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004

MEASURING THE SPEED AND ALTITUDE OF AN AIRCRAFT USING SIMILAR TRIANGLES*

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Abstract. In this note a simple and sometimes practical method is devised with which the average speed and the cruising altitude of an aircraft in flight can be computed from a window seat inside the aircraft. No knowledge of the mechanical structure of the vehicle is assumed. Rather, either two separate time measurements or a time and a length measurement are used allowing one to construct sets of similar triangles. These triangles can then be used to derive the formulas for speed and altitude. This method can also be used to determine the average speed of moving ground vehicles such as trains and buses.

Key words. similar triangles, speed, altitude, time, length, measurements

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1. Introduction.

In his novel *The Mysterious Island* [3, p. 99], Jules Verne tells us how the engineer Cyrus Harding determined the height of a cliff using only a wooden stick and similar triangles. The mathematics involved in that method is elementary and accessible to high school students and college freshmen. In a similar, though somewhat more elaborate fashion this note presents a way of computing both the speed and the altitude of an aircraft by a passenger using only a watch and possibly a ruler. The key is to make two separate measurements: a time measurement, followed by either another time measurement or a length measurement. The accuracy of the results depends critically on the accuracy of the measurements, and we will consider ways of dealing with the problem of measurement-taking below. Although we will be using a bit more mathematics than Cyrus Harding did, this note is still quite accessible to students in mathematics and physics courses at late high school or early college levels, and may prove to be a source of lively class discussions.

We point out for reference that the air-speed is usually measured with the aid of a *pitot-static tube* and the altitude with an *altimeter* [1, pp. 71, 102]. Both instruments operate by measuring the pressure on a volume of mercury and are not bound by our restrictions (throughout this note, it is assumed that the aircraft has attained a fixed "cruising" altitude and is flying at a constant velocity above and parallel to a sufficiently flat, visible ground surface). On the other hand, neither of the above instruments (particularly, the air-speed indicator) can be operated in isolation from the supporting mechanisms by an untrained passenger, and both instruments are sophisticated mechanical devices that are not normally available in the general market.

2. The Time-Time method.

We begin by describing the simpler of the two procedures mentioned in the Introduction. First, choose a convenient point on the ground for reference. Then position yourself so that your eye is at the point E_1 a short distance behind the upper right-hand "corner" of the aircraft window (see Fig. 1), and wait for the reference point to come into view at the edge of the glass (point A). When the point appears in the window (at the ground position P_1) immediately begin measuring time. Suppose that the reference point takes t_1 seconds to move across the window and disappear behind the other edge (point B) at the ground position P_2. At that time, quickly move your eye parallel to the window to the position E_2, directly

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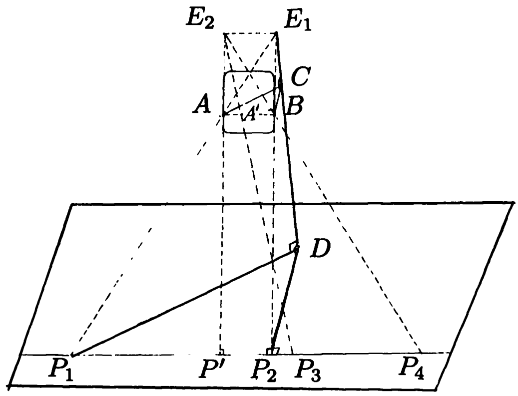

FIG. 1

behind the upper left-hand corner of the window, while continuing to measure the elapsed time. When your eye reaches E_2, you will find the reference point at ground position P_3. The reference point disappears once again at B (ground position P_4). Suppose that it takes t_2 seconds after the first t_1 for the point to clear the window. Now measure the width a and the distance b from A (or B) to the top of the window using a small ruler. If the window does not have a definite width (e.g., if it is circular), then a could represent a suitable horizontal cross section.

Having completed the required measurements, let us now derive the formulas for altitude and speed. From Fig. 1, using similar triangles, obtain the following equal ratios:

$$\frac{P_1P_2}{AB} = \frac{E_1D}{E_1C} = \frac{E_1P_2}{E_1B}$$

and

$$\frac{P_2P_4}{E_1E_2} = \frac{P_2B}{E_1B}.$$

Let v denote the (constant) velocity and $h = CD$ the cruising altitude of the aircraft. Note that $AB = E_1E_2 = a$ and

$$E_1P_2 = E_1B + P_2B, \qquad P_1P_2 = vt_1, \qquad P_2P_4 = vt_2.$$

Substituting these into the above ratios, we obtain

$$\frac{vt_1}{a} = \frac{vt_2}{a} + 1 = \frac{h}{b} + 1.$$

These immediately yield

$$v = \frac{a}{t_1 - t_2}, \qquad h = \frac{bt_2}{t_1 - t_2}.$$

It is easy to see that the same method and formula can be used to determine the speed of a train or a bus that is moving with a constant velocity (h and b may be set equal to zero in this case, and the reference point may be any convenient *distant* point). Note that our approach here does not require any knowledge of the external specifications of the vehicle or of its internal mechanical structure.

Remark. The velocity equation above implies that $t_1 - t_2$ is the time it takes the vehicle itself to travel the distance a. In the case of aircrafts (particularly jet aircrafts), $t_1 - t_2$ may be about 0.001 second (due to high velocities [2]). This can be a problem, since conventional timepieces and chronometers available on the market are usually accurate to only 0.01 second. One way of remedying this situation is by enlarging a (and thus also the time scale). To do this, simply choose *two* windows sufficiently far apart and let A be a point on the left-hand side of the left window and B the corresponding point on the right-hand side of the right window. Then $AB = qa$ for some rational number q, so that

$$v = \frac{a}{t_1 - t_2} = \frac{qa}{qt_1 - qt_2}$$

requires less accuracy in time measurements by a factor of q. In other words, if ordinarily (using one window) t_1 and t_2 have to be measured with an accuracy of 0.001 second, then with, say, $q = 20$, we need to measure t_1 and t_2 with an accuracy of only $20(0.001) = 0.02$ second. In an aircraft with $a = 1$ ft. and consecutive windows 1 ft. apart, $q = 20$ translates into 20 ft., or about 11 windows (of course, only the first and the last window are used in our measurements). Since in this case it may be hard to see clearly through the farthest window and mark the appearance of the reference point, it may be helpful to use a small mirror fastened to a seat adjacent to that window.

In § 3 below, we develop an alternative form of the Time-Time method in which we need to measure a length accurately. A discussion of measurement errors follows after that.

3. The Time-Length method. Through a refinement of the Time-Time method, we arrive at the *Time-Length* method. As its name indicates, this method requires both a time and a length measurement. For this reason, this method may seem inferior to the Time-Time method. However, the Time-Length method removes the requirement for accuracy in time measurements by requiring accurate length measurements instead, and is, therefore, a somewhat different approach. We will elaborate on this aspect after describing the measurement procedures.

After measuring t_1 as described in § 2, let t seconds elapse (for example, t could be the time that it takes you to move your eye from E_1 to E_2). Make sure that t is small enough so that you can still see the image of the reference point in the window from E_2. Mark the spot A' on the window where your line of sight from E_2 to P_3 intersects the window, and let $x = AA'$. By symmetry and similarity we have

$$\frac{P'P_3}{AA'} = \frac{P'E_2}{E_2A} = \frac{P_2E_1}{E_1B} = \frac{vt_1}{a}.$$

Since $P'P_3 = P'P_2 + P_2P_3 = a + vt$, we obtain

$$\frac{a + vt}{x} = \frac{vt_1}{a} = \frac{h}{b} + 1$$

so that

$$v = \frac{a^2}{xt_1 - at}, \qquad h = \frac{(a-x)bt_1 + abt}{xt_1 - at}.$$

It may be worth noting here that for $x = a$ (and hence, $t = t_2$) these formulas yield the speed and altitude formulas of § 2 (the Time-Time formulas). Only note that here x is measured while t is fixed (known). Also observe that a comparison of the Time-Time formula and the above formula for v reveals that

$$\frac{a^2/x}{t_1 - (a/x)t} = v = \frac{(a/x)a}{(a/x)(t_1 - t_2)}.$$

If x is chosen small relative to a, then the coefficient a/x can boost the value of the time difference $t_1 - t_2$ significantly. Hence the measurement of t_1 in this (Time-Length) case is not required to be as accurate as the Time-Time case. Indeed, with $a =$ 1 ft., if we take $t = 1$ second and $b = 1$ ft., then for a typical Boeing 747 "Jumbo Jet" cruising at $v = 900$ ft per second (approximately, 600 miles per hour) and $h = 45000$ ft. [2, p. 378], we obtain $x = 0.24$ in. and $a/x = 49.95$ (x can be determined from the Time-Length formulas for v and h above after eliminating t_1). This is equivalent to a q value of almost 50 (a 25-window enlargement) in the above remark. Similarly, for the smaller and slower Beechcraft 1900C "turboprop," $v = 450$ ft. per second and $h = 16000$ ft. [2, p. 358], we arrive at $x = 0.34$ in. and $a/x = 35.45$. Naturally, we can still apply the method of enlargements in § 2 to enlarge a by a factor of q according to

$$v = \frac{a^2}{xt_1 - at} = \frac{(qa)^2}{qx(qt_1) - qa(qt)}$$

to magnify the measurement scale for both x and t_1.

4. A note on errors. From a practical point of view, the accuracy of results in both the Time-Time and the Time-Length method is limited by the sensitivity of the corresponding formulas to measurement errors. Thus without some additional effort (such as an adequate enlargement of a as mentioned in §§ 2 and 3 above) or without precise measuring instruments and methods with which to measure t_1 and t_2 (or x), only poor results can be expected.

However, in the presence of partial information (i.e., if one of the two values v or h is known) it is possible to reduce the influence of measurement errors considerably by using formulas that make use of this partial information. For typical commercial aircrafts (v values of a few hundred feet per second and h values of several thousand feet) x is approximately equal to bvt/h. From this, we obtain

$$\frac{v}{h} = \frac{x}{bt}$$

so that

$$\frac{|\Delta(v/h)|}{v/h} = \frac{h}{btv}|\Delta x| = \frac{|\Delta x|}{x}.$$

Hence the *relative error* in v/h is equal to the relative measurement error in x. From this we can conclude that if the value of h is given (or measured exactly), then the relative error in the calculated value of v is approximately equal to the relative error in the measured value of x. Conversely, if v is known or measured exactly, then

$$\Delta\left(\frac{v}{h}\right) = v\left(\frac{1}{h'} - \frac{1}{h}\right)$$

where h denotes the exact value and h' the computed value. Thus

$$\frac{\Delta x}{x} = h\left(\frac{h - h'}{h'h}\right) = -\frac{\Delta h}{h'} = -\frac{h}{h'}\left(\frac{\Delta h}{h}\right).$$

Therefore,

$$\frac{\Delta h}{h} = -\frac{h'}{h}\left(\frac{\Delta x}{x}\right) = -\left(\frac{\Delta h}{h} + 1\right)\left(\frac{\Delta x}{x}\right).$$

Solving for $\Delta h/h$ and taking absolute values, we obtain the following estimate for the relative error in h in terms of the relative error in x:

$$\frac{|\Delta h|}{h} = \left(\frac{1}{|1 + \frac{\Delta x}{x}|}\right)\frac{|\Delta x|}{x}.$$

Hence, if v is known, the relative error in the value of h is also approximately the same as the relative measurement error for x. A similar analysis involving t_1 or t_2 instead of x reveals essentially the same error behavior in the Time-Time case.

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