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J. Math. Anal. Appl. 296 (2004) 538–540
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

The Li–Yorke theorem and infinite discontinuities

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Received 19 February 2004
Available online 8 July 2004
Submitted by G. Ladas

Abstract

An unbounded mapping of the half-line which has no 3-cycles is continuously extended to the one-point compactification of the set of nonnegative real numbers. The extension is shown to have a 3-cycle so that the Li–Yorke theorem may be applied to the extension to efficiently obtain a scrambled set that is then relativized to the original domain. More generally, this approach may invoke other known chaos theorems and can be applied to various types of mappings with infinite discontinuities.

In [3] the continuous, piecewise smooth mapping

$$\phi(r) = \left|1 - \frac{1}{r}\right|, \quad r > 0,$$

was used to study various properties of the solutions of the second-order equation in the title. The dynamical properties of ϕ govern the dynamics of the sequence of ratios of consecutive terms x_n/x_{n-1} for each solution $\{x_n\}$ of the second-order equation. Of particular interest here is Theorem 3 of [3] where the unique fixed point of ϕ, namely, $\bar{r} = (\sqrt{5} - 1)/2$, was shown to be a snap-back repeller (in the general nonsmooth sense) and Marotto's theorem from [2] was then used to establish the chaotic nature of ϕ on the set of positive irrationals which is invariant under ϕ. In this note, we consider a simpler approach using a continuous extension of ϕ that uses the Li–Yorke theorem in [1].

This is an indirect application of the Li–Yorke theorem because ϕ itself does not have a period-3 point. Interestingly, ϕ does have period-p points for all positive integers $p \neq 3$ and in [3] these points were explicitly determined using the Fibonacci numbers. Our approach

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2004.04.016

here not only simplifies the proof of Theorem 3 in [3] considerably and avoids technical difficulties associated with a direct application of Marotto's theorem, but in a sense it also completes the list of periodic solutions for ϕ by adding a "3-cycle that passes through ∞."

For reference, we say that a mapping f of the real line is *chaotic in the sense of* [1] if it has a *scrambled set*; i.e., a set S with the following properties:

(i) S is uncountable, contains no periodic points of f and $f(S) \subset S$.

(ii) For every $x, y \in S$ and $x \neq y$,

$$\limsup_{k\to\infty} \| f^k(x) - f^k(y) \| > 0, \qquad \liminf_{k\to\infty} \| f^k(x) - f^k(y) \| = 0.$$

(iii) For every $x \in S$ and periodic y,

$$\limsup_{k\to\infty} \| f^k(x) - f^k(y) \| > 0.$$

Here now is the aforementioned theorem from [3] and its shorter proof.

Theorem. *The mapping ϕ is chaotic in the sense of* [1].

Proof. Let $[0, \infty]$ be the one-point compactification of $[0, \infty)$ and define ϕ^* on $[0, \infty]$ as follows:

$$\phi^*(r) = \phi(r), \quad 0 < r < \infty, \ \phi^*(0) = \infty, \ \phi^*(\infty) = 1.$$

Note that ϕ^* extends ϕ continuously to $[0, \infty]$ and furthermore, ϕ^* has a 3-cycle $\{1, 0, \infty\}$. Since $[0, \infty]$ is homeomorphic to [0,1], by the Li–Yorke Theorem ϕ^* is chaotic on $[0, \infty]$ in the sense of [1]. To show that ϕ is chaotic on $(0, \infty)$, we find a scrambled set for it. Let S^* be a scrambled set for ϕ^* and define

$$S = S^* - \left[\{\infty\} \cup \bigcup_{n=0}^{\infty} \phi^{-n}(0) \right] \subset (0, \infty).$$

Note that S is uncountable because S^* is uncountable and because each inverse image $\phi^{-n}(0)$ is countable for all $n = 0, 1, 2, \ldots$ (easy to see, and in fact it is shown in [3] that $\bigcup_{n=0}^{\infty} \phi^{-n}(0)$ is the set of all nonnegative rational numbers). Further, since $\phi(S) \subset S$ and $\phi|_S = \phi^*|_S$ it follows that S is a scrambled set for ϕ and the proof is complete. □

Remarks. (1) The idea behind the above proof can obviously be extended to more general maps that have infinite discontinuities. In particular, the infinite discontinuity of a map to which the above procedure applies may occur in the interior of an interval. An example is provided by the one-parameter family

$$f_a(r) = \frac{1}{|r|} - a, \quad 0 < a < 2, \ r \in [-a, \infty), \ r \neq 0.$$

Unlike ϕ, we note that the maps f_a are smooth on the domain given above. It can be shown that f_a has a 3-cycle for each $a > 1$ and at $a = 1$ the continuous extension $f_1^* = f^*$ to the one-point compactification $[-a, \infty]$ defined as

$$f^*(r) = f_1(r) \quad \text{for } r \in [-a, \infty),\ r \neq 0,\ f^*(0) = \infty,\ f^*(\infty) = -a = -1$$

has a 3-cycle $\{-1, 0, \infty\}$. Therefore, the above proof can be repeated here for $a \geqslant 1$.

With regard to the existence of full orbits, note that if a is rational then the set of positive irrationals is invariant under f_a. Hence, the singularity at 0 is never visited if the initial value r_0 is irrational; chaotic orbits and scrambled sets then exist within the set of irrational numbers. Note that the scrambled sets are unbounded.

(2) In the absence of 3-cycles for the extended maps, it is possible that an unstable fixed point is a snap-back repeller (in the general sense used in [3] based on a remark in [2]). If so, then a scrambled set exists by the main result of [2]. For example, the unique fixed point of f_a is a snap-back repeller for the extension f_a for $a \geqslant 1/\sqrt{a}$. Finally, it is of interest that the concept of snap-back repellers applies not only to maps of the line that have no periodic points at the top of the Sharkovski ordering, but also to maps of higher dimensional Euclidean spaces with infinite discontinuities. So in principle the procedure outlined in this paper can be extended to these higher dimensional maps.

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