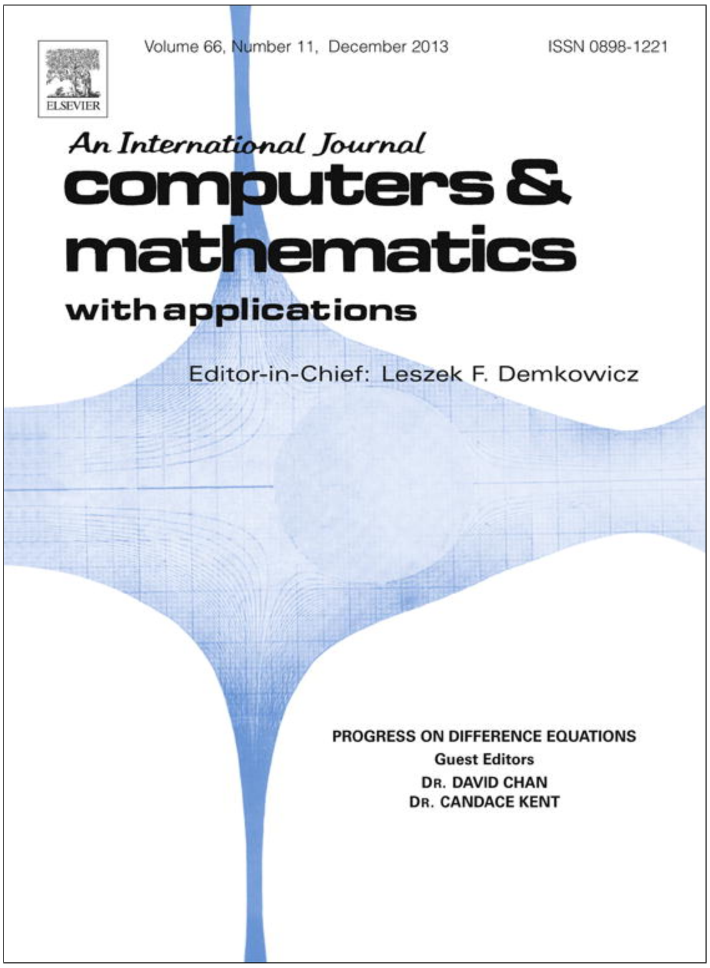
Volume 66, Number 11, December 2013
ISSN 0898-1221
ELSEVIER
An International Journal
computers &
mathematics
with applications
Editor-in-Chief: Leszek F. Demkowicz
PROGRESS ON DIFFERENCE EQUATIONS
Guest Editors
DR. DAVID CHAN
DR. CANDACE KENT

Computers and Mathematics with Applications 66 (2013) 2231–2238

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Reduction of order, periodicity and boundedness in a class of nonlinear, higher order difference equations

H. Sedaghat

Department of Mathematics, Virginia Commonwealth University, Richmond, VA, 23284-2014, USA

ARTICLE INFO

Keywords:
Reduction of order
Boundedness
Periodic solutions
Oscillations
Non-autonomous
Higher order

ABSTRACT

Non-autonomous, higher order difference equations of type

$$x_{n+1} = \sum_{i=0}^{k} a_i x_{n-i} + g_n\left(\sum_{i=0}^{k} b_i x_{n-i}\right)$$

with real variables and parameters have appeared frequently in the literature. We extend some recent results on semiconjugate factorization and reduction of order to cases where characteristic polynomials of the linear expressions $\sum_{i=0}^{k} a_i u_i$ and $\sum_{i=0}^{k} b_i u_i$ have complex roots. This extension yields new results on boundedness and existence of periodic solutions for equations of order 3 or greater.

1. Introduction

Special cases of the following type of higher order difference equation have frequently appeared in the literature:

$$x_{n+1} = \sum_{i=0}^{k} a_i x_{n-i} + g_n\left(\sum_{i=0}^{k} b_i x_{n-i}\right), \quad n = 0, 1, 2, \ldots. \tag{1}$$

We assume here that k is a fixed positive integer and for each n, the function $g_n : \mathbb{R} \to \mathbb{R}$ is defined on the real line. The parameters a_i, b_i are fixed real numbers such that $a_k \neq 0$ or $b_k \neq 0$. Upon iteration, Eq. (1) generates a unique sequence of points $\{x_n\}$ in $\mathbb{R}$ (its solution) from any given set of $k+1$ initial values $x_0, x_{-1}, \ldots, x_{-k} \in \mathbb{R}$. The number $k+1$ is the order of (1).

Special cases of Eq. (1) appeared in the classical economic models of the business cycle in the twentieth century in the works of Hicks [1], Puu [2], Samuelson [3] and others; see [4, Section 5.1] for some background and references. Other special cases of (1) occurred later in mathematical studies of biological models ranging from whale populations to neuron activity; see, e.g., Clark [5], Fisher and Goh [6], Hamaya [7] and Section 2.5 in Kocic and Ladas [8].

The dynamics of special cases of (1) have been investigated by several authors. Hamaya uses Lyapunov and semicycle methods in [7] to obtain sufficient conditions for the global attractivity of the origin for the following special case of (1)

$$x_{n+1} = \alpha x_n + a \tanh\left(x_n - \sum_{i=1}^{k} b_i x_{n-i}\right)$$

with $0 \leq \alpha < 1$, $a > 0$ and $b_i \geq 0$. These results can also be obtained using only the contraction method in [9,10]; also see [11] for a discussion of alternative methods. The results in [10] are used in [4, Section 4.3D], to prove the global asymptotic stability of the origin for an autonomous special case of (1) with $a_i, b_i \geq 0$ for all i and $g_n = g$ for all n, where g is a

E-mail address: hsedagha@vcu.edu.

0898-1221/$ – see front matter © 2013 Elsevier Ltd. All rights reserved.
http://dx.doi.org/10.1016/j.camwa.2013.06.019

continuous, non-negative function. The study of global attractivity and stability of fixed points for other special cases of (1) appears in [12,13]; also see Section 6.9 in [8].

The second-order case ($k = 1$) has been studied in greater depth. Kent and Sedaghat obtain sufficient conditions in [14] for the boundedness and global asymptotic stability of

$$x_{n+1} = cx_n + g(x_n - x_{n-1}). \tag{2}$$

In [15], El-Morshedy improves the convergence results of [14] for (2) and also gives necessary and sufficient conditions for the occurrence of oscillations. The boundedness of solutions of (2) is studied in [16] and periodic and monotone solutions of (2) are discussed in [17]. Li and Zhang study the bifurcations of solutions of (2) in [18]; their results include the Neimark–Sacker bifurcation (discrete analog of Hopf).

A more general form of (2), i.e., the following equation

$$x_{n+1} = ax_n + bx_{n-1} + g_n(x_n - cx_{n-1}) \tag{3}$$

is studied in [19] where sufficient conditions for the occurrence of periodic solutions, limit cycles and chaotic behavior are obtained using reduction of order and factorization of the above difference equation into a pair of equations of lower order. These methods are used in [20] to determine sufficient conditions on parameters for occurrence of limit cycles and chaos in those rational difference equations of the following type

$$x_{n+1} = \frac{ax_n^2 + bx_{n-1}^2 + cx_nx_{n-1} + dx_n + ex_{n-1} + f}{\alpha x_n + \beta x_{n-1} + \gamma} \tag{4}$$

that can be reduced to special cases of (3).

In this paper, we consider the possible occurrence of complex roots for the characteristic polynomials associated with the linear expressions $\sum_{i=0}^{k} a_iu_i$ and $\sum_{i=0}^{k} b_iu_i$ in (1). Complex, non-real roots that are common to both polynomials may occur when (1) has order 3 or greater ($k \geq 2$), a situation that cannot occur in the second-order equations of [20] or [19]. The results obtained here extend certain results in [19] to equations of order 3 and greater for the first time.

2. Reduction of order

The following result is proved as Theorem 5.6 in [21]. Its generalization to algebras over fields is proved in essentially the same way; see [22].

Lemma 1. *Let $g_n : \mathcal{F} \to \mathcal{F}$ be a sequence of functions on a field $\mathcal{F}$. If for $a_i, b_i \in \mathcal{F}$ the polynomials*

$$P(u) = u^{k+1} - \sum_{i=0}^{k} a_iu^{k-i}, \qquad Q(u) = \sum_{i=0}^{k} b_iu^{k-i}$$

have a common, nonzero root $\rho \in \mathcal{F}$ then each solution $\{x_n\}$ of (1) in $\mathcal{F}$ satisfies

$$x_{n+1} = \rho x_n + t_{n+1} \tag{5}$$

where the sequence $\{t_n\}$ is the unique solution of the equation:

$$t_{n+1} = -\sum_{i=0}^{k-1} p_it_{n-i} + g_n\left(\sum_{i=0}^{k-1} q_it_{n-i}\right) \tag{6}$$

in $\mathcal{F}$ with initial values $t_{-i} = x_{-i} - \rho x_{-i-1}$ for $i = 0, 1, \dots, k-1$ and coefficients

$$p_i = \rho^{i+1} - a_0\rho^i - \cdots - a_i \quad \text{and} \quad q_i = b_0\rho^i + b_1\rho^{i-1} + \cdots + b_i$$

in $\mathcal{F}$. Conversely, if $\{t_n\}$ is a solution of (6) with initial values $t_{-i} \in \mathcal{F}$ then the sequence $\{x_n\}$ that it generates in $\mathcal{F}$ via (5) is a solution of (1).

This result shows that Eq. (1) splits into the equivalent pair of Eqs. (5) and (6) provided that the polynomials P and Q have a common nonzero root ρ. We call the pair of Eqs. (5) and (6) a *semiconjugate factorization* of (1). Eq. (6), whose order is one less than the order of (1) is the *factor equation* and Eq. (5) which bridges the order (or dimension) gap between (1) and (6) is the *cofactor equation*. Since Eq. (6) is of the same type as (1) applying Lemma 1 to (6) yields a further reduction of order.

Lemma 2. *Let $k \geq 2$ and assume that the coefficients of (1) are complex, i.e., $a_i, b_i \in \mathbb{C}$. Let $\mathcal{F} = \mathbb{C}$ in Lemma 1 and suppose that $g_n : \mathbb{C} \to \mathbb{C}$ are complex functions for all n. If the polynomials P, Q in Lemma 1 have two common, nonzero roots $\rho, \gamma \in \mathbb{C}$ then (6) has a factor equation*

$$r_{n+1} = -\sum_{j=0}^{k-2} p'_jr_{n-j} + g_n\left(\sum_{j=0}^{k-2} q'_jr_{n-j}\right) \tag{7}$$

with coefficients

$$p'_j = \gamma^{j+1} + p_0\gamma^j + \cdots + p_j \quad \text{and} \quad q'_j = q_0\gamma^j + q_1\gamma^{j-1} + \cdots + q_j$$

where the numbers p_j, q_j are as defined in Lemma 1 in terms of the root ρ. There are two cofactor equations

$$t_{n+1} = \gamma t_n + r_{n+1} \tag{8}$$

$$x_{n+1} = \rho x_n + t_{n+1} \tag{9}$$

the second of which is just (5) from Lemma 1. The triangular system of three equations (7)–(9) is equivalent to (1) in the sense of Lemma 1; i.e., they generate the same set of solutions.

Proof. Consider the polynomials associated with the factor equation (6), i.e.,

$$P_1(u) = u^k + \sum_{j=0}^{k-1} p_j u^{k-j-1}, \qquad Q_1(u) = \sum_{j=0}^{k-1} q_j u^{k-j-1}.$$

Let ρ be a root of P. We claim that $(u-\rho)P_1(u) = P(u)$. This is established by a straightforward calculation:

$$\begin{aligned}(u-\rho)P_1(u) &= (u-\rho)\left(u^k + \sum_{j=0}^{k-1} p_j u^{k-j-1}\right)\\ &= u^{k+1} - \rho p_{k-1} + \sum_{j=0}^{k-1}(p_j - \rho p_{j-1})u^{k-j}\end{aligned}$$

where we define $p_{-1} = 1$ to simplify the notation. Using the definition of the numbers p_i in Lemma 1 we obtain

$$p_j - \rho p_{j-1} = -a_j$$

and further, since $P(\rho) = 0$ we obtain

$$\begin{aligned}\rho p_{k-1} &= \rho(\rho^k - a_0\rho^{k-1} - \cdots - a_{k-1})\\ &= P(\rho) + a_k\\ &= a_k\end{aligned}$$

which completes the proof of the claim. A similar argument shows that if ρ is a root of Q then

$$(u-\rho)Q_1(u) = Q(u).$$

Now, suppose that γ is also a common root of P and Q. If $\gamma \neq \rho$ then clearly $P_1(\gamma) = Q_1(\gamma) = 0$ so γ is a common root of P_1 and Q_1. Otherwise, $\gamma = \rho$ and ρ is a double root, hence a zero of the derivatives P' and Q', i.e.,

$$P'(\rho) = Q'(\rho) = 0.$$

In addition, we find that

$$\begin{aligned}P_1(\rho) &= \rho^k + \sum_{j=0}^{k-1}(\rho^{j+1} - a_0\rho^j - \cdots - a_{j-1}\rho - a_j)\rho^{k-j-1}\\ &= (k+1)\rho^k - \sum_{j=0}^{k-1}(k-j)a_j\rho^{k-j-1}\\ &= P'(\rho)\end{aligned}$$

so that $\rho = \gamma$ is a root of P_1. Similarly, $\rho = \gamma$ is also seen to be a root of Q_1. Now applying Lemma 1 to (6) yields a factor equation (7) and a cofactor (8).

Finally, the last assertion follows from Theorem 3.1 in [21]. ■

The next result on factorization of polynomials is also needed.

Lemma 3. *Suppose that $\gamma, \rho \in \mathbb{C}$ are roots of the polynomial $c_0u^m + c_1u^{m-1} + \cdots + c_{m-1}u + c_m$ of degree $m \geq 2$ with coefficients $c_j \in \mathbb{R}$. Then*

$$\sum_{j=0}^{m} c_j u^{m-j} = (u^2 - (\gamma+\rho)u + \gamma\rho)\sum_{j=0}^{m-2}\alpha_j u^{m-j-2} \tag{10}$$

where $\alpha_0 = c_0$,

$$\alpha_j = c_j + (\gamma+\rho)\alpha_{j-1} - \gamma\rho\alpha_{j-2}, \quad j = 1, 2, \ldots, m-2, \qquad \alpha_{-1} \doteq 0, \tag{11}$$

and the following equalities hold:

$$c_{m-1} + (\gamma+\rho)\alpha_{m-2} - \gamma\rho\alpha_{m-3} = 0, \tag{12}$$

$$c_m - \gamma\rho\alpha_{m-2} = 0. \tag{13}$$

Further, if γ *and* ρ *are either both real or they are complex conjugates then the numbers* α_j, $j = 1, 2, \ldots, m-2$ *that satisfy the recursions (11) are real and found to be:*

$$\alpha_j = \sum_{i=0}^{j} \frac{\gamma^{i+1} - \rho^{i+1}}{\gamma - \rho} c_{j-i}, \quad \text{if } \gamma \neq \rho \tag{14}$$

$$\alpha_j = \sum_{i=0}^{j} (i+1)\rho^i c_{j-i}, \quad \text{if } \gamma = \rho. \tag{15}$$

Conversely, let $\sum_{j=0}^{m} c_j u^{m-j}$ *be a polynomial with real coefficients* c_j*. If* $\gamma, \rho \in \mathbb{C}$ *and there are real numbers* α_j *satisfying (11)–(13) then (10) holds and* γ, ρ *are roots of* $\sum_{j=0}^{m} c_j u^{m-j}$*.*

Proof. Assume that $\gamma, \rho \in \mathbb{C}$ are roots of $\sum_{j=0}^{m} c_j u^{m-j}$. Then this polynomial is evenly divided by the quadratic polynomial

$$(u-\gamma)(u-\rho) = u^2 - (\gamma+\rho)u + \gamma\rho \tag{16}$$

with a resulting quotient polynomial $\sum_{j=0}^{m-2} \alpha_j u^{m-j-2}$; i.e., (10) holds. To determine the coefficients α_j of the quotient, multiply the polynomials on the right hand side of (10) and rearrange terms to obtain the identity

$$\begin{aligned}\sum_{j=0}^{m} c_j u^{m-j} &= \alpha_0 u^m + (\alpha_1 - (\gamma+\rho)\alpha_0)u^{m-1} + (\alpha_2 - (\gamma+\rho)\alpha_1 + \gamma\rho\alpha_0)u^{m-2} \\ &\quad + \cdots + (\alpha_{m-2} - (\gamma+\rho)\alpha_{m-3} + \gamma\rho\alpha_{m-4})u^2 + (-(\gamma+\rho)\alpha_{m-2} + \gamma\rho\alpha_{m-3})u + \gamma\rho\alpha_{m-2}.\end{aligned}$$

Now, matching coefficients on the two sides yields (11)–(13).

Next, if γ and ρ are either both real or they are complex conjugates then $(\gamma+\rho)$ and $\gamma\rho$ are both real. In this case, the numbers α_j defined by the recursions (11) are also real. Finally, (14) and (15) may be proved by induction. First, suppose that $\gamma \neq \rho$. For $j = 1$ we have

$$c_1 + \frac{\gamma^2 - \rho^2}{\gamma - \rho} c_0 = c_1 + (\gamma+\rho)\alpha_0 = \alpha_1$$

so (14) is true if $j = 1$. Suppose next that for $1 \le j \le m-3$, (14) is true for $1, 2, \ldots, j$. Then for $j+1$

$$\begin{aligned}\alpha_{j+1} &= c_{j+1} + (\gamma+\rho)\alpha_j - \gamma\rho\alpha_{j-1} \\ &= c_{j+1} + (\gamma+\rho)\sum_{i=0}^{j} \frac{\gamma^{i+1} - \rho^{i+1}}{\gamma - \rho} c_{j-i} - \gamma\rho \sum_{i=1}^{j} \frac{\gamma^i - \rho^i}{\gamma - \rho} c_{j-i} \\ &= c_{j+1} + (\gamma+\rho)c_j + \sum_{i=1}^{j} \left[(\gamma+\rho)\frac{\gamma^{i+1} - \rho^{i+1}}{\gamma - \rho} - \gamma\rho\frac{\gamma^i - \rho^i}{\gamma - \rho} \right] c_{j-i}.\end{aligned}$$

Since for each $i = 1, \ldots, j$

$$(\gamma+\rho)(\gamma^{i+1} - \rho^{i+1}) - \gamma\rho(\gamma^i - \rho^i) = \gamma^{i+2} - \rho^{i+2}$$

we obtain

$$\alpha_{j+1} = c_{j+1} + \frac{\gamma^2 - \rho^2}{\gamma - \rho} c_j + \sum_{i=1}^{j} \frac{\gamma^{i+2} - \rho^{i+2}}{\gamma - \rho} c_{j-i}$$

which verifies the induction step. If $\gamma = \rho$ then for $j = 1$

$$c_1 + 2\rho c_0 = c_1 + 2\rho\alpha_0 = \alpha_1$$

so (15) is true if $j = 1$. Suppose next that for $1 \le j \le m-3$, (15) is true for $1, 2, \ldots, j$. Then for $j+1$

$$\begin{aligned}
\alpha_{j+1} &= c_{j+1} + 2\rho\alpha_j - \rho^2\alpha_{j-1} \\
&= c_{j+1} + 2\rho\sum_{i=0}^{j}(i+1)\rho^i c_{j-i} - \rho^2\sum_{i=1}^{j} i\rho^{i-1}c_{j-i} \\
&= c_{j+1} + 2\rho c_j + \sum_{i=1}^{j}[2(i+1)\rho^{i+1} - i\rho^{i+1}]c_{j-i} \\
&= c_{j+1} + 2\rho c_j + \sum_{i=1}^{j}(i+2)\rho^{i+1}c_{j-i}
\end{aligned}$$

which verifies the induction step.

Conversely, if $\gamma, \rho \in \mathbb{C}$ and $\alpha_j \in \mathbb{R}$ satisfy (11)–(13) then by the definition of α_j the quadratic polynomial (16) divides $\sum_{j=0}^{m} c_j u^{m-j}$ evenly. Therefore, γ, ρ are roots of $\sum_{j=0}^{m} c_j u^{m-j}$. ■

If the coefficients a_i, b_i in (1) are real and a common root ρ of P and Q is complex then these polynomials also share another complex root, namely, the conjugate $\bar{\rho}$; thus, Lemma 2 is applicable. However, if the functions $g_n : \mathbb{R} \to \mathbb{R}$ are real functions then a direct application of Lemma 2 is problematic since the coefficients p_i, q_i of the factor equation (6) are complex. The next result shows that this difficulty does not actually arise since the coefficients p_i', q_i' of the secondary factor equation (7) are in fact, real and furthermore, the two complex cofactor equations in Lemma 2 were combined into a single second-order cofactor equation in $\mathbb{R}$.

Theorem 4. *Let $k \ge 2$ in (1) and assume that the coefficients a_i, b_i are all real and $g_n : \mathbb{R} \to \mathbb{R}$ for $n \ge 0$. If the polynomials P, Q in Lemma 1 have a common complex root $\rho = \mu e^{i\theta} \notin \mathbb{R}$ then the following statements are true.*

(a) *The coefficients p_j', q_j' of the factor equation (7) in Lemma 2 are real numbers that may be written in terms of the original coefficients a_i, b_i of (1) as*

$$p_j' = \mu^{j+1}\frac{\sin(j+2)\theta}{\sin\theta} - \frac{1}{\sin\theta}\sum_{m=0}^{j} a_m\,\mu^{j-m}\sin(j-m+1)\theta, \tag{17}$$

$$q_j' = \frac{1}{\sin\theta}\sum_{m=0}^{j} b_m\,\mu^{j-m}\sin(j-m+1)\theta \tag{18}$$

for $j = 0, 1, \ldots, k-2$.

(b) *The pair of first-order cofactor equations in Lemma 2 with complex coefficients ρ and $\gamma = \bar{\rho}$ combine into one equivalent, second-order, non-homogeneous linear equation with real coefficients*

$$x_{n+1} - 2\mu\cos\theta\, x_n + \mu^2 x_{n-1} = r_{n+1} \tag{19}$$

where the sequence $\{r_n\}$ is a solution of the factor equation (7) in $\mathbb{R}$.

(c) *The system of Eqs. (7) and (19) is equivalent to (1); i.e., the set of solutions of (19) with $\{r_n\}$ satisfying (7) is identical with the set of solutions of (1).*

Proof. (a) Let $\rho = \mu e^{i\theta} = \mu\cos\theta + i\mu\sin\theta$ and $\gamma = \bar{\rho}$ be complex conjugate roots of both P and Q with $\sin\theta \ne 0$ since $\rho \notin \mathbb{R}$. Recall from the proof of Lemma 2 that

$$P(u) = (u-\rho)P_1(u), \qquad Q(u) = (u-\rho)Q_1(u).$$

Applying the same argument to the polynomials P_1 and Q_1 using their common root $\bar{\rho}$ yields

$$\begin{aligned}
P(u) &= (u-\rho)(u-\bar{\rho})P_2(u) = (u^2 - (\rho+\bar{\rho})u + \rho\bar{\rho})P_2(u), \\
Q(u) &= (u-\rho)(u-\bar{\rho})Q_2(u) = (u^2 - (\rho+\bar{\rho})u + \rho\bar{\rho})Q_2(u)
\end{aligned}$$

where

$$P_2(u) = u^{k-1} - \sum_{j=0}^{k-2} p_j' u^{k-j-2}, \qquad Q_2(u) = \sum_{j=0}^{k-2} q_j' u^{k-j-2}.$$

Applying Lemma 3 to each of P and Q we obtain (17) and (18) from (14) since for every positive integer m,

$$\frac{\gamma^m - \rho^m}{\gamma - \rho} = \frac{-\mu^m(e^{i\theta m} - e^{-i\theta m})}{-\mu(e^{i\theta} - e^{-i\theta})} = \mu^{m-1}\frac{\sin m\theta}{\sin\theta}.$$

(b) Eliminate t_{n+1} and t_n from (8) using (9) to obtain

$$x_{n+1} - \rho x_n = \gamma(x_n - \rho x_{n-1}) + r_{n+1}, \quad \text{or:}$$
$$x_{n+1} - (\rho + \gamma)x_n + \rho\gamma x_{n-1} = r_{n+1}$$

which is the same as (19). Now if $\{x_n\}$ is a solution of (19) with a given sequence $\{r_n\}$ then by the preceding argument, the sequence $\{x_n - \rho x_{n-1}\}$ satisfies (8). Further, with $t_n = x_n - \rho x_{n-1}$ it is clear that $\{x_n\}$ satisfies (9) so that the sequence of pairs $\{(t_n, x_n)\}$ is a solution of the system of Eqs. (8) and (9). Conversely, if $\{(t_n, x_n)\}$ is a solution of the system then the above construction shows that $\{x_n\}$ satisfies (19). Therefore, the same set of solutions $\{x_n\}$ is obtained; i.e., the system is equivalent to the second-order equation.

(c) The equivalence of the system of Eqs. (7) and (19) to (1) is a consequence of Theorem 3.1 in [21]. ■

Remarks. 1. Theorem 4 shows that the existence of a common *complex* (non-real) root ρ for the polynomials P, Q leads to a reduction of order for (1) over the real numbers. Since the coefficients are real, the complex root requires the cofactor equation to have order two; i.e., the order reduction is type-$(k-1, 2)$ in the language of [21]. By contrast, over the field of complex numbers $\mathbb{C}$ (thinking of the real coefficients as special complex numbers) this reduction is equivalent to repeated type-$(k, 1)$ reductions as outlined in Lemmas 1 and 2.

2. The parameters $a_j, b_j, j = k-1, k$ which affect ρ but do not appear in (17) and (18) are not free. They satisfy (12) and (13) in Lemma 3 and for the complex conjugate pair of roots in Theorem 4 they take the forms

$$a_k = p'_{k-2}\mu^2, \qquad a_{k-1} = p'_{k-3}\mu^2 - 2p'_{k-2}\mu\cos\theta; \tag{20}$$

$$b_k = q'_{k-2}\mu^2, \qquad b_{k-1} = q'_{k-3}\mu^2 - 2q'_{k-2}\mu\cos\theta. \tag{21}$$

Here we assume that $p'_{-1} = 1$ and $q'_{-1} = 0$ when $k = 2$.

3. Boundedness and periodicity

In this section we use reduction of order and factorization methods of the preceding section to prove the existence of oscillations in the real solutions of certain difference equations of type (1). Convergence and global attractivity issues regarding this equation are discussed in [22] in at a much more general level.

We quote the next result from the literature as a lemma; see [19] or Section 5.5 in [21]. This result pertains to Eq. (9) whose solution may be written in the following way:

$$x_n = \rho^n x_0 + \sum_{j=1}^{n} \rho^{n-j} t_j. \tag{22}$$

Lemma 5 (*Periodicity, Limit Cycles, Boundedness*)**.** *Let p be a positive integer and let $\rho \in \mathbb{C}$ with $\rho \neq 0$.*

(a) *If for a given sequence $\{t_n\}$ of complex numbers Eq. (22) has a solution $\{x_n\}$ of period p then $\{t_n\}$ is periodic with period p.*

(b) *Let $\{t_n\}$ be a periodic sequence of complex numbers with prime (or minimal) period p and assume that ρ is not a p-th root of unity; i.e., $\rho^p \neq 1$. If $\{\tau_0, \ldots, \tau_{p-1}\}$ is one cycle of $\{t_n\}$ and*

$$\xi_i = \frac{1}{1-\rho}\sum_{j=0}^{p-1} \rho^{p-j-1}\tau_{(i+j) \bmod p} \quad i = 0, 1, \ldots, p-1 \tag{23}$$

then the solution $\{x_n\}$ of Eq. (22) with $x_0 = \xi_0$ and $t_1 = \tau_0$ has prime period p and $\{\xi_0, \ldots, \xi_{p-1}\}$ is a cycle of $\{x_n\}$.

(c) *If $|\rho| < 1$ and $\{t_n\}$ is a sequence that converges to a p-cycle then the sequence $\{x_n\}$ that is generated by (22) converges to a p-cycle. If $\{\tau_0, \ldots, \tau_{p-1}\}$ is one cycle of the limit of $\{t_n\}$ then $\{\xi_0, \ldots, \xi_{p-1}\}$ is a cycle of the limit of $\{x_n\}$ where ξ_i is defined by (23).*

(d) *If $|\rho| < 1$ and $\{t_n\}$ is a bounded sequence with $|t_n| \leq M$ for all n then the sequence $\{x_n\}$ that is generated by (22) is also bounded and there is a positive integer N such that*

$$|x_n| \leq |\rho| + \frac{M}{1-|\rho|} \quad \text{for all } n \geq N.$$

Lemma 5 and Theorem 4 imply the following result.

Corollary 6. *Let $k \geq 2$ in (1) and assume that the coefficients a_i, b_i are real and $g_n : \mathbb{R} \to \mathbb{R}$ for $n \geq 0$. If the polynomials P, Q in Lemma 1 have a common complex root $\rho = \mu e^{i\theta} \notin \mathbb{R}$ then the following statements are true.*

(a) *If ρ is not a p-th root of unity then for every periodic solution of (7) of prime period p (1) has a periodic solution of prime period p that is given by (23).*

(b) *If modulus $|\rho| < 1$ then for every limit cycle (attracting a periodic solution) of (7) of period p (1) has a limit cycle of period p.*

(c) *If modulus $|\rho| < 1$ then for every bounded solution of (7) the corresponding solution of (1) is bounded.*

Proof. We prove only (a) since the proofs of (b) and (c) use similar reasoning using Lemma 5. Recall that the second-order cofactor equation (19) in Theorem 4 is equivalent to the pair of first-order cofactor equations (8) and (9). Let $\{r_n\}$ be a solution of (7) having prime period p. If ρ is not a p-th root of unity then by Lemma 5(b) Eq. (8) has a solution $\{t_n\}$ in $\mathbb{C}$ with prime period p. Another application of Lemma 5 to Eq. (9) shows that the solution $\{x_n\}$ of (19) in $\mathbb{R}$ and hence, of (1) also has prime period p. ■

It is worth pointing out that if $|\rho| \geq 1$ then the periodic solution of (1) in Corollary 6(a) is not attracting even if the corresponding solution $\{r_n\}$ of (7) is attracting. Therefore, such solutions may be difficult to identify numerically. Only when $|\rho| < 1$ and the homogeneous part of the cofactor equation (19) fades away do the solutions of the factor equation (7) determine the asymptotic behavior of solutions of (1).

In closing, we discuss the solutions of a third-order version of (1), i.e., $k = 2$ to illustrate the various aspects of the preceding results. Consider the autonomous difference equation

$$x_{n+1} = a_0 x_n + a_1 x_{n-1} + a_2 x_{n-2} + g(x_n + b_1 x_{n-1} + b_2 x_{n-2}) \tag{24}$$

where $a_0, a_1, a_2, b_1, b_2 \in \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$. If $a_2 = b_2 = 0$ then (24) reduces to an autonomous version of the second-order equation (3). We assume here that $b_2 \neq 0$.

The polynomial Q of (24) is the quadratic $u^2 + b_1 u + b_2$ whose roots are complex if and only if $b_1^2 < 4b_2$. These complex conjugate roots are shared by the polynomial P if and only if conditions (20) hold. Since $k = 2$ we calculate

$$\begin{aligned} &p_0 = \rho - a_0, \qquad p_1 = \rho^2 - a_0\rho - a_1 \\ &q_0 = 1, \qquad q_1 = \rho + b_1 \\ &p'_0 = \bar{\rho} + p_0 = -b_1 - a_0, \qquad q'_0 = 1. \end{aligned}$$

Note that $\rho + \bar{\rho} = -2\mu\cos\theta = -b_1$ and $\rho\bar{\rho} = \mu^2 = b_2$. Thus conditions (20) in this case are

$$a_1 = b_1(a_0 + b_1) + b_2, \qquad a_2 = -b_2(a_0 + b_1). \tag{25}$$

We may alternatively obtain (25) using (12) and (13). If $b_1^2 < 4b_2$ and conditions (25) hold then (24) is equivalent to the pair of equations

$$r_{n+1} = (a_0 + b_1) r_n + g(r_n), \tag{26}$$

$$x_{n+1} = -b_1 x_n - b_2 x_{n-1} + r_{n+1} \tag{27}$$

for $n \geq 0$ where the initial value of (26) is $r_0 = x_0 + b_1 x_{-1} + b_2 x_{-2}$ for a given triple of real initial values x_0, x_{-1}, x_{-2} for (24).

Next, suppose that g is a rational function of the following type

$$g(u) = \frac{A}{u} + B + Cu, \quad A, B, C \in \mathbb{R}, A \neq 0. \tag{28}$$

With this g the difference equation is an example of a third-order rational recursive equation. A second-order version of this equation is a rational equation of type (4) that is studied in [20].

If $B = 0$ and $C = -a_0 - b_1$ then the factor equation (26) reduces to

$$r_{n+1} = \frac{A}{r_n}. \tag{29}$$

Every solution of (29) has period 2 with cycles $\{r_0, A/r_0\}$ as long as $r_0 \neq 0$. Since $\rho \neq \pm 1$, corresponding to each solution of (29) with period two, the solution of (24) whose triple of initial values (x_{-2}, x_{-1}, x_0) is not on the plane $u + b_1 v + b_2 w = 0$ (so that $r_0 \neq 0$) has period 2. This plane which passes through the origin is in fact the singularity (or forbidden) set of (24) in this case. The aforementioned periodic solutions of (24) have cycles $\{\xi_0, \xi_1\}$ that we calculate in two stages using (23). First, for (8) with $\gamma = \bar{\rho}$ we calculate the cycles $\{\tau_0, \tau_1\}$ in $\mathbb{C}$ as

$$\tau_0 = \frac{r_0 + \bar{\rho}A/r_0}{1 - \bar{\rho}^2}, \qquad \tau_1 = \frac{\bar{\rho} r_0 + A/r_0}{1 - \bar{\rho}^2}.$$

Next, using $\{\tau_0, \tau_1\}$ in (23) we calculate the cycles $\{\xi_0, \xi_1\}$ for (24) as

$$\xi_0 = \frac{\rho\tau_0 + \tau_1}{1 - \rho^2} = \frac{\rho r_0 + \rho\bar{\rho}A/r_0 + \bar{\rho} r_0 + A/r_0}{(1 - \rho^2)(1 - \bar{\rho}^2)} = \frac{(1 + \rho\bar{\rho})A/r_0 + (\rho + \bar{\rho}) r_0}{1 - (\rho^2 + \bar{\rho}^2) + \rho^2\bar{\rho}^2}.$$

Since $\rho\bar{\rho} = b_2$ and $\rho + \bar{\rho} = -b_1$ it follows that

$$\rho^2 + \bar{\rho}^2 = (\rho + \bar{\rho})^2 - 2\rho\bar{\rho} = b_1^2 - 2b_2$$

and thus,

$$\xi_0 = \frac{(1+b_2)A/r_0 - b_1 r_0}{1-(b_1^2-2b_2)+b_2^2} = \frac{A(1+b_2)-b_1 r_0^2}{r_0[(1+b_2)^2-b_1^2]}.$$

As expected, $\xi_0 \in \mathbb{R}$. A similar calculation yields

$$\xi_1 = \frac{r_0^2(1+b_2)-Ab_1}{r_0[(1+b_2)^2-b_1^2]}.$$

If $0 < b_2 < 1$ then $|\rho| = \mu = \sqrt{b_2} < 1$. In this case, every solution of (24) converges to a 2-cycle ξ_0 and ξ_1. These limit cycles depend on r_0 and thus, on the initial values x_{-2}, x_{-1}, x_0 in the sense that all initial points on the plane $u+b_1v+b_2w = r_0$ converge to the same limit cycle. However, if $b_2 \geq 1$ then other types of solutions, including unbounded solutions are possible for (24) that are driven by the homogeneous part of (27). To observe the 2-cycles numerically it is necessary to use the initial values

$$x_{-2} = \xi_0, \qquad x_{-1} = \xi_1, \qquad x_0 = r_0 - b_1\xi_1 - b_2\xi_0 = \xi_0.$$

Going in a different direction, if the function g in (28) has a 3-cycle then as is well-known, it has cycles of all possible lengths. In this case, if $0 < b_2 < 1$ then (24) also has cycles of all possible lengths. A set of parameter values that imply this situation is $A = 1$, $C = 1 - a_0 - b_1$ and $B = -\sqrt{3}$; see [20]. In this case, (26) is

$$r_{n+1} = \frac{1}{r_n} - \sqrt{3} + r_n$$

and its 3-cycle is found to be

$$\sigma_0 = \frac{2}{\sqrt{3}}\left(1+\cos\frac{\pi}{9}\right), \qquad \sigma_1 = g(\sigma_0), \quad \sigma_2 = g(\sigma_1).$$

In summary, we extended a number of previous results on the semiconjugate factorizations of second-order equations of type (1) to cases of order three and greater. Since in this case the associated polynomials of (1) have degree 3 or greater a mix of real and complex roots may occur. We showed that the decomposition of (1) into lower-order equations via a complex root and its conjugate yields a factor–cofactor pair in the sense described in Section 5.6 of [21] within the $r$eal number system thus bypassing the need for complex coefficients. We used this factorization to study boundedness and the occurrence of periodic solutions and limit cycles for (1).

Acknowledgments

The author thanks the referees for their many useful comments.

References

[1] J.R. Hicks, A Contribution to the Theory of the Trade Cycle, second ed., Clarendon Press, Oxford, 1965.
[2] T. Puu, Nonlinear Economic Dynamics, third ed., Springer, New York, 1993.
[3] P.A. Samuelson, Interaction between the multiplier analysis and the principle of acceleration, Rev. Econ. Stat. 21 (1939) 75–78.
[4] H. Sedaghat, Nonlinear Difference Equations: Theory with Applications to Social Science Models, Kluwer Academic, Dordrecht, 2003.
[5] C.W. Clark, A delayed recruitment model of population dynamics with an application to baleen whale populations, J. Math. Biol. 3 (1976) 381–391.
[6] M.E. Fisher, B.S. Goh, Stability results for delayed recruitment in population dynamics, J. Math. Biol. 19 (1984) 147–156.
[7] Y. Hamaya, On the asymptotic behavior of solutions of neuronic difference equations, in: Proc. Int'l. Conf. Difference Eq., Special Func. Appl, World Scientific, Singapore, 2007, pp. 258–265.
[8] V. Kocic, G. Ladas, Global Behavior of Nonlinear Difference Equations of Higher Order with Applications, Kluwer Academic, Dordrecht, 1993.
[9] R. Memarbashi, Sufficient conditions for the exponential stability of nonautonomous difference equations, Appl. Math. Lett. 21 (2008) 232–235.
[10] H. Sedaghat, Geometric stability conditions for higher order difference equations, J. Math. Anal. Appl. 224 (1998) 255–272.
[11] E. Liz, Stability of non-autonomous difference equations: simple ideas leading to useful results, J. Difference Equ. Appl. 17 (2011) 221–234.
[12] I. Győri, G. Ladas, P.N. Vlahos, Global attractivity in a delayed difference equation, Nonlinear Anal. 17 (1991) 473–479.
[13] G. Karakostas, C.G. Philos, Y.G. Sficas, The dynamics of some discrete population models, Nonlinear Anal. 17 (1991) 1069–1084.
[14] C.M. Kent, H. Sedaghat, Global stability and boundedness in $x_{n+1} = cx_n + f(x_n - x_{n-1})$, J. Difference Equ. Appl. 10 (2004) 1215–1227.
[15] H.A. El-Morshedy, On the global attractivity and oscillations in a class of second-order difference equations from macroeconomics, J. Difference Equ. Appl. 17 (2011) 1643–1650.
[16] H. Sedaghat, A class of nonlinear second-order difference equations from macroeconomics, Nonlinear Anal. 29 (1997) 593–603.
[17] H. Sedaghat, On the Equation $x_{n+1} = cx_n + f(x_n - x_{n-1})$, Fields Inst. Commun. 42 (2004) 323–326.
[18] S. Li, W. Zhang, Bifurcations in a second-order difference equation from macroeconomics, J. Difference Equ. Appl. 14 (2008) 91–104.
[19] H. Sedaghat, Periodic and chaotic behavior in a class of second order difference equations, Adv. Stud. Pure Math. 53 (2009) 311–318.
[20] M. Dehghan, C.M. Kent, R. Mazrooei-Sebdani, N. Ortiz, H. Sedaghat, Dynamics of rational difference equations containing quadratic terms, J. Difference Equ. Appl. 14 (2008) 191–208.
[21] H. Sedaghat, Form Symmetries and Reduction of Order in Difference Equations, Chapman & Hall/CRC Press, Boca Raton, 2011.
[22] H. Sedaghat, Global attractivity in a class of non-autonomous, nonlinear, higher order difference equations, J. Difference Equ. Appl. (2012) preprint on arXiv.org:1203.0227.