable limit cycle $r = 1$. As may be directly verified, this system is not factorable in rectangular coordinates. The results of this note, of course, apply only to those systems that are factorable in rectangular coordinates.

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