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A CLASS OF NONLINEAR SECOND ORDER DIFFERENCE EQUATIONS FROM MACROECONOMICS

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1. INTRODUCTION

Consider second order difference equations of the type

$$x_{n+1} = cx_n + f(x_n - x_{n-1}), \qquad n = 1, 2, 3, \ldots \tag{1}$$

where x_0, x_1 are given real numbers (the initial condition), $c \geq 0$ is constant and $f: \mathbf{R} \to \mathbf{R}$ is a real function. Equations of type (1) and similar, arose in some of the earliest mathematical models of the macroeconomic "trade cycle". For example, equation (1) generalizes the classic Hansen–Samuelson's accelerator–multiplier model [1], namely,

$$Y_n = cY_{n-1} + \alpha c(Y_{n-1} - Y_{n-2}) + A_0$$

where the constant $A_0 = C_0 + I_0 + G_0$ represents the sum of the minimum consumption, the "autonomous" investment and the fixed government spending in period n, and Y_n is the output—GNP or national income—in period n. The net investment amount in the same period is given as $I_n = \alpha c(Y_{n-1} - Y_{n-2})$. The constant $c \in (0, 1)$ represents Keynes' "marginal propensity to consume" or the MPC, while the coefficient $\alpha > 0$ is the "accelerator".

The linear model above improved the earlier Keynesian models and resulted in substantial new research. However, this model was soon found to be unsatisfactory, since αc can exceed unity in typical economic settings (see, [2, Chap. 9]). This fact results in exponentially divergent solutions for the linear equation, which of course, is not observed in reality. Certain nonlinear models were subsequently proposed to resolve this anomaly. For instance, rather than the linear Keynesian consumption $C(Y) = cY + C_0$, Samuelson considered a nonlinear consumption function [3]. Samuelson's assumptions amount to a third order difference equation and an MPC which is itself a decreasing *function* of output. Some years later, Hicks proposed a model in which consumption was linear, but investment and output were both piecewise linear [4]. Hicks' model results in a second order difference equation with constant MPC, but the accelerator is not defined (or could mathematically be set equal to zero) for a certain range of output differences; the model also happens to be nonautonomous because of a time dependent "hard ceiling" on output (as well as the induced investment "floor"). There have been several other models such as *continuous time* models of Goodwin and Kaldor, or the more recent *stochastic* models which we have not mentioned because of the attributes italicized here; [5] contains brief discussions of some of these models and comprehensive bibliography.

A special form of equation (1), namely,

$$Y_n = cY_{n-1} + I(Y_{n-1} - Y_{n-2}) + C_0 + G_0 \tag{2}$$

where $I\colon \mathbf{R} \to \mathbf{R}$ is a nondecreasing *induced investment* function (not necessarily continuous) thus represents an autonomous deterministic model that incorporates elements of both Hicks' and Samuelson's approaches. For instance, we obtain the original linear model when $I(u) = \alpha cu + I_0$. Hicks' model with the hard ceiling for output set at infinity, also becomes a special case of (2). The hard ceiling—proposed by Hicks to check the unlimited growth of output in his model—is unnecessary *as such* is the accelerator is a *function* of output differences (somewhat like Samuelson's MPC). Indeed, it may be argued that monetary effects (rising interest rates at large values of output change to discourage inflation; see [4, Chap. XII]) or an "investment ceiling" (see [4, Chap. X] or [6]) cause the accelerator coefficient (defined as the rate of change of I with respect to output change $Y_{n-1} - Y_{n-2}$) to decrease, thus bending the graph of I downwards as $Y_{n-1} - Y_{n-2}$ increases. The constant accelerator obviously cannot account for such nonlinear stabilizing effects.

From a mathematical point of view, since the function $f(u)$ in (1) is defined on the line $\mathbf{R}$, equation (1) enjoys a simplicity that is lacking in a general autonomous second order equation $x_{n+1} = F(x_n, x_{n-1})$, where the function $F(u, v)$ is defined on the plane $\mathbf{R}^2$. Nevertheless, few of the existing results in the literature seem directly applicable to (1) in settling basic questions such as boundedness and convergence. On the other hand, the first order initial value problem

$$v_{n+1} = f(v_n), \qquad v_1 = x_1 - x_0 \tag{3}$$

relates naturally to (1) and is comparatively easier to analyze. In particular, various properties of the solutions of (3)—such as convergence and the existence of means—are intimately related to the boundedness of solutions of the second order equation.

The objective of this paper is to investigate some of the mathematical properties of (1). For $0 \leq c < 1$ we obtain sufficient conditions for the permanence of (1) by exploring the relationship between the latter equation and (3) for certain classes of function f. The degenerate case $c = 1$ is discussed in some detail and convergence results are obtained again with the aid of (3) which, in particular, show the tendency of (1) to not be permanent in this case. The use of first order equations in connection with the higher order ones is not new to this paper; for other examples see [7–10]. For $c \neq 1$, linear stability results near the unique fixed point $\bar{x} = f(0)/(1 - c)$ are omitted, since such results are routine applications of the standard theory (as presented in [11]).

2. BOUNDEDNESS AND PERMANENCE

Our first result also establishes a useful link between the solutions of (1) and the solutions of (3).

LEMMA 1. Let $\mathbf{R} \to \mathbf{R}$ be a nondecreasing function, and let $0 \leq c < 1$. (a) If $\{x_n\}$ is a nonnegative solution of (1) then

$$x_n \leq c^{n-1}x_0 + \sum_{k=1}^{n} c^{n-k} v_k \tag{4}$$

for all n, where $\{v_n\}$ is a solution of (3).

(b) If $\{x_n\}$ is a nonpositive solution of (1), then

$$x_n \geq c^{n-1}x_0 + \sum_{k=1}^{n} c^{n-k}v_k. \tag{5}$$

(c) If $c \geq 1$, then the inequalities in Parts (a) and (b) are reversed.

Proof. (a) Observe that $x_1 = x_0 + v_1$ and that

$$x_2 = cx_1 + f(v_1) = cx_0 + cv_1 + v_2.$$

Hence (4) holds for $n = 1, 2$. So assume that (4) holds for all integers less than or equal to some integer n. Then

$$x_{n+1} = cx_n + f(x_n - x_{n-1}) \leq c^n x_0 + \sum_{k=1}^{n} c^{n-k+1}v_k + f(x_n - x_{n-1})$$

so it remains to show that $f(x_n - x_{n-1}) \leq v_{n+1}$. To this end, note that

$$x_n - x_{n-1} = (c-1)x_{n-1} + f(x_{n-1} - x_{n-2}) \leq f(x_{n-1} - x_{n-2})$$

which together with the assumption that f is nondecreasing implies that

$$f(x_n - x_{n-1}) \leq f(f(x_{n-1} - x_{n-2})) \doteq f^2(x_{n-1} - x_{n-2}).$$

Continuing inductively in this fashion, it is evident that

$$f(x_n - x_{n-1}) \leq f^n(x_1 - x_0) = v_{n+1}$$

thus proving (4).

(b) Let $y_n = -x_n$, and note that

$$y_{n+1} = cy_n + g(y_n - y_{n-1})$$

where $g(u) = -f(-u)$ is nondecreasing. Now apply Part (a) to obtain (5). The proof of Part (c) is the same as that in (a) and (b) except that all inequalities in the proof are to be reversed. ■

COROLLARY 1. Let f be nondecreasing and nonnegative, and assume that f has an invariant set S—i.e. S is nonempty and $f(S) \subset S$. If S is a bounded set and $0 \leq c < 1$, then every solution of (1) with $x_1 - x_0 \in S$ is bounded.

Proof. If $x_k \geq 0$ for any k, then $x_n \geq 0$ for all $n \geq k$, since

$$x_{k+1} = cx_k + f(x_k - x_{k-1}) \geq cx_k$$

and it follows by induction that for all $m \geq 0$, $x_{k+m} \geq c^m x_k \geq 0$. Lemma 1 now applies and implies that $\{x_n\}$ is bounded. Now suppose that $x_n < 0$ for all n. Then for every n it is true that

$$x_{n-1} = cx_n + f(x_n - x_{n-1}) \geq cx_n > x_n \geq x_1$$

from which it follows that $\{x_n\}$ is again bounded. ■

Remarks. The case $c = 1$ is different and discussed later. Corollary 1 is false if f is not nonnegative and nondecreasing (see Example 4). We emphasize here that *the function f is not assumed to be continuous* in Lemma 1 or in Corollary 1. Generally, continuity is not assumed unless explicitly stated. More often in this paper we let f be bounded on the compact subsets of $\mathbf{R}$. This is true, in particular, if f is monotone on $\mathbf{R}$ (as in the next theorem), or if f is bounded by continuous functions (Theorem 4). An exception to this rule is Theorem 2.

Recall that (1) is said to be *permanent* if there are real number a, b such that $a < b$ and every solution of (1) is eventually in $[a, b]$; see [10].

THEOREM 1. Let f be nondecreasing and bounded from below on the reals, and let $0 \leq c < 1$. If there exists $\alpha \in (0, 1)$ and $u_0 > 0$ such that $f(u) \leq \alpha u$ for all $u \geq u_0$, then (1) is permanent.

Proof. If we define $w_n \doteq f(x_n - x_{n-1})$ for all $n \geq 1$, then it follows inductively from (1) that

$$x_n = c^{n-1}x_1 + c^{n-2}w_1 + \cdots + cw_{n-2} + w_{n-1}. \tag{6}$$

Let L_0 be a lower bound for $f(u)$, and without loss of generality assume that $L_0 \leq 0$. As $w_k \geq L_0$ for all k, we conclude from (6) that

$$x_n \geq c^{n-1}x_1 + \left(\frac{1 - c^{n-1}}{1 - c}\right)L_0$$

for all n, and therefore, $\{x_n\}$ is bounded from below. In fact, it is clear that there is a positive integer n_0 such that for all $n \geq n_0$,

$$x_n \geq L \doteq \frac{L_0}{1 - c} - 1.$$

We now show that $\{x_n\}$ is bounded from above as well. Define $z_n \doteq x_{n+n_0} - L$ for all $n \geq 0$, so that $z_n \geq 0$ for all n. Now for each $n \geq 1$ we note that

$$\begin{aligned} z_{n+1} &= cx_{n+n_0} + f(x_{n+n_0} - x_{n+n_0-1}) - L \\ &= cz_n + f(z_n - z_{n-1}) - L(1 - c). \end{aligned}$$

Define $g(u) \doteq f(u) - L(1 - c)$, and led $\delta \in (\alpha, 1)$. It is readily verified that $g(u) \leq \delta u$ for all $u \geq u_1$ where

$$u_1 \doteq \max\left\{u_0, \frac{-L(1 - c)}{\delta - \alpha}\right\}.$$

If $\{r_n\}$ is a solution of the first order problem

$$r_{n+1} = g(r_n), \qquad r_1 = z_1 - z_0$$

then since g is bounded from below by $L_0 - (1 - c)L = 1 - c$, we have

$$r_n = g(r_{n-1}) \geq 1 - c$$

for all $n \geq 2$. Thus $\{r_n\}$ is bounded from below. Also, if $r_k \geq u_1$ for some $k \geq 1$, then

$$r_{k+1} = g(r_k) \leq \delta r_k < r_k.$$

If $r_{k+1} \geq u_1$ also, then $\delta r_k \geq r_{k+1} \geq u_1$ and since g is nondecreasing,

$$r_{k+2} = g(r_{k+1}) \leq g(\delta t_k) \leq \delta^2 r_k.$$

It follows inductively that

$$r_{k+l} \leq \delta^l r_k$$

as long as $r_{k+l} \geq u_1$. Clearly there is $m \geq k$ such that $r_m < u_1$. Then

$$r_{m+1} = g(r_m) \leq g(u_1) \leq \delta u_1 < u_1$$

by the definition of u_1. By induction $r_n < u_1$ for all $n \geq m$. Now Lemma 1(a) implies that for all such n,

$$\begin{aligned} z_n &\leq c^{n-1} z_0 + c^{n-1} r_1 + \cdots + c^{n-m+1} r_{m-1} + \sum_{k=m}^{n} c^{n-k} r_k \\ &< c^{n-m+1}(z_0 c^{m-2} + \cdots + r_{m-1}) + u_1 \sum_{k=0}^{n-m} c^k \\ &= c^{n-m+1} K_0 + u_1 (1-c)^{-1}(1 - c^{n-m+1}). \end{aligned}$$

Thus there exists $n_1 \geq m$ such that

$$z_n \leq \frac{u_1}{1-c} + 1$$

for all $n \geq n_1$. Hence, for all $n \geq n_0 + n_1$ *we have* $x_n \in [L, M]$ where

$$M \doteq \frac{u_1}{1-c} + 1 - L.$$

It follows that (1) is permanent. ■

The next corollary follows from Theorem 1 by considering the negative of f.

COROLLARY 2. Let f be nonincreasing and bounded from above on the reals, and let $0 \leq c < 1$. If there exists $\alpha \in (0, 1)$ and $u_0 > 0$ such that $f(u) \geq -\alpha u$ for all $u \geq u_0$, then (1) is permanent.

Remarks. The part of the proof of Theorem 1 that is based on (6) can be easily expanded to establish the following assertion:

(A) Let f be bounded from below (or from above) on **R**. If $0 \leq c < 1$, then every solution of (1) is bounded from below (respectively, from above). In particular, if f is bounded, then every solution of (1) is bounded.

Note that the function f in the preceding statement is *not* assumed to be monotone (in fact no new results are needed for proving (A) which follows from standard comparison results; see, e.g. [11, Section 1.6]). On the other hand, if $c = 1$, then by choosing $f(u) \equiv K$ for all $u \in \mathbf{R}$ where K is a nonzero constant, we see that $x_n = x_1 + K(n-1)$, so that *every* solution of (1) is unbounded from above or below, depending on the sign of K. The same choice of f shows that this conclusion holds for $c > 1$ as well. Thus both Theorem 1 and (A) fail if $c \geq 1$.

Since in equation (2) the function I is assumed nondecreasing, Theorem 1 is a comprehensive boundedness result concerning that equation. The next result shows that under the hypotheses of Theorem 1, the average rate of change of output is zero in the long-run (i.e. over a long time, if the output is not monotonically converging to a limit then it is increasing on average as often as it is decreasing).

COROLLARY 3. Assume that f and c satisfy the hypotheses of Theorem 1 or Corollary 2. Then the second order equation

$$y_{n+1} = cy_n + f(y_n) - f(y_{n-1}) \tag{7}$$

is permanent and each of its solutions has a zero mean.

Proof. Let $\{y_n\}$ be a solution of (7), and for $n \geq 1$ define

$$x_n = x_0 + \sum_{k=0}^{n-1} y_k$$

for some real number x_0. We wish to show that $\{x_n\}$ satisfies (1). Note that $x_{n+1} = x_n + y_n$, so in particular,

$$x_2 = x_1 + y_1 = x_0 + y_0 + y_1.$$

To satisfy (1), x_0 must be defined so that the right-hand side of the last equality above equals $cx_1 + f(x_1 - x_0) = c(x_0 + y_0) + f(y_0)$. Solving for x_0 we obtain

$$x_0 = \frac{f(y_0) - y_1}{1 - c} - y_0. \tag{8}$$

Now for $n \geq 2$,

$$\begin{aligned} x_{n+1} &= x_0 + y_0 + y_1 + \sum_{k=2}^{n} [cy_{k-1} + f(y_{k-1}) - f(y_{k-2})] \\ &= (1 - c)(x_0 + y_0) + y_1 - f(y_0) + cx_n + f(x_n - x_{n-1}) \\ &= cx_n + f(x_n - x_{n-1}) \end{aligned}$$

where the last equality follows from (8) and the fact that $y_{n-1} = x_n - x_{n-1}$. Therefore, $\{x_n\}$ is a solution of (1). Since $|y_n| \leq |x_{n+1}| + |x_n|$, it follows from Theorem 1 (or Corollary 2) that (7) is permanent. Furthermore,

$$\frac{1}{n}\sum_{k=0}^{n-1} y_k = \frac{1}{n}\sum_{k=0}^{n-1} (x_{k+1} - x_k) = \frac{x_n - x_0}{n}$$

for every n, so that by the boundedness of $\{x_n\}$ the mean $\lim_{n\to\infty}(1/n)\sum_{k=0}^{n-1} y_k$ of $\{y_n\}$ is zero. ■

We close this section with the following general fact regarding $c > 1$.

THEOREM 2. Let f be nonnegative. If $c > 1$, then every solution of (1) with $x_1 > 0$ is unbounded.

Proof. From (6), $x_n \geq c^{n-1}x_1$, so that $\{x_n\}$ is unbounded for $x_1 > 0$. ■

Example 1. The restriction $x_1 > 0$ is required in the preceding theorem. Let

$$f(u) = \begin{cases} -cu & \text{if } u < 0 \\ 0 & \text{if } u \geq 0. \end{cases}$$

Then f is nonnegative for all $c \geq 0$. It is easily seen that if $x_0 = 0$, then for any choice of $x_1 \leq 0$, it follows that $x_n = 0$ for all $n \geq 2$.

3. THE CASE $c = 1$: CONVERGENCE AND OTHER RESULTS

Now we turn to the structurally different case $c = 1$. In contrast to the results of the previous section, the boundedness of solutions of (1) in this case depends critically on the *detailed properties* of solutions of (3), including such properties as periodicity (Theorem 3), oscillations (Theorem 5) and the rate of convergence (Theorems 4 and 5). Throughout this section, it is assumed that $c = 1$, unless otherwise stated.

THEOREM 3. Let $\{x_n\}$ be a solution of the second order equation (1) and let $\{\upsilon_n\}$ be the corresponding solution of the first order equation (3).

(a) If υ^* is a fixed point of (3) then $x_n = x_0 + \upsilon^* n$ is a solution of (1)

(b) If $\{\upsilon_1, \ldots, \upsilon_p\}$ is a periodic solution of (3) with period p, then

$$x_n = x_0 - \omega_n + \bar{\upsilon} n \tag{9}$$

is a solution of (1) with $\bar{\upsilon} = p^{-1} \sum_{j=0}^{p} \upsilon_i$ the average solution, and

$$\omega_n = \bar{\upsilon}\rho_n - \sum_{j=0}^{\rho_n} \upsilon_j, \qquad (\upsilon_0 \doteq 0)$$

where ρ_n is the remainder resulting from the division of n by p. The sequence $\{\omega_n\}$ is periodic with period at most p.

(c) If $\{x_n\}$ is bounded, then $\{\upsilon_n\}$ is bounded and has a zero mean.

Proof. Part (a) follows immediately from the identity

$$x_n = x_0 + \sum_{i=1}^{n} \upsilon_i \tag{10}$$

which also establishes the useful fact that solutions of the second order equation are essentially the partial sums of solutions of the first order equation. To prove (b), observe that in (10), after every p iterations we add a fixed sum $\sum_{i=1}^{p} \upsilon_i$ to the previous total. Therefore, since n may generally take on any one of the values $pk + \rho_n$, where $0 \leq \rho_n \leq p - 1$, we have

$$x_n = x_0 + k \sum_{i=1}^{p} \upsilon_i + \sum_{j=0}^{\rho_n} \upsilon_j. \tag{11}$$

Now substituting $k = n/p - \rho_n/p$ in (11) and rearranging terms we obtain (9). Also ω_n is periodic since ρ_n is periodic, and the period of ω_n cannot exceed p, since $\omega_{pk} = 0$ for each nonnegative integer k. The proof of Part (c) uses (10) and an argument similar to that given in Corollary 3. ■

COROLLARY 4. (a) If $\{\upsilon_n\}$ is periodic with period $p \geq 1$, then the sequence $\{x_n - \bar{\upsilon} n\}$ is also periodic with period at most p. In particular, $\{x_n\}$ is periodic (hence bounded) if and only if $\bar{\upsilon} = 0$.

(b) If $\{\upsilon_n\}$ has a nonzero mean (in particular, if it converges to a nonzero limit), or if $\{\upsilon_n\}$ does not have a mean, then $\{x_n\}$ is unbounded.

Example 2. The converse of Theorem 3(c) or Corollary 4(b) is false. Let $f(u) = u(1 + u^2)^{-1/2}$. If $v_1 = 1$, then $v_n = n^{-1/2}$ is a solution of (3) that converges to zero. However, from (10) we obtain

$$x_n = x_0 + \sum_{k=1}^{n} \frac{1}{\sqrt{k}}$$

which diverges for all x_0 as $n \to \infty$. Hence, $\{x_n\}$ may be unbounded even when $\{v_n\}$ converges to zero.

In spite of Example 2, partial converses to Theorem 3(c) do exist. Such results yield sufficient conditions for the boundedness of $\{x_n\}$ by establishing an adequately rapid rate of convergence to zero for $\{v_n\}$. The next two theorems are examples of such converses, the first of which may be compared to Theorem 1.

THEOREM 4. Assume that there exists a constant $\alpha \in (0, 1)$ such that $|f(u)| \leq \alpha|u|$ for all u. Then every $\{v_n\}$ converges to zero and every $\{x_n\}$ is bounded and converges to a real number that is determined by the initial conditions x_0, x_1.

Proof. Note that $|v_{n+1}| = |f(v_n)| \leq \alpha|v_n|$ for all $n \geq 1$. It follows inductively that $|v_n| \leq \alpha^n|v_1|$, and hence,

$$\sum_{k=1}^{n} |v_k| \leq |v_1| \sum_{k=1}^{n} \alpha^k \leq \frac{\alpha|v_1|}{1-\alpha}$$

which implies that the series $\sum_{n=1}^{\infty} |v_n|$ converges. It follows at once that $\{v_n\}$ is bounded and in fact converges to the real number $x_0 + \sum_{n=1}^{\infty} v_n$. ■

Examples 3. The most obvious examples of functions that satisfy the hypotheses of Theorem 4 are the linear function $f(u) = au$ (monotone) and the absolute value function $f(u) = a|u|$ (unimodal) with $|a| < 1$. More generally, f may be any Lipschitz function with constant α and with $f(0) = 0$. On the other hand, the function f in Example 2 satisfies $|f(u)| < |u|$ for all $u \neq 0$; hence the condition $\alpha < 1$ is necessary in Theorem 4 (but also see the next theorem with regard to continuous monotone function). Of course, af works in Theorem 4 if $|a| < 1$ and f is the function in Example 2. We point out that even though we have only cited examples in which f is continuous, clearly in Theorem 4 f need not be continuous, except of course, at the origin.

In the next theorem f^2 denotes the function composition $f \circ f$.

THEOREM 5. Suppose that f is continuous, nonincreasing, and $f(0) = 0$. Then the conclusions of Theorem 4 about $\{v_n\}$ and $\{x_n\}$ hold if at least one of the following conditions is also satisfied:

(a) $|f(u)| < |u|$ for all $u \neq 0$;
(b) f is bounded from below on **R**, and $f^2(u) \neq u$ if $u \neq 0$;
(c) f is bounded from above on **R**, and $f^2(u) \neq u$ if $u \neq 0$.

Proof. For nontriviality, assume that $v_n \neq 0$ for all n. If $v_1 < 0$ then $v_2 = f(v_1) > f(0) = 0$, with the reverse inequalities holding for positive v_1. It follows inductively that the sequence $\{v_n\}$ is alternatively negative and positive. Next we show that the origin is globally asymptotically stable in (3) if any one of the conditions (a)–(c) holds. We consider the case $v_1 < 0$ (the case $v_1 > 0$ being similar). If (a) holds, then

$$|v_{n+1}| = |f(v_n)| < |v_n|$$

showing that $\{|v_n|\}$ is a decreasing sequence. Define

$$L \doteq \lim_{n\to\infty} |v_n|. \tag{12}$$

Observe that L is the monotone limit of $\{v_{2n}\}$ or of $\{v_{2n-1}\}$, whichever is positive, in this case $\{v_{2n}\}$. Since f^2 is continuous,

$$\lim_{n\to\infty} f^2(v_{2n}) = f^2(L).$$

Now from the inequality

$$|f^2(L) - L| \leq |f^2(L) - f^2(v_{2n})| + |v_{2n+2} - L| \qquad n = 1, 2, 3, \ldots$$

it follows that $f^2(L) = L$; i.e. L is a fixed point of f^2. On the other hand, the origin, which happens to be the *unique* fixed point of f, is also a fixed point of f^2. Since

$$|f^2(u)| = |f(f(u))| < |f(u)| < |u|$$

for all $u \neq 0$, we conclude that $f^2(u) = u$ if and only if $u = 0$. It follows that $L = 0$.

In cases (b) and (c) the origin by assumption is the unique fixed point of both f and f^2. Hence $v_3 \neq v_1$, and $v_4 \neq v_2$. If $v_3 < v_1$, then

$$v_4 = f(v_3) > f(v_1) = v_2. \tag{13}$$

If (b) above holds with $f(u) > M_1$ for all u, then applying (13) inductively we have

$$M_1 < v_{2n+1} < v_{2n-1} < 0 < v_{2n} < v_{2n+2} < f(M_1) \qquad n = 1, 2, 3, \ldots,$$

implying the existence of number $\bar{L}$ and $\underline{L}$ such that $\underline{L} < 0 < \bar{L}$ and

$$\lim_{n\to\infty} v_{2n} = \bar{L} \qquad \lim_{n\to\infty} v_{2n-1} = \underline{L}.$$

But then by continuity $\bar{L} = f(\underline{L}) = f^2(\bar{L})$, which is not possible. Therefore, it must be true that $v_3 > v_1$. In this case, it is easily verified that

$$0 < v_4 < v_2$$

so $v_3 = f(v_2) < 0$, and by induction

$$v_{2n-1} < v_{2n+1} < 0 < v_{2n+2} < v_{2n} \qquad n = 1, 2, 3, \ldots$$

from which it follows that L defined by (12) exists and is zero. Now suppose that (c) holds with $f(u) < M_2$ for all u. If $v_3 < v_1$, then applying (13) inductively again, we have

$$f(M_2) < v_{2n+1} < v_{2n-1} < 0 < v_{2n} < v_{2n+2} < M_2 \qquad n = 1, 2, 3, \ldots.$$

Arguing as in the case of (b) we once again find that we must have $v_3 > v_1$ and then it follows that $L = 0$, where L is the limit defined in (12).

Hence, if any one of (a)–(c) holds, then $|v_n| \to 0$ as $n \to \infty$. Therefore, the alternating series test implies that $\{x_n\}$ converges to the real number $x_0 + \sum_{n=1}^{\infty} v_n$ for all choices of x_0, x_1. ■

Remarks. The proof of the part of Theorem 5 concerning $\{v_n\}$ in cases (b) and (c) is similar to the proof of [10, Lemma 1.6.5], which originally appeared in [12]. We may also point out that since they are convergence results, Theorems 4 and 5 of course imply boundedness; however, they clearly do not imply permanence as in Theorem 1.

Part (a) of the preceding theorem may be compared with Examples 2, 3, Theorem 4, and the next example.

Example 4. Theorems 4 and 5 are false if $c \neq 1$. Indeed, unbounded solutions appear in both theorems even when $0 \leq c < 1$, demonstrating in particular, that *the boundedness potential does not improve in all cases by reducing the value of c.*

Let $f(u) = -au$ where $a \in (0, 1)$. Then the solution of (3) with arbitrary v_1 is $v_n = (-a)^{n-1} v_1$. Clearly, $v_n \to 0$ as $n \to \infty$, so in particular $\{v_n\}$ is bounded and the origin is globally asymptotically stable. On the other hand, using this linear choice of f in (1) yields

$$x_{n+1} = (c - a)x_n + ax_{n-1}.$$

Solving by linear methods [11, Section 2.3], we obtain

$$x_n = C_1 \lambda_1^n + C_2 \lambda_2^n$$

where C_1, C_2 are constants depending on the initial conditions, and the eigenvalues λ_1 and λ_2 are solutions of the quadratic equation $\lambda^2 + (a - c)\lambda - a = 0$; i.e.

$$\lambda_1 = \frac{-(a-c) - \sqrt{(a-c)^2 + 4a}}{2}, \qquad \lambda_2 = \frac{-(a-c) + \sqrt{(a-c)^2 + 4a}}{2}.$$

Both eigenvalues are real, but it is easy to see that $\lambda_1 < -1$ when

$$\frac{c+1}{2} < a < 1.$$

If $c < 1$, then such a value of a always exists and results in $\{x_n\}$ undergoing unbounded, expanding oscillations.

In closing, we point out that our study of equation (1) is far from complete. In fact, this paper constitutes only an introduction to this intriguing equation. Going beyond this point, results on the global stability of the fixed point, as well as on stable oscillations and limit cycles are certainly desirable for $0 \leq c < 1$. We also mention that in the case $c = 1$ highly complex behavior arises from certain smooth unimodal functions possessing a compact invariant set. As an example, $f(u) = 1 - 2u^2$ results in highly complex behavior in (3) over its invariant interval $[-1, 1]$; see [13]. For (1), such complexity in detail translates into difficulties even at the stage of determining whether there are any *bounded* solutions. For instance, the mean ergodic theorem implies that except on a subset of $[-1, 1]$ of Lebesgue measure zero, every solution of the first order equation has zero mean; is this an indication that almost every solution of the second order equation is bounded? Naturally, not every solution is bounded (consider the nonzero fixed points) and in fact an infinite set of initial conditions can be found inductively in $[-1, 1]$ that has Lebesgue measure zero and each point of it leads to an unbounded solution.

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