Asymptotic Stability for A Higher Order Rational Difference Equation

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Abstract

For the rational difference equation

$$x_n = \frac{\alpha + \sum_{i=1}^{m} \; a_i x_{n-i}}{\beta + \sum_{i=1}^{m} \; b_i x_{n-i}}, \quad n = 1\,, 2\,, \ldots$$

we obtain sufficient conditions for the asymptotic stability of a unique fixed point relative to an invariant interval. We focus on negative values for the coefficients a_i, a range that has not been considered previously.

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1 Introduction.

Consider the higher order difference equation

$$x_n = f(x_{n-1}, \ldots, x_{n-m}), \quad n = 1, 2, \ldots \tag{1}$$

where m is a non-negative integer and $f : \mathbb{R}^m \to \mathbb{R}$ is a given function. In the literature on difference equations, problems involving the asymptotic stability of fixed points of (1) in the case in which f is monotonic (non-increasing or non-decreasing) in each of its arguments or coordinates arise frequently. In particular, the general rational difference equation

$$x_n = \frac{\alpha + \sum_{i=1}^{m} \; a_i x_{n-i}}{\beta + \sum_{i=1}^{m} \; b_i x_{n-i}}, \quad n = 1\,, 2\,, \ldots \tag{2}$$

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and various special cases of it have been studied extensively in the literature; see, e.g., [3] and [6] for a discussion of (2) in its general form and [4] for a discussion of the second order case. The bibliographies in these books contain numerous references to additional results that discuss asymptotic stability for various special cases.

In this paper we consider (2) in its general form above and give conditions for the asymptotic stability of a fixed point relative to an invariant interval that contains the fixed point. Our results concern a range of parameters, including negative coefficients, that extend those previously considered elsewhere.

2 The Main Results.

We first quote a basic result from [1] as a lemma. This result concerns the general equation (1) and was inpired by the study of pulse propagation in a ring of excitable media in [8] which involved an equation of type (1).

Lemma 1. *Let r_0, s_0 be extended real numbers where $-\infty \leq r_0 < s_0 \leq \infty$ and consider the following hypotheses:*

(H1) $f(u_1, \ldots, u_m)$ is non-increasing in each $u_1, \ldots, u_m \in I_0$ where $I_0 = (r_0, s_0]$ if $s_0 < \infty$ and $I_0 = (r_0, \infty)$ otherwise;

(H2) $g(u) = f(u, \ldots, u)$ is continuous and decreasing for $u \in I_0$;

(H3) There is $r \in [r_0, s_0)$ such that $r < g(r) \leq s_0$. If $r_0 = -\infty$ or $\lim_{t \to r_0^+} g(t) = \infty$ then we assume that $r \in (r_0, s_0)$.

(H4) There is $s \in [r, x^*)$ such that $g^2(s) \geq s$, where $g^2(s) = g(g(s))$.

(H5) There is $s \in [r, x^*)$ such that $g^2(u) > u$ for all $u \in (s, x^*)$.

Then the following is true:

(a) *If (H2) and (H3) hold then Equation (1) has a unique fixed point x^* in the open interval $(r, g(r))$.*

(b) *Let $I = [s, g(s)]$. If (H1)-(H4) hold then I is an invariant interval for* (1) *and $x^* \in I$.*

(c) *If (H1)-(H3) and (H5) hold then x^* is stable and attracts all solutions of (1) with initial values in $(s, g(s))$.*

(d) *If (H1)-(H3) hold then x^* is an asymptotically stable fixed point of (1) if it is an asymptotically stable fixed point of the mapping g; e.g., if g is continuously differentiable with $g'(x^*) > -1$.*

Remarks. 1. If f is continuous on $[s, g(s)]^m$, then the attractivity of x^* in Lemma 1(c) also follows from the general Theorem 1.15 in [2]; see [1] for additional comments in this regard.

2. Condition (H5) is equivalent to x^* being an asymptotically stable fixed point of the function g relative to the interval $(s, g(s))$; see Theorem 2.1.2 in [6]. Hence Lemma 1(d) follows from Lemma 1(c).

Now we consider the rational difference equation (2) which we re-write for notational convenience as follows:

$$x_n = \frac{\alpha - \sum_{i=1}^m a_i x_{n-i}}{\beta + \sum_{i=1}^m b_i x_{n-i}} \tag{3}$$

where

$$\begin{aligned} &\alpha > 0\,,\ a_i, b_i \geq 0,\ i = 1\,, 2\,, \dots\,, m, \\ &a = \sum_{i=1}^m a_i > 0,\ b = \sum_{i=1}^m b_i > 0,\ \beta > a. \end{aligned} \tag{4}$$

We note that the special case where $a_i = 0$ for all i is discussed in [1] and [7] so we will not consider that case here. The functions f and g in Lemma 1 take the following forms for Eq.(3)

$$f(u_1, \dots, u_m) = \frac{\alpha - \sum_{i=1}^m a_i u_i}{\beta + \sum_{i=1}^m b_i u_i}, \quad g(u) = \frac{\alpha - au}{\beta + bu},\ u, u_i \in \mathbb{R}. \tag{5}$$

Theorem 1. *Assume that f, g are given by (5) and that conditions (4) hold. If $s = -\,\alpha\,(\beta - a)\,/(a^2 + \alpha b)$ then:*

(a) *$g(s) = \alpha/a$ and $s > -\beta/b$;*

(b) *$I = [s, g(s)]$ is an invariant interval for Eq.(3);*

(c) *Every solution of (3) with initial values in $(s, g(s))$ converges to the fixed point*

$$x^* = \frac{-(a+\beta) + \sqrt{(a+\beta)^2 + 4\alpha b}}{2b} \in (0, g(s)) \subset I.$$

Proof. (a) The first assertion is easily verified by substitution, and the second follows from the observation that the value of s is an increasing function of a when $\beta > a$ and the infimum of s is $-\beta/b$.

(b) and (c) In Lemma 1, set $r_0 = -\beta/b$ and let $s_0 = \alpha/a$. For $u \in (r_0, s_0] = I_0$, we have $\alpha - au \geq 0$ and $\beta + bu > 0$. Thus, g has a decreasing numerator and an increasing denominator on I_0, so g is decreasing on I_0. Similarly, if $(u_1, \ldots, u_m) \in I_0^m$ then

$$\alpha - \sum_{i=1}^{m} a_i u_i \geq \alpha - a \max\{u_1, \ldots, u_m\} \geq \alpha - a s_0 = 0,$$
$$\beta + \sum_{i=1}^{m} b_i u_i \geq \beta + b \min\{u_1, \ldots, u_m\} > \beta + b r_0 = 0$$

so that $f(u_1, \ldots, u_m) \geq 0$. Thus, for $(u_1, \ldots, u_m) \in I_0^m$, the numerator of f is a decreasing function of u_j and its denominator an increasing function of u_j for each $j = 1, \ldots, m$ with u_i fixed for $i \neq j$. Therefore, f is a decreasing function on I_0^m in each of its coordinates. Therefore, Hypotheses (H1) and (H2) are satisfied in Lemma 1, and (H3) also holds since for $r = s \in (r_0, 0)$ it is true that

$$r = s < 0 < \frac{\alpha}{a} = s_0 = g(s) = g(r).$$

Further, the interval I is invariant because g is decreasing with $g(g(s)) = 0 \in I$ so $g(I) \subset I$ and Part (b) is established. To complete the proof of Part (c), we now establish (H5). First, we may verify by a straightforward calculation that x^* is a solution of the equation $g(u) = u$ so that x^* is a fixed point of (3). Also, under conditions (4) $x^* > 0$ and $x^* < g(s)$ if and only if

$$\begin{aligned}
\frac{-(a+\beta) + \sqrt{(a+\beta)^2 + 4\alpha b}}{2b} &< \frac{\alpha}{a} \text{ iff} \\
a\sqrt{(a+\beta)^2 + 4\alpha b} &< 2\alpha b + a(a+\beta) \text{ iff} \\
a^2(a+\beta)^2 + 4a^2\alpha b &< [2\alpha b + a(a+\beta)]^2, \text{ iff} \\
0 &< \alpha^2 b^2 + \alpha\beta a b
\end{aligned}$$

The last inequality is true under conditions (4), so it follows that $x^* \in (0, g(s)) \subset I$. Since g is decreasing on $I_0 \supset I$, we conclude that x^* is the only fixed point of g in I. Now the inequality $g^2(u) > u$ can be written as

$$\frac{\alpha(\beta - a) + (\alpha b + a^2)\, u}{(\beta^2 + \alpha b) + b\,(\beta - a)\, u} > u$$

or equivalently, as

$$b(\beta - a)u^2 + (\beta^2 - a^2)u + \alpha(\beta - a) > 0.$$

If $\beta > a$ then dividing by $\beta - a$ gives

$$bu^2 + (\beta + a)u - \alpha < 0\,. \tag{6}$$

For this last inequality to hold, we need $u \in (u_-\,, u_+)$ where u_- and u_+ are the two roots of the equation

$$bu^2 + (\beta + a)\,u - \alpha = 0\,.$$

But $u_+ = x^*$ and

$$u_- \;=\; \frac{-(a+\beta) - \sqrt{(a+\beta)^2 + 4\alpha b}}{2b} < \frac{-(a+\beta) - (a+\beta)}{2b} \;< -\frac{\beta}{b}.$$

Therefore, (6) holds for $u \in (-\beta/b, x^*)$ and in particular, for $u \in (s, x^*)$. Thus (H5) holds and by Lemma 1 x^* is a stable attractor of all solutions in $(s, g(s))$. This completes the proof.

Remarks. The function g above is in fact decreasing on $(-\beta/b, \infty)$ and iterates of g starting from an initial value $u_0 \in (-\beta/b, \infty)$ converge to x^* if $\beta > a$. This inequality assures that $(-\beta/b, \infty)$ is an invariant interval for g in addition to (H5) holding on $(-\beta/b, x^*)$ as shown in the proof of Theorem 1. However, the fixed point x^* above will *not* in general attract solutions of the higer order equation (3) that start from initial values outside the interval $I = [s, g(s)]$. One reason for this is that (H1) does not hold if the numerator of f can be negative, which is possible if some of the coordinates of the point $(u_1, \ldots, u_m)$ are large and positive.

For example, consider the following special case of Eq.(3):

$$x_n = \frac{1 - ax_{n-2}}{1 + b_1 x_{n-1} + b_2 x_{n-2}} \tag{7}$$

where $\alpha = \beta = 1$, $b_1, b_2 > 0$ and $a_1 = 0$ so $a = a_2$. The second order equation (7) has a 2-cycle $\{p, q\}$ if

$$p = \frac{1 - ap}{1 + b_1 q + b_2 p} \text{ and } q = \frac{1 - aq}{1 + b_1 p + b_2 q}. \tag{8}$$

If, e.g.

$$b_2 < b_1 < 4b_2^2 + b_2 \text{ and } a > \frac{2b_2}{\sqrt{b_1 - b_2}} - 1$$

then the system of equations (8) has real solutions

$$p = \frac{c \pm \sqrt{c^2 - 4(b_1 - b_2)}}{2(b_1 - b_2)}, \quad q = -p - \frac{1+a}{b_2} = \frac{c \mp \sqrt{c^2 - 4(b_1 - b_2)}}{2(b_1 - b_2)} \qquad (9)$$

where $c = (1 + a)(1 - b_1/b_2)$. In particular, if $b_2 = 1/4$, $b_1 = 1$ and $a = 0.8$, then $c = -5.4$ and from (9) we obtain $p \approx -7.01$ and $q \approx -0.19$. Also, here $\alpha/a = 1/0.8 = 1.25$ and the invariant interval is $I \approx [-0.106, 1.25]$. Choosing at least one initial condition greater than 1.25, may cause a trajectory of (7) to reach the 2-cycle $\{p, q\}$. With e.g. $x_0 = 0.4$ and $x_{-1} = 5$ as initial values, we obtain $x_1 \approx -1.13$ which is less than $-\beta/b = -0.8$, and after this the solutions oscillate about the point of discontinuity $-\beta/b$.Of course, if both initial conditions are in the interval of Theorem 1 then the corresponding solution of (7) converges to the fixed point $x^* \approx 0.45$.

References.

[1] Chan, D., E.R.Chang, M. Dehghan, C.M. Kent, R. Mazrooei-Sebdani, H. Sedaghat, 2005, Asymptotic stability for difference equations with decreasing arguments, to appear.
[2] Grove, E.A., G. Ladas, 2005, *Periodicities in Nonlinear Difference Equations*, Chapman & Hall/CRC, Boca Raton.
[3] Kocic, V.L., G. Ladas, 1993, *Global Behavior of Nonlinear Difference Equations of Higher Order with Applications*, Kluwer Academic, Dordrecht.
[4] Kulenovic, M.R.S., G. Ladas, 2001, *Dynamics of Second Order Rational Difference Equations with Open Problems and Conjectures*, Chapman & Hall/CRC, Boca Raton.
[6] Sedaghat, H., 2003, *Nonlinear Difference Equations: Theory with Applications to Social Science Models,* Kluwer Academic, Dordrecht.
[7] Sedaghat, H., 2005, Asymptotic stability in a class of nonlinear, monotone difference equations, *International J. of Pure and Appl. Math.*, **21**, 167-174.
[8] Sedaghat, H., C.M. Kent, M.A. Wood, 2005, Criteria for the convergence, oscillations and bistability of pulse circulation in ring of excitable media, SIAM J. of Applied Mathematics, in press.