Journal of Difference Equations and Applications

ISSN: 1023-6198 (Print) 1563-5120 (Online) Journal homepage: http://www.tandfonline.com/loi/gdea20

Difference equations by differential equations methods

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JOURNAL OF DIFFERENCE EQUATIONS AND APPLICATIONS, 2016
VOL. 22, NO. 5, 724–726
http://dx.doi.org/10.1080/10236198.2015.1099637

Taylor & Francis
Taylor & Francis Group

BOOK REVIEW

Difference equations by differential equations methods, by Peter E. Hydon, Cambridge, Cambridge University Press, 2014, 206 pp., $65 (hardback), ISBN 9780521878524

From a historical perspective, difference equations have a longer history than differential equations, dating back to the work of Fibonacci if not earlier. But the invention of calculus in the 17th century and its unparalleled success in solving engineering problems over the next three centuries, led to the near universal use of differential equations in modeling scientific phenomena and resulting in the accumulation of a substantial body of research on differential equations and related concepts (function spaces, topology, etc). Until the appearance of digital computers there was little incentive outside of pure mathematics to study difference equations. These devices and the discrete methods needed to utilize them have transformed science and engineering but have not yet resulted in scientific modeling with difference equations on a scale that is comparable to differential equations.

In addition to the inertia gained by centuries-long research in differential equations, the continued widespread modeling of scientific phenomena in terms of differential equations and systems today leads to the growth of this field at a substantially higher rate than difference equations. It is sensible then to use available methods for differential equations to gain a better understanding of difference equations.

Linear stability is perhaps the most familiar method that applies equally well to both difference and differential equations. Linear stability is algebraic in nature since the behaviors of solutions (locally) are determined by the eigenvalues and eigenvectors of the Jacobian or derivative matrix. Though less familiar and containing some analysis content, the methods of algebraic geometry involving Lie groups and algebras also apply to both differential and difference equations. The book by Hydon is in this category.

This book has six chapters that are evenly divided between ordinary and partial difference equations (abbreviated OΔE and PΔE, respectively in analogy with ODE and PDE). After discussing linear difference equations in Chapter 1, an introduction to Lie symmetry for OΔE is given in Chapter 2. In Section 2.2 symmetries of geometrical objects and the groups that they generate are distinguished from Lie symmetries of differential and difference equations. An important distinction between Lie symmetries and "discrete symmetries" is that the latter do not involve infinitesimals.

Sections 2.3 and 2.4 discuss how to find the characteristic and canonical coordinates for certain OΔE of order 1 or 2 and use this information to find the solutions of these types of difference equations. Section 2.5 gives a discussion of how Lie symmetries can lead to the reduction of order of a difference equation and Sections 2.6–2.8 discuss first integrals, a concept inherited from differential equations but fundamental to the theory here.

Chapter 3 offers extensions of the basic methods introduced in Chapter 2, including systems of OΔE, multiple or repeated reductions of order of an OΔE, the commutator and variational OΔEs and first integrals. Chapter 4 lays the ground work for PΔEs; the concept of lattice $\mathbb{Z}^N$ as the "domain" of a PΔE ($N = 1$ is OΔE) and fibers $\mathbb{R}^K$ and "total space" $\mathbb{Z}^N \times \mathbb{R}^K$ and lattice transformations (a restricted meaning of this term) and related concepts such as order, initial value problems and lattice symmetries are defined in this chapter.

Chapter 5 discusses some solutions methods for PΔE, including factorizability (Section 5.2) which is analogous to the OΔE case, separation of variables and wave like solutions (Sections 5.3

and 5.4) that are inherited from differential equations as well as finding and using symmetries to solve PΔE (Sections 5.5 and 5.6). Chapter 6 on "conservation laws" has the flavor of theoretical physics, which was the early motivator for Lie symmetry methods for differential equations about a century ago. Two Theorems of Noether for PΔEs are discussed in Chapter 6.

Overall, the writing style of this book is intuitive and informal. Definitions of even fundamental ideas are not explicitly presented and unless one reads the book cover to cover in linear fashion it is not easy to spot the definitions quickly. Fortunately, the main terms are usually highlighted in italic type so with some going back and forth it is possible to find the definitions of various items. A large number of examples and a sprinkling of well-constructed diagrams help improve the reading experience. The book is suitable for students at the graduate level who may find the exercises helpful in consolidating their understanding of topics. Also researchers in differential equations who may be curious about the discrete analogs of familiar topics will find the book accessible. Researchers in difference equations who may be interested in learning about the Lie symmetry method should be prepared for unfamiliar notation and arrangement of topics (e.g. the definition of *autonomous* equation appears off-handedly in an example on page 128 and given for PΔEs only).

The range of topics covered is quite wide for a small book of about 200 pages. A presumably unintended consequence is the large amount of concepts and notation that need to be introduced and absorbed. Thus having a solid background in both ODE and PDE is helpful, and a background in the Lie symmetry method for the continuous case is highly recommended for mastering the material in this book. Such a background also helps motivate the reader better since no specific scientific models are discussed explicitly here that substantially benefit from using the methods discussed.

The absence of applications in this book is not necessarily an indication of the uselessness of the Lie symmetries in the discrete case but the reader is left wondering. Many existing publications on difference equations provide some indication of scientific applicability (and thus added motivation) by discussing applications of their methods to scientific models. See, for example, [1] for methods applicable to exploring chaos, [2,3] for semicycles or [4,7] for monotone systems.

Closer to this book, at least in spirit, is [6] in which the *form symmetries* of difference equations (rather than their solution spaces) are studied. Lie symmetries and form symmetries both apply to all linear difference equations and systems (non-autonomous) and each method independently applies to large classes of nonlinear difference equations and systems. As far as the discrete case is concerned, form symmetries do not require an underlying continuum and are thus meaningful not only in algebraic and analytical settings but also in the context of discrete spaces in computer science (e.g. cellular automata) and increasingly, in other areas as well [5].

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http://dx.doi.org/10.1080/10236198.2015.1099637