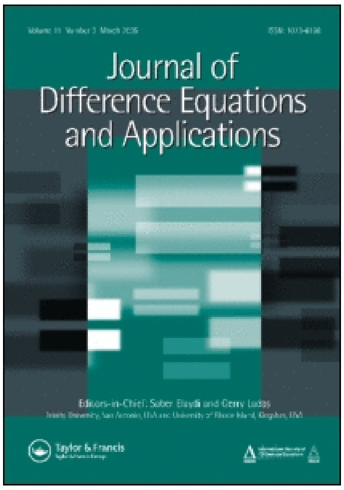

Journal of Difference Equations and Applications
Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t713640037

Global behaviours of rational difference equations of orders two and three with quadratic terms

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Online Publication Date: 01 March 2009

To cite this Article Sedaghat, H.(2009)'Global behaviours of rational difference equations of orders two and three with quadratic terms',Journal of Difference Equations and Applications,15:3,215 — 224

To link to this Article: DOI: 10.1080/10236190802054126

URL: http://dx.doi.org/10.1080/10236190802054126

Journal of Difference Equations and Applications
Vol. 15, No. 3, March 2009, 215–224

Global behaviours of rational difference equations of orders two and three with quadratic terms

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(*Received 10 March 2008; final version received 13 March 2008*)

We determine the global behaviours of all solutions of the following rational difference equations

$$x_{n+1} = \frac{ax_{n-1}}{x_n x_{n-1} + b}, \quad x_{n+1} = \frac{ax_n x_{n-1}}{x_n + bx_{n-2}}, \quad a, b > 0.$$

These equations are related to each other via semiconjugate relations that also let us reduce them to first-order equations. Using this approach, we determine the forbidden sets of each equation explicitly and show that for initial values outside the forbidden sets, their solutions may converge to 0, or to a positive fixed point, or they may be periodic of period 2 or unbounded. In some cases, different types of solutions coexist depending on the initial values.

Keywords: rational; quadratic; second-order; third-order; semiconjugate; periodic

2000 Mathematics Subject Classification: 39A10; 39A11

1. Introduction

Rational difference equations containing polynomial terms of degree 2 (or quadratic terms) either in their numerators or their denominators have not not been widely or systematically studied, although some such equations have been discussed previously in the literature; see Ref. [5] and the references listed therein. In particular, different types of second-order rational equations with quadratic terms are discussed in Refs. [4,6,7].

The equations that we study in this paper belong to this category. Specifically, we investigate the global behaviour of all solutions of the second-order rational difference equation

$$x_{n+1} = \frac{ax_{n-1}}{x_n x_{n-1} + b}, \quad a, b > 0, \ \ x_0, x_{-1} \in \mathbb{R}, \tag{1}$$

and also of the third-order equation

$$x_{n+1} = \frac{ax_n x_{n-1}}{x_n + bx_{n-2}}, \quad a, b > 0, \ \ x_0, x_{-1}, x_{-2} \in \mathbb{R}. \tag{2}$$

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ISSN 1023-6198 print/ISSN 1563-5120 online

DOI: 10.1080/10236190802054126
http://www.informaworld.com

These equations are related in the sense that after extracting a first-order linear equation from (2), it reduces to (1). This process is a type of semiconjugate factorisation; our discussion of semiconjugates here is self-contained, but [9,10] contain additional details if desired. Equation (1) can further be reduced to a (nonhomogeneous) linear difference equation after a first-order rational equation is factored out of it. Using these procedures, we determine the forbidden sets (see Ref. [8]) of singular points of (2) and (1) explicitly and give global characterisations of all solutions of these equations.

2. The second-order equation

The results in this section update and extend those in Ref. [2] (from non-negative solutions to all real solutions) using a different approach in proofs that is based on order reduction. For reference, we observe that if $\{x_n\}$ is any solution of equation (1) then so is $\{-x_n\}$; further, each of the quadrants $(0, \infty)^2$ and $(-\infty, 0)^2$ (i.e. first and third quadrants of $\mathbb{R}^2$, respectively) is invariant under (1).

Equation (1) has an equivalent, one parameter representation which is slightly easier to work with. Substituting $x_n/\sqrt{b}$ for x_n and c for a/b in (1) gives the equivalent equation

$$x_{n+1} = \frac{cx_{n-1}}{x_n x_{n-1} + 1}. \tag{3}$$

If $c \leq 1$ then the origin is the unique fixed point of (3). If $c > 1$ then in addition to the origin (3) also has a pair of fixed points

$$\bar{x}_{\pm} = \pm\sqrt{c-1}.$$

It is relevant at this point to mention that the linearization of (3) at each of these fixed points has two real eigenvalues $1/c$, -1. Therefore, the nonzero fixed points are nonhyperbolic. Further, the function

$$f(u, v) = \frac{cv}{uv + 1}, \tag{4}$$

that defines our equation is decreasing in u and increasing in v. The following general result from Ref. [3] outlines the kinds of behaviour we can expect of the solutions of equation (3).

LEMMA 1. *Let I be a set of real numbers and let $F: I \times I \to I$ be a function $F(u, v)$ that decreases in u and increases in v. Then for every solution $\{x_n\}$ of the equation*

$$x_{n+1} = F(x_n, x_{n-1}),$$

the subsequences x_{2n} of even terms and x_{2n+1} of odd terms do exactly one of the following:

(1) *they are both increasing;*
(2) *they are both decreasing; and*
(3) *eventually, one of them is increasing and the other is decreasing.*

The next result is needed in the sequel; its straightforward proof is omitted.

LEMMA 2. (a) *If y_n satisfies the first-order, nonautonomous equation*

$$y_n = \rho_n y_{n-1},$$

where ρ_n is a given sequence of real numbers, then

$$y_n = y_0 \rho_1 \rho_2 \ldots \rho_n.$$

(b) *If y_n satisfies the first-order, nonautonomous equation*

$$y_n = \frac{\sigma_n}{y_{n-1}},$$

where $y_0 \neq 0$ and σ_n is a given sequence of nonzero real numbers, then

$$y_{2n} = \frac{y_0 \sigma_2 \sigma_4 \ldots \sigma_{2n}}{\sigma_1 \sigma_3 \ldots \sigma_{2n-1}}, \quad y_{2n+1} = \frac{\sigma_1 \sigma_3 \ldots \sigma_{2n+1}}{y_0 \sigma_2 \sigma_4 \ldots \sigma_{2n}}.$$

We now consider solutions of (3) from initial points (x_0, x_{-1}) other than (0,0) that are located on one of the coordinate axes. These solutions have a different global character than those generated by initial points off the coordinate axes.

LEMMA 3. *Let $c > 0$ and the initial values satisfy*

$$x_0 x_{-1} = 0, \quad x_0 + x_{-1} = \delta \neq 0. \tag{5}$$

Then the terms of a solution of (3) *alternate between* 0 *and $c^n \delta$. In particular, if $c > 1$, then solutions satisfying* (5) *are unbounded, and if $c = 1$ then such solutions are periodic.*

Proof. If $x_0 = \delta, x_{-1} = 0$ then from (3) it follows that $x_{2n-1} = 0$ for all $n \geq 1$. For the even terms, we have

$$x_{2n} = \frac{c x_{2n-2}}{x_{2n-1} x_{2n-2} + 1} = c x_{2n-2} = c^2 x_{2n-4} = \cdots = c^n x_0 = c^n \delta,$$

as desired. The case $x_0 = 0$ which results in the even-indexed terms being zeros is proved similarly.

A key feature of (1) is that off the coordinate axes it is reducible to a linear nonhomogeneous equation. If $x_0 x_{-1} \neq 0$ and we multiply both sides of (1) by x_n and substitute

$$\frac{1}{t_n} = x_n x_{n-1}, \tag{6}$$

in the result, then we obtain the first-order, linear nonhomogeneous equation

$$t_{n+1} = \frac{1}{c} t_n + \frac{1}{c}, \quad t_0 = \frac{1}{x_0 x_{-1}}. \tag{7}$$

The explicit solution of (7) is obtained by a straightforward induction argument as

$$t_n = \begin{cases} \frac{1+\beta c^{-n}}{c-1}, & \beta \doteq \frac{c-1}{x_0 x_{-1}} - 1, \quad \text{if } c \neq 1 \\ n+\alpha, & \alpha \doteq \frac{1}{x_0 x_{-1}}, \quad \text{if } c = 1 \end{cases}, \quad n = 0, 1, 2, \ldots \tag{8}$$

Before discussing the solutions of (3), we determine the subset of $\mathbb{R}^2$ that must be excluded. This 'forbidden set' is in fact the set of all pre-images of the complement of the domain of the function f in (4) under the unfolding of f (i.e. the mapping $(u, v) \rightarrow [f(u, v), u]$ of $\mathbb{R}^2$). Such pre-images make up the backward orbits of points in the complement of the domain under the unfolding of f, so that if the initial values are chosen outside the forbidden set then the solution will never enter that set. The next result shows in particular that for equation (3) the forbidden set is contained entirely in the interiors of the second and fourth quadrants of $\mathbb{R}^2$. □

LEMMA 4. *The forbidden set F_1 of equation* (3) *is a sequence of hyperbolas as follows:*

$$F_1 = \bigcup_{n=0}^{\infty} \{(u, v) : uv = -\mu_n\}, \tag{9}$$

where for $n = 0,1,2,\ldots,$

$$\mu_n = \begin{cases} \frac{(c-1)}{c^{n+1}-1}, & \text{if } c \neq 1 \\ \frac{1}{n+1}, & \text{if } c = 1 \end{cases}$$

Proof. We show that starting from an initial point (y_0, y_{-1}) with $y_0 y_{-1} = -1$, the backward orbits of (3) stay in F_1. This fact is established by first noting that for the backward orbits, the product reciprocals $1/(y_k y_{k-1}) = s_k$ satisfy the equation

$$s_k = g^{-1}(s_{k-1}), \quad g(t) = \frac{1}{c} t + \frac{1}{c}, \quad s_0 = \frac{1}{y_0 y_{-1}} = -1.$$

This is true because product reciprocals satisfy equation (7) for the forward orbits of (3). Next, the inverse of g is easily calculated as $g^{-1}(s) = cs - 1$ and by a straightforward induction

$$\frac{1}{y_k y_{k-1}} = s_k = g^{-k}(s_0) = -\sum_{j=0}^{k} c^j.$$

It follows that the points (y_k, y_{k-1}) are on the hyperbolas (9).

We note that the closure $\bar{F}_1$ consists of F_1 together with the two coordinate axes. Lemma 3 discusses solutions of (3) that start from a point on the coordinate axes. The next result considers the remaining solutions of (3). □

THEOREM 1. *Assume that $(x_0, x_{-1}) \notin \bar{F}_1$ and let $\{x_n\}$ be the solution of* (3) *starting from this initial point.*

(1) *If $c \leq 1$ then $\{x_n\}$ converges to* 0.
(2) *If $c > 1$ then $\{x_n\}$ converges to a cycle $\{\xi_0, \xi_1\}$ of period* 2 (*not necessarily prime*)

that satisfies the relation

$$\xi_0 \xi_1 = c - 1,$$

i.e. period 2 *solutions are located on the hyperbola* $uv = c - 1$ *in* $\mathbb{R}^2$. *Further, if* β *is as in* (8) *and*

$$P_n = \prod_{k=1}^{n} \frac{1 + \beta c^{-2k+1}}{1 + \beta c^{-2k}}, \tag{10}$$

then $P = \lim_{n \to \infty} P_n \in (0, \infty)$. *Furthermore,* $\xi_0 = \xi_1 = \pm\sqrt{c-1} = \bar{x}_\pm$ *if and only if*

$$x_0 P = \bar{x}_\pm, \tag{11}$$

where it is $\bar{x}_+$ *or* $\bar{x}_-$ *depending on whether* $x_0 P > 0$ *or* $x_0 P < 0$*, respectively. Thus, exceptional solutions from initial values satisfying* (11) *converge to* $\bar{x}_\pm$.

Proof. (a) Let $c < 1$ and first assume that $x_0 x_{-1} > 0$ in which case (7) implies that $x_n x_{n-1} = (1/t_n) > 0$ for all n. We may also assume that $x_0, x_{-1} > 0$ so that $x_n > 0$ for all n (the case $x_0, x_{-1} < 0$ is treated in essentially the same way or made redundant by noticing that the purely negative solutions are in one to one correspondence with the purely positive ones and with the same absolute values). From (3), we have

$$x_{n+1} = \frac{c x_{n-1}}{x_n x_{n-1} + 1} < c x_{n-1}.$$

Proceeding as in the proof of Lemma 3, it follows that the even and odd terms satisfy the relations:

$$x_{2n} < c^n x_0, \quad x_{2n-1} < c^n x_{-1}.$$

It follows that $x_n \to 0$ as $n \to \infty$. Next, consider the case $x_0 x_{-1} < 0$. In this case, since the negative fixed point $-1/(1-c)$ of (7) is repelling, every non-constant solution of (7) will approach ∞ or $-\infty$ monotonically. Hence $x_n x_{n-1} = (1/t_n) \to 0$ and there is $k \geq 1$ such that $x_n x_{n-1} > -1$ for all $n > k$. For the constant solution $t_n = -1/(1-c)$ of (7) we have $x_n x_{n-1} = -(1-c)$ so $x_n x_{n-1} > -1$ for all n. Thus $x_n x_{n-1} + 1 > 0$ for all sufficiently large n and for all such n,

$$|x_{n+1}| = \frac{c|x_{n-1}|}{x_n x_{n-1} + 1} < c|x_{n-1}|.$$

Now, the preceding argument can be applied again with minor modifications to complete the proof for the case $c < 1$.

Next, let $c = 1$. In this case, from either (7) or (8) we see that $t_n \to \infty$ as $n \to \infty$ so that there is $k \geq 1$ such that $x_n x_{n-1} = (1/t_n) > -1$ for all $n > k$. Thus as argued in the case $c < 1$, we may assume without loss of generality that $x_0, x_{-1} > 0$. From (6) and (8), we infer that $\{x_n\}$ satisfies the first-order rational equation

$$x_n = \frac{1/t_n}{x_{n-1}} = \frac{1}{(n+\alpha)x_{n-1}}, \quad \alpha = \frac{1}{x_0\, x_{-1}} > 0. \tag{12}$$

By Lemma 2(b), the even and odd terms of the sequence generated by the iteration of (12) may be written as

$$x_{2n} = x_0 \frac{(1+\alpha)(3+\alpha)\cdots(2n-1+\alpha)}{(2+\alpha)(4+\alpha)\cdots(2n+\alpha)}, \quad x_{2n+1} = \theta_0 \frac{(2+\alpha)(4+\alpha)\cdots(2n+\alpha)}{(3+\alpha)\cdots(2n+1+\alpha)},$$

where $\theta_0 = 1/[x_0(1+\alpha)]$. To show that $x_{2n} \to 0$ it suffices to show that

$$Q_k = \left(\frac{1+\alpha}{2+\alpha}\right)\left(\frac{3+\alpha}{4+\alpha}\right)\cdots\left(\frac{2k-1+\alpha}{2k+\alpha}\right) \to 0 \quad \text{as } k \to \infty. \tag{13}$$

A similar argument holds for $x_{2n+1} \to 0$. Equation (13) is true if $\ln Q_k \to -\infty$, or equivalently if $\ln(1/Q_k) \to \infty$; i.e.

$$\sum_{j=1}^{\infty} \ln\left(\frac{2j+\alpha}{2j-1+\alpha}\right) = \infty. \tag{14}$$

Equality (14) is true by a comparison test given that $\sum_{j=1}^{\infty} 1/(2j+\alpha) = \infty$ and

$$\lim_{j\to\infty} \frac{\ln[(2j+\alpha)/(2j-1+\alpha)]}{1/(2j+\alpha)} = 1.$$

(b) First, from either (7) or (8) we see that $t_n \to 1/(c-1) > 0$ as $n \to \infty$. Hence $x_n x_{n-1} = (1/t_n) > 0$ for all sufficiently large n and as in the case $c = 1$ we may assume without loss of generality that $x_0, x_{-1} > 0$. This implies that $\beta \in (-1, \infty)$ in (8) with $\beta \neq 0$ for non-constant solutions. By Lemma 2(b), (8) and (10),

$$x_{2n} = x_0 \prod_{k=1}^{n} \left(\frac{1/t_{2k}}{1/t_{2k-1}}\right) = x_0 \prod_{k=1}^{n} \frac{1+\beta c^{-2k+1}}{1+\beta c^{-2k}} = x_0 P_n.$$

Clearly, x_{2n} converges to a finite, positive limit if P_n does. To prove the convergence of P_n, it is sufficient to show that

$$\sum_{k=1}^{\infty} \left| \ln\left(\frac{1+\beta c^{-2k+1}}{1+\beta c^{-2k}}\right) \right| < \infty. \tag{15}$$

Since the logarithms in the above sum are negative if and only if $\beta < 0$, (15) is proved by considering two cases and using the same comparison test in each case. For $\beta < 0$, the series in (15) is finite because the series $\sum_{k=1}^{\infty} \beta c^{-2k}$ converges for $c > 1$ and

$$\lim_{k\to\infty} \frac{\ln[(1+\beta c^{-2k+1})/(1+\beta c^{-2k})]}{\beta c^{-2k}} = c - 1.$$

For $\beta < 0$, the same argument using the convergent series $\sum_{k=1}^{\infty} (-\beta) c^{-2k}$ establishes the finiteness of the series in (15). Also, a similar argument establishes that x_{2n+1}

converges to a finite positive limit. Thus, there are $\xi_0, \xi_1 > 0$ such that

$$\lim_{n\to\infty} x_{2n} = \xi_0, \quad \lim_{n\to\infty} x_{2n+1} = \xi_1.$$

Further, the values ξ_0, ξ_1 are related as follows:

$$\xi_0 \xi_1 = \lim_{n\to\infty} x_{2n+1} x_{2n} = \lim_{n\to\infty} \frac{1}{t_{2n+1}} = c - 1.$$

From the preceding argument, we also infer that P_n converges to a positive finite limit P and

$$x_{2n} = x_0 P_n, \quad x_{2n+1} = \frac{1}{x_0 P_n t_{2n+1}}.$$

Taking limits as $n \to \infty$ and setting $\xi_0 = \xi_1$ completes the proof. □

Remarks.

(1) It may be worth mentioning at this stage that for $c > 1$, the hyperbola $uv = c - 1$ is just the invariant level set containing the fixed points $\bar{x}_\pm$ of equation (7) with respect to the semiconjugate link (6). Theorem 1 states in particular that this invariant set is attracting, and further, the solutions of (3) within the invariant set itself are 2-periodic (except for the constant solutions $\bar{x}_\pm$). For general remarks about semiconjugate geometry see Ref. [9].

(2) To gain a better understanding of the effects of the quadratic term in (3), at least for the non-negative solutions, let us consider what happens when the term $x_n x_{n-1}$ in the denominator is made linear by dropping either x_n or x_{n-1}. The resulting equations are still second-order and of the variety discussed in Ref. [8]:

$$\text{(a)}\ \ x_{n+1} = \frac{c x_{n-1}}{x_n + 1}, \quad \text{(b)}\ \ x_{n+1} = \frac{c x_{n-1}}{x_{n-1} + 1}. \tag{16}$$

The next result about equations (16) suggests that the occurrence of x_n in equation (3) promotes oscillatory and unbounded behaviour in that equation whereas by contrast the occurrence of x_{n-1} in the denominator of (3) tends to curb unboundedness and oscillations. The proofs of some of the statements below are straightforward; for proofs of the rest and some related results, see Refs. [1,8].

Proposition.

(1) *If* $c < 1$ *then the origin is a stable global attractor of all non-negative solutions of each of the equations in* (16).

(2) *If* $c = 1$ *then all non-negative solutions of* (16)(*a*) *satisfying* (5) *are periodic with either even or odd terms* 0, *whereas all positive solutions* (*that are not strictly decreasing*) *converge to a periodic solution with alternating* 0 *and positive terms. For Equation* (16)(*b*), *all non-negative solutions converge to* 0, *whether or not they satisfy* (5).

(3) *If* $c > 1$ *then both equations in* (16) *have the same positive fixed point* $\bar{x} = c - 1$. *Locally, for* (16)(*a*) $\bar{x}$ *is a saddle point and for* (16)(*b*) $\bar{x}$ *is a stable node.*

(4) *If* $c > 1$ *then all positive solutions of* (16)(*b*) *converge to* $\bar{x}$, *whereas all solutions*

of (16)(*a*) *that do not start from an initial point on the stable manifold of* $\bar{x}$ *are unbounded and oscillatory, with the subsequences of even terms and odd terms one converging to* 0 *and the other to* ∞.

(5) *If* $c > 1$ *then all non-negative solutions of* (16)(*a*) *satisfying* (5) *have their subsequences of even terms and odd terms one being* 0 *and the other converging to* ∞. *The non-negative solutions of* (16)(*b*) *satisfying* (5) *have their subsequences of even terms and odd terms one being* 0 *and the other converging to* $\bar{x}$. *In particular,* (16)(*b*) *has no unbounded, non-negative solutions.*

3. The third-order equation

We now consider the solutions of the third-order equation (2). This equation is homogeneous of degree 1 with respect to the multiplicative group of nonzero real numbers so we can use the idea in Ref. [10]. Dividing both sides of (2) by x_n and rearranging terms gives

$$\frac{x_{n+1}}{x_n} = \frac{ax_{n-1}}{x_n + bx_{n-2}} = \frac{ax_{n-1}/x_{n-2}}{(x_n/x_{n-1})(x_{n-1}/x_{n-2}) + b}.$$

Now substituting $r_n = x_n/x_{n-1}$ in the above gives

$$r_{n+1} = \frac{ar_{n-1}}{r_n r_{n-1} + b}, \quad x_n = r_n x_{n-1}. \tag{17}$$

We recognise the first of the above equations as equation (1); thus the ratios of consecutive terms of (2) satisfy (1). The second equation in (17), which is linear of order 1, has already been discussed in Lemma 2(a). The initial values are determined by those of equation (2) as

$$r_0 = \frac{x_0}{x_{-1}}, \quad r_{-1} = \frac{x_{-1}}{x_{-2}} \quad \text{if } x_{-1}x_{-2} \neq 0.$$

Equation (2) has no isolated fixed points and in particular, its domain does not include the origin. For intial points $(x_0, x_{-1}, x_{-2}) \in \mathbb{R}^3$, (2) is in fact undefined on the plane $x_0 = -bx_{-2}$. In what follows, it is convenient to state the analogs of (7) and (8) for equation (1):

$$t_{n+1} = \frac{b}{a}t_n + \frac{1}{a}, \quad t_0 = \frac{1}{r_0 r_{-1}} = \frac{x_{-2}}{x_0}. \tag{18}$$

The explicit solution of (18) is obtained by a straightforward induction argument as

$$t_n = \begin{cases} \theta\left(\frac{b}{a}\right)^n + \frac{1}{a-b}, \quad \theta \doteq \frac{x_{-2}}{x_0} - \frac{1}{a-b}, & \text{if } a \neq b \\ \frac{n}{b} + \alpha, \quad \alpha \doteq \frac{x_{-2}}{x_0}, & \text{if } a = b \end{cases}, \quad n = 0, 1, 2, \ldots \tag{19}$$

Lemma 5. *The forbidden set* F_2 *of* (2) *is a sequence of planes containing the origin in* $\mathbb{R}^3$ *as follows:*

$$F_2 = \bigcup_{n=0}^{\infty} \{(u, v, w) : u = -\gamma_n w\} \cup \{(u, v, w) : u = 0\} \cup \{(u, v, w) : v = 0\}, \tag{20}$$

where for $n = 0, 1, 2, \ldots,$

$$\gamma_n = \begin{cases} \frac{(a-b)}{(a/b)^{n+1}-1}, & \text{if } a \neq b \\ \frac{b}{n+1}, & \text{if } a = b \end{cases}$$

Proof. We note that $x_{n+1} = 0$ if and only if $x_n x_{n-1} = 0$ and $x_n + bx_{n-2} \neq 0$. Thus as long as the latter inequality is true and $x_n \neq 0$, then $x_{n+1} \neq 0$. Next, suppose that none of the initial values is zero. Then, we see that x_{n+1} cannot be defined if $x_n + bx_{n-2} = 0$. This is true if and only if

$$-b = \frac{x_n}{x_{n-2}} = \frac{x_n}{x_{n-1}} \frac{x_{n-1}}{x_{n-2}} = r_n r_{n-1}. \tag{21}$$

Now, we proceed as in the proof of Lemma 4 but use the inverse of (18) with $s_0 = -1/b$ since the denominator of (1) is zero when $r_0 r_{-1} = -b$. The straightforward calculations show that starting from nonzero inital values, (21) holds if and only if $x_0 = -\gamma_n x_{-2}$ for some n.

Next, suppose that $x_0 x_{-1} \neq 0$. If $x_{-2} = 0$ then form (2) we calculate $x_1 = ax_{-1} \neq 0$ so we may drop x_{-2} and start with nonzero initial values x_{-1}, x_0, x_1 to which the preceding argument applies. So finally suppose that $x_0 x_{-1} = 0$. If $x_{-1} = 0$ then x_2 is undefined, and if $x_0 = 0$ then x_3 is undefined. Thus if the initial point (x_0, x_{-1}, x_{-2}) is in either of the planes $u = 0$ or $v = 0$ then x_n is undefined for some $n \geq 1$. Thus, the set F_2 in (20) is the forbidden set of equation (2).

We note that in contrast to F_1, the set F_2 is closed and includes all three coordinate axes as well as two of the three coordinate planes. Hence we do not expect an analog of Lemma 3 for equation (2). □

THEOREM 2. *Let* $\{x_n\}$ *be a solution of* (2) *with initial point* $(x_0, x_{-1}, x_{-2}) \notin F_2$.

(1) *If* $a < b + 1$ *then* $\lim_{n\to\infty} x_n = 0$.
(2) *If* $a = b + 1$ *then* $\{x_n\}$ *converges to a cycle* $\{\zeta_0, \zeta_1\}$ *of period* 2 (*not necessarily prime*) *where* $\zeta_1 = \xi_1 \zeta_0$ *with* ξ_1 *as in Theorem* 1 *and*

$$\zeta_0 = x_0 \prod_{n=1}^{\infty} \frac{1}{1 + \theta_1 (1 + 1/b)^{-n}}, \quad \theta_1 = \frac{bx_{-1}}{x_1} - 1. \tag{22}$$

In particular, if $\xi_1 = 1$ *then* $\{x_n\}$ *converges to the single number* ζ_0.

(3) *If* $a > b + 1$ *then each of the secquences* $\{x_{2n}\}$ *and* $\{x_{2n+1}\}$ *is unbounded.*

Proof. (a) If $(x_0, x_{-1}, x_{-2}) \notin F_2$ then we may assume without loss of generality that $x_{-2} \neq 0$. From the second equation in (17) and Lemma 2(a) we have

$$x_n = x_0 r_1 r_2 \ldots r_n.$$

If $a \le b$ then by Theorem 1, $r_n \to 0$ as $n \to \infty$; therefore, $\lim_{n\to\infty} x_n = 0$. Now, suppose that $b < a < b + 1$. In this case, we can write

$$x_{2n} = x_0(r_1 r_2)(r_3 r_4) \cdots (r_{2n-1} r_{2n}) = \frac{x_0}{t_2 t_3 \cdots t_n}, \tag{23}$$

$$x_{2n+1} = r_{2n+1} x_{2n} = \frac{x_0}{t_2 t_3 \ldots t_n t_{2n+1} r_{2n}}. \tag{24}$$

From (19), it follows that $t_n \to 1/(a - b) > 1$ so Theorem 1 implies that $r_{2n} \to \xi_0 \neq 0$. Therefore, both x_{2n} and x_{2n+1} approach 0 as $n \to \infty$. This conlcudes the proof of (a).

(b) If $a - b = 1$ then from (19) and (23) we obtain

$$x_{2n} = x_0 \prod_{k=1}^{n} \frac{1}{1 + \theta_1 [b/(b+1)]^k} = x_0 \prod_{k=1}^{n} \frac{1}{1 + \theta_1 (1 + 1/b)^{-k}},$$

where θ_1 is as in (22). Since $1 + \theta_1(1 + (1/b))^{-k} > 0$ for all large k, the convergence of the product sequence is established by showing that

$$\sum_{n=1}^{\infty} |\ln(1 + B^{-k}A)| < \infty, \quad |A| < 1, \ B > 1. \tag{25}$$

Since (25) is trivially true for $A = 0$ and since for $A \neq 0$ we have

$$\lim_{k\to\infty} \frac{|\ln(1 + B^{-k}A)|}{B^{-k}|A|} = 1,$$

comparison with the convergent series $\sum_{n=1}^{\infty} B^{-k}|A|$ shows that (25) is true. Thus $x_{2n} \to \zeta_0$ where ζ_0 is as in (22). Further, since by Theorem 1, $\lim_{n\to\infty} r_{2n+1} = \xi_1$ it follows from (24) that $x_{2n+1} \to \xi_1 \zeta_0$ as $n \to \infty$.

(c) If $a - b > 1$ then from (19) it follows that $1/t_n \to a - b > 1$ as $n \to \infty$. Hence (23) and (24) imply that both of the even and odd indexed sequences x_{2n} and x_{2n+1} are unbounded. □

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