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# Semiconjugate factorizations of higher order linear difference equations in rings

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# Semiconjugate factorizations of higher order linear difference equations in rings

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We use a new nonlinear method to study linear difference equations with variable coefficients in a non-trivial ring $R$. If the homogeneous part of the linear equation has a solution in the unit group of a ring with identity (a unitary solution), then we show that the equation decomposes into two linear equations of lower orders. This decomposition, known as a semiconjugate factorization in the nonlinear theory, is based on sequences of ratios of consecutive terms of a unitary solution. Such sequences, which may be called eigensequences, are well suited to variable coefficients; for instance, they provide a natural context for the expression of the Poincaré–Perron theorem. As applications, we obtain new results for linear difference equations with periodic coefficients and for linear recurrences in rings of functions (e.g. the recurrence for the modified Bessel functions).

**Keywords:** linear; semiconjugate factorization; eigensequence; unitary solution; rings of functions; periodic coefficients

**AMS Subject Classification**: 39A06; 39A10; 39B52

## 1. Introduction

A linear, non-homogeneous difference equation with variable coefficients is defined as

$$x_{n+1} = a_{0,n}x_n + a_{1,n}x_{n-1} + \cdots + a_{k,n}x_{n-k} + b_n, \tag{1}$$

where $\{b_n\}$ and $\{a_{j,n}\}$ are given sequences in a non-trivial ring $R$ for $j = 0, 1, \ldots, k$.

Equation (1) may be unfolded in the standard way to a map of a $(k+1)$-dimensional $R$-module (or vector space, if $R$ is a field). In this form there is extensive published literature extending the classical theory on the real line to a diverse selection of rings that include finite rings, rings of polynomials and rings of functions; see, e.g. [1–3,6,9,10,18].

We pursue a different approach in this paper, using a method originally developed for nonlinear difference equations. Specifically, we explore the existence of a semiconjugate factorization (SC-factorization) of (1) in its underlying ring $R$ into two or more linear difference equations of lower orders. While a linear equation may have many different SC-factorizations, the equations with lower orders are generally nonlinear. In this study we obtain a SC-factorization where the lower order equations are also linear. The key requirement for the existence of such a SC-factorization is the existence of a special type of sequence that we call an *eigensequence* of (1); eigenvalues are essentially constant eigensequences.

We show that in a ring $R$ with identity, quotients of the consecutive terms of a unitary solution (invertible terms) form an eigensequence in $R$. In particular, this leads to a simple

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and natural expression for the classical *Poincaré–Perron theorem* in this context; see Section 4.6. SC-factorization generalizes the classical notion of operator factorization of a homogeneous equation with constant coefficients over the real or complex numbers. SC-factorizations of linear equations over arbitrary non-trivial fields are studied in chapter 7 of [15]. The results in this study substantially extend those in [15] to rings and add several new results, including results for equations with periodic coefficients and for recurrences in rings of functions; see Sections 5 and 6.

In particular, rings of functions on a given set are typically not fields but the SC-factorization method applies to linear difference equations in such rings in essentially the same way (finding an eigensequence) as it does to linear difference equations in a field. We study SC-factorizations of linear difference equations in rings of functions and apply the results to obtain explicit formulas for the solutions of some known higher order functional difference equations; e.g. second-order recurrences that define modified Bessel functions, but with arbitrary initial functions.

## 2. Preliminaries

A (forward) *solution* of (1) is defined, as usual, to be any sequence $\{x_n\}$ in $R$ that satisfies the equation for $n \geq 0$. Given the recursive nature of (1) it is clear that with any $k+1$ given initial values $x_j \in R$ for $j = 0, 1, \dots, k$ (the number $k+1$ being the order of the difference equation) (1) generates a unique solution in $R$ through iteration, since $R$ is closed under its addition and multiplication. If $b_n = 0$ for all $n$ then (1) is *homogeneous* and in this case, the constant sequence $x_n = 0$ for $n \geq 0$ is a solution of (1), namely, the trivial solution.

In the classical theory of higher order linear difference equations in the field of real numbers, linear operators such as the those used to define (1) may be 'factored' using their eigenvalues; see, e.g. section 2.3 in [4]. This elementary procedure yields both a reduction of order for a linear difference equation and a symbolic 'operator method' for obtaining its solutions. For discussions of these basic classical notions, including operator methods, see [4] or [8].

In this section, for the reader's convenience we present some general results from [15] that are valid for all difference equations of recursive type, not just the linear equations.

Let $\mathcal{G}$ be a non-trivial group and consider the recursive difference equation, or recurrence

$$x_{n+1} = f_n(x_n, x_{n-1}, \dots, x_{n-k}), \tag{2}$$

where $f_n : \mathcal{G}^{k+1} \to \mathcal{G}$ is a given function for each $n \geq 0$. Starting from a set of $k+1$ initial values $x_j \in \mathcal{G}$ for $j = 0, 1, \dots, k$ a unique solution of (2) is obtained by iteration.

Equation (2) may be unfolded in the usual way to a first-order recurrence

$$X_{n+1} = \mathfrak{F}_n(X_n)$$

on $\mathcal{G}^{k+1}$, where $\mathfrak{F}_n : \mathcal{G}^{k+1} \to \mathcal{G}^{k+1}$. Let $k \geq 1$, $1 \leq m \leq k$. Suppose that there is a sequence of maps $\Phi_n : \mathcal{G}^m \to \mathcal{G}^m$ and a sequence of surjective maps $H_n : \mathcal{G}^{k+1} \to \mathcal{G}^m$ that satisfy the SC *relation*

$$H_{n+1} \circ \mathfrak{F}_n = \Phi_n \circ H_n \tag{3}$$

for a given pair of function sequences $\{\mathfrak{F}_n\}$ and $\{\Phi_n\}$. Then we say that $\mathfrak{F}_n$ is SC to $\Phi_n$ for each $n$ and that the sequence $\{H_n\}$ is a *form symmetry* of (2). Since $m < k+1$, the form symmetry $\{H_n\}$ is *order-reducing*.

We state the next core result from [15] as a lemma here.

LEMMA 1. (SC-FACTORIZATION). *Let $\mathcal{G}$ be a non-trivial group and let $k \geq 1$, $1 \leq m \leq k$ be integers. If $h_n : \mathcal{G}^{k-m+1} \to \mathcal{G}$ is a sequence of functions and the functions $H_n : \mathcal{G}^{k+1} \to \mathcal{G}^m$ are defined by*

$$H_n(u_0, \ldots, u_k) = [u_0 * h_n(u_1, \ldots, u_{k+1-m}), \ldots, u_{m-1} * h_{n-m+1}(u_m, \ldots, u_k)]$$

*where $*$ denotes the group operation in $\mathcal{G}$, then the following statements are true:*

(a) *The function $H_n$ is surjective for every $n \geq 0$.*

(b) *If $\{H_n\}$ is an order-reducing form symmetry then the difference equation (2) is equivalent to the system of equations*

$$t_{n+1} = \phi_n(t_n, \ldots, t_{n-m+1}), \tag{4}$$

$$x_{n+1} = t_{n+1} * h_{n+1}(x_n, \ldots, x_{n-k+m})^{-1} \tag{5}$$

*whose orders $m$ and $k+1-m$, respectively, add up to the order of (2).*

(c) *The map $\Phi_n$ in (3) is the standard unfolding of equation (4) to $\mathcal{G}^m$ for each $n \geq 0$.*

*Remark* 2. Part (c) above permits us to stay within the context of higher order difference equations. Being able to work within this context is especially beneficial in the case of *non-recursive* equations (including some linear equations) which do not in general unfold to maps on modules over rings (or vector spaces over fields) and therefore, their solutions are not determined via group actions. Extensions of the method of this study to non-recursive equations no longer rely on semiconjugacy but they do retain the basic concepts of form symmetry and factor-cofactor pairs; see [15], chapter 8.

DEFINITION 3. *The pair of equations (4) and (5) constitutes the* SC-factorization *of* (2). *This pair of equations is a triangular system* (*see* [19]) *since* (4) *is independent of* (5). *We call* (4) *the* factor *equation of* (2) *and* (5) *its* cofactor *equation.*

Note that (4) has order $m$ and (5) has order $k+1-m$. Consider the following special case of $H_n$ in Lemma 1 with $m = k$

$$H_n(u_0, u_1, \ldots, u_k) = [u_0 * h_n(u_1), u_1 * h_{n-1}(u_2), \ldots, u_{k-1} * h_{n-k+1}(u_k)], \tag{6}$$

where $h_n : \mathcal{G} \to \mathcal{G}$ is a given sequence of maps. The SC-factorization of (2) in this case is

$$t_{n+1} = \phi_n(t_n, \ldots, t_{n-k+1}), \tag{7}$$

$$x_{n+1} = t_{n+1} * h_{n+1}(x_n)^{-1} \tag{8}$$

in which the factor equation has order $k$ and the cofactor equation has order 1.

The next result gives a necessary and sufficient condition for the existence of a form symmetry of type (6); see [15] for the proof.

Lemma 4 (Invertible-map criterion). *Let $\mathcal{G}$ be a non-trivial group and assume that $h_n : \mathcal{G} \to \mathcal{G}$ is a sequence of bijections. For arbitrary elements $u_0, v_1, \ldots, v_k \in \mathcal{G}$ and every $n \geq 0$ define $\zeta_{0,n}(u_0) \equiv u_0$ and for $j = 1, \ldots, k$ define*

$$\zeta_{j,n}(u_0, v_1, \ldots, v_j) = h_{n-j+1}^{-1}(\zeta_{j-1,n}(u_0, v_1, \ldots, v_{j-1})^{-1} * v_j). \tag{9}$$

*Then (2) has the form symmetry (6) if and only if the quantity*

$$f_n(\zeta_{0,n}, \zeta_{1,n}(u_0, v_1), \ldots, \zeta_{k,n}(u_0, v_1, \ldots, v_k)) * h_{n+1}(u_0) \tag{10}$$

*is independent of $u_0$ for every $n \geq 0$. In this case (2) has a SC-factorization into (7) and (8) where the factor functions in (7) are given by*

$$\phi_n(v_1, \ldots, v_k) = f_n(\zeta_{0,n}, \zeta_{1,n}(u_0, v_1), \ldots, \zeta_{k,n}(u_0, v_1, \ldots, v_k)) * h_{n+1}(u_0). \tag{11}$$

In the context of rings, the group $\mathcal{G}$ in the preceding result is the additive group of the ring so that $*$ denotes addition and thus (9), (10) and (8) read, respectively, as follows:

$$\zeta_{j,n}(u_0, v_1, \ldots, v_j) = h_{n-j+1}^{-1}(v_j - \zeta_{j-1,n}(u_0, v_1, \ldots, v_{j-1})),$$
$$f_n(\zeta_{0,n}, \zeta_{1,n}(u_0, v_1), \ldots, \zeta_{k,n}(u_0, v_1, \ldots, v_k)) + h_{n+1}(u_0) \quad \text{and} \quad x_{n+1} = t_{n+1} - h_{n+1}(x_n).$$

A basic class of maps $h_n$ in rings is defined next.

Definition 5. *Let $R$ be a ring and let $\{\alpha_n\}$ be a sequence in $R$ such that $\alpha_n \neq 0$ for all $n$. A* linear form symmetry *is defined as the special case of (6) with $h_n(u) = -\alpha_n u$; i.e.*

$$[u_0 - \alpha_n u_1, u_1 - \alpha_{n-1} u_2, \ldots, u_{k-1} - \alpha_{n-k+1} u_k]. \tag{12}$$

If $\alpha$ is *not* a zero divisor then $h(u) = -\alpha u$ is one-to-one or injective since for every $u, v \in R$

$$h(u) = h(v) \Rightarrow -\alpha(u - v) = 0 \Rightarrow u - v = 0.$$

In general, $h$ is not surjective even if $R$ contains no zero divisors (consider $\alpha \in \mathbb{Z}$, $\alpha \neq \pm 1$). But if $R$ has an identity and each $\alpha$ is a unit then each $h$ is a bijection with inverse $h^{-1}(u) = -\alpha^{-1} u$.

*Remark* 6. Many nonlinear difference equations possess the linear form symmetry (12); see [15,16]. The SC-factorizations of such nonlinear equations always have a linear cofactor.

## 3. SC-factorization in rings

Assume that the underlying ring $R$ of (1) has a (multiplicative) identity denoted by 1. For such a ring, the set of all units (elements having multiplicative inverses or reciprocals) is a group, namely the *unit group*, that we denote by $G$.

### *3.1. Unitary sequences*

A *unitary sequence* is any sequence in $G$. If $\{u_n\}$ is a unitary sequence then the sequence $\{u_{n+1}u_n^{-1}\}$ of *right ratios* of $\{u_n\}$ is well defined and unitary. Similarly, the sequence $\{u_n^{-1}u_{n+1}\}$ of *left ratios* is well defined and unitary. If $R$ is commutative then the sequences $\{u_{n+1}u_n^{-1}\}$ and $\{u_n^{-1}u_{n+1}\}$ are the same, representing the *ratios sequence* of $\{u_n\}$.

Call two sequences $\{x_n\}$ and $\{y_n\}$ in $R$ *right equivalent* (or *left equivalent*) if there is a unit $u$ such that $y_n = x_n u$ (or $y_n = ux_n$) for all $n$. These two relations on the set of sequences in $R$ are indeed equivalence relations, and if $\{x_n\}$ is unitary then so are $\{x_n u\}$ and $\{ux_n\}$.

The next result has the same flavour as the result in calculus which states that differentiable functions having the same derivative are equal up to a constant.

Lemma 7. *Let $R$ have an identity and $\{x_n\}$ and $\{y_n\}$ be unitary sequences. Then $\{x_n\}$ and $\{y_n\}$ are right (or left) equivalent if and only if their sequences of right (or left) ratios are equal.*

*Proof*. Suppose that $\{x_n\}$ and $\{y_n\}$ are right equivalent. Then $y_n = x_n u$ for some unit $u$ and all $n$ so that

$$y_{n+1}y_n^{-1} = x_{n+1}u(x_n u)^{-1} = x_{n+1}u\left(u^{-1}x_n^{-1}\right) = x_{n+1}x_n^{-1},$$

i.e. the sequences of right ratios are the same. Conversely, suppose that $y_{n+1}y_n^{-1} = x_{n+1}x_n^{-1}$ for all $n \geq 0$ and define $u = x_0^{-1}y_0$. Then

$$x_1 u = x_1 x_0^{-1} y_0 = y_1 y_0^{-1} y_0 = y_1.$$

This equality also implies that $u = x_1^{-1}y_1$ so the preceding argument may be repeated to show that $x_n u = y_n$ for all $n \geq 0$. A similar argument proves the left-handed case. □

### *3.2. The SC-factorization theorem*

Recall that the existence of a linear form symmetry (12) implies that (2) has a SC-factorization with a first-order, linear non-homogeneous cofactor equation

$$x_{n+1} = t_{n+1} + \alpha_{n+1}x_n. \tag{13}$$

The following necessary and sufficient condition for the existence of a linear form symmetry is a consequence of the invertible-map criterion with $\mathcal{G}$ being the additive group of the ring.

Lemma 8. *Let $R$ be a ring with identity. Equation (2) has the linear form symmetry (12) if and only if there is a unitary sequence $\{\alpha_n\}$ in $R$ such that the quantity*

$$f_n(u_0, \zeta_{1,n}(u_0, v_1), \ldots, \zeta_{k,n}(u_0, v_1, \ldots, v_k)) - \alpha_{n+1}u_0 \tag{14}$$

*is independent of $u_0$ for all n, where for $j = 1, \dots, k$,*

$$\zeta_{j,n}(u_0, v_1, \dots, v_j) = \left(\prod_{i=0}^{j-1} \alpha_{n-i}\right)^{-1} u_0 - \sum_{i=1}^{j} \left(\prod_{m=i}^{j} \alpha_{n-m+1}\right)^{-1} v_i. \tag{15}$$

*Proof*. Define $h_n(u) = -\alpha_n u$ for each $n$ so that $h_n^{-1}(u) = -\alpha_n^{-1} u$ for all $n$. If we define $\zeta_{j,0} = u_0$ and for $j = 1, \dots, k$ set

$$\zeta_{j,n}(u_0, v_1, \dots, v_j) = \alpha_{n-j+1}^{-1}[\zeta_{j-1,n}(u_0, v_1, \dots, v_{j-1}) - v_j]$$

recursively, then the first assertion of the lemma is true by the invertible-map criterion. To prove (15), observe that

$$\zeta_{1,n}(u_0, v_1) = \alpha_n^{-1}(u_0 - v_1) = \alpha_n^{-1} u_0 - \alpha_n^{-1} v_1$$

which proves (15) if $j = 1$. Suppose that (11) is true for indices less than $j$ where $j \leq k$. Then

$$\begin{aligned}
\zeta_{j,n}(u_0, v_1, \dots, v_j) &= \alpha_{n-j+1}^{-1}[\zeta_{j-1,n}(u_0, v_1, \dots, v_{j-1}) - v_j] \\
&= \alpha_{n-j+1}^{-1}\left[\left(\prod_{i=0}^{j-2} \alpha_{n-i}\right)^{-1} u_0 - \sum_{i=1}^{j-1} \left(\prod_{m=i}^{j-1} \alpha_{n-m+1}\right)^{-1} v_i - v_j\right] \\
&= \left(\prod_{i=0}^{j-1} \alpha_{n-i}\right)^{-1} u_0 - \sum_{i=1}^{j-1} \left(\prod_{m=i}^{j} \alpha_{n-m+1}\right)^{-1} v_i - \alpha_{n-j+1}^{-1} v_j \\
&= \left(\prod_{i=0}^{j-1} \alpha_{n-i}\right)^{-1} u_0 - \sum_{i=1}^{j} \left(\prod_{m=i}^{j} \alpha_{n-m+1}\right)^{-1} v_i,
\end{aligned}$$

and the proof is complete. □

THEOREM 9. *Let R be a ring with identity. The linear equation (1) has the linear form symmetry (12) with unit coefficients if there is a unitary sequence $\{\alpha_n\}$ that satisfies the relation*

$$\alpha_{n+1} = a_{0,n} + \sum_{j=1}^{k} a_{j,n} \left(\prod_{i=0}^{j-1} \alpha_{n-i}\right)^{-1}. \tag{16}$$

*The corresponding SC-factorization of (1) is*

$$t_{n+1} = a'_{0,n} t_n + a'_{1,n} t_{n-1} + \dots + a'_{k-1,n} t_{n-k+1} + b_n, \tag{17}$$

$$x_{n+1} = \alpha_{n+1} x_n + t_{n+1}, \tag{18}$$

*where for* $m = 0, \ldots, k-1$, $t_{m+1} = x_{m+1} - \alpha_{m+1}x_m$ *and*

$$a'_{m,n} = -\sum_{i=m+1}^{k} a_{i,n}\left(\prod_{j=m+1}^{i} \alpha_{n-j+1}\right)^{-1}.$$

*Proof.* By Lemma 8 it is only necessary to determine a unitary sequence $\{\alpha_n\}$ in $R$ such that for each $n$ (14) is independent of $u_0$ for the following functions:

$$f_n(u_0, \ldots, u_k) = a_{0,n}u_0 + a_{1,n}u_1 + \cdots + a_{k,n}u_k + b_n.$$

For arbitrary $u_0, v_1, \ldots, v_k \in R$ and $j = 1, \ldots, k$ define $\zeta_{j,n}(u_0, v_1, \ldots, v_j)$ as in Lemma 8. Then expression (14) is

$$-\alpha_{n+1}u_0 + b_n + a_{0,n}u_0 + a_{1,n}\zeta_{1,n}(u_0, v_1) + \cdots + a_{k,n}\zeta_{k,n}(u_0, v_1, \ldots, v_k)$$
$$= b_n + \left[-\alpha_{n+1} + a_{0,n} + \sum_{j=1}^{k} a_{j,n}\left(\prod_{i=0}^{j-1} \alpha_{n-i}\right)^{-1}\right]u_0 - \sum_{j=1}^{k} a_{j,n}\sum_{i=1}^{j}\left(\prod_{m=i}^{j} \alpha_{n-m+1}\right)^{-1} v_i.$$

The above quantity is independent of $u_0$ if and only if the coefficient of $u_0$ is zero for all $n$; i.e. if and only if $\{\alpha_n\}$ satisfies the difference equation (16). Dropping the $u_0$ terms leaves the following:

$$b_n - \sum_{j=1}^{k} a_{j,n}\left[\sum_{i=1}^{j}\left(\prod_{m=i}^{j} \alpha_{n-m+1}\right)^{-1} v_i\right] = b_n - \sum_{j=1}^{k}\left[\sum_{i=j}^{k} a_{i,n}\left(\prod_{m=i}^{j} \alpha_{n-m+1}\right)^{-1}\right] v_j. \quad (19)$$

From this expression, we obtain the SC-factorization of (1). The cofactor equation (18) is simply (13) while the factor equation is obtained using (7), the above calculations and (19). Finally, (17) is obtained by slightly adjusting the summation indices to simplify notation. □

### 3.3. *Complete SC-factorizations*

Equation (17) is once again linear but with order less than (1) by one. If (17) also possesses a unitary solution in $R$ then it is reducible in order by a second application of Theorem 9. If this process can be repeated $k$ times, then a triangular system of $k+1$ first-order equations is obtained that is equivalent to (1). A complete SC-factorization of this type is calculated for equation (31); see chapter 7 of [15] for further information.

### 3.4. *The inversion form symmetry*

Equation (16) is not only a consequence of the invertible map criterion but it is also related to a different SC-factorization. Think of $\{\alpha_n\}$ as a solution of the following $k$th order rational difference equation on the (multiplicative) unit group $G$ of $R$

$$r_{n+1} = a_{0,n} + \sum_{j=1}^{k} a_{j,n} \left( \prod_{i=0}^{j-1} r_{n-i} \right)^{-1}. \tag{20}$$

If in the homogeneous part of (1), i.e. the linear equation

$$x_{n+1} = a_{0,n}x_n + a_{1,n}x_{n-1} + \cdots + a_{k,n}x_{n-k} \tag{21}$$

$a_{j,n} \in G$ for all $j$, then (20) is in fact the factor equation of a SC-factorization in $G$ of (21) and the cofactor equation is $x_{n+1} = r_{n+1}x_n$; see [14]. The *inversion form symmetry* that yields this SC-factorization is $[u_0u_1^{-1}, u_1u_2^{-1}, \ldots, u_{k-1}u_k^{-1}]$. This (nonlinear) form symmetry is characteristic of all difference equations that are *homogeneous of degree 1*, linear or not; see [14] or chapter 4 in [15]. On the other hand, notice that the factor equation (20) of (21) relative to the inversion form symmetry is nonlinear, whereas the factor equation (17) relative to the linear form symmetry is again linear.

## 4. Characteristic equation and eigensequences

The sequence $\{\alpha_n\}$ in Theorem 9 is a solution of the nonlinear difference equation (20). Since $\{\alpha_n\}$ plays a fundamental role in the SC-factorization of (1), it is necessary to examine (20) closely.

### *4.1. The characteristic difference equation*

Equation (20) may be written in a way that does not involve inversion. Multiply it on both sides by the quantity $r_n r_{n-1} \cdots r_{n-k+1}$

$$\begin{aligned} r_{n+1}r_n r_{n-1}\cdots r_{n-k+1} &= a_{0,n}(r_n r_{n-1}\cdots r_{n-k+1}) + a_{1,n}(r_{n-1}\cdots r_{n-k+1}) \\ &\quad + a_{2,n}(r_{n-2}\cdots r_{n-k+1}) + \cdots + a_{k-1,n}r_{n-k+1} + a_{k,n} \end{aligned}$$

which may be written more succinctly as

$$\prod_{i=0}^{k} r_{n-i+1} - \sum_{j=0}^{k-1} a_{j,n} \left( \prod_{i=j}^{k-1} r_{n-i} \right) - a_{k,n} = 0. \tag{22}$$

This equation is not as esoteric as it may appear at first glance. To clarify, consider the special homogeneous case with constant coefficients, i.e.

$$x_{n+1} = a_0x_n + a_1x_{n-1} + \cdots + a_kx_{n-k}. \tag{23}$$

Then (22) reduces to the following difference equation:

$$r_{n+1}r_n\cdots r_{n-k+1} - a_0(r_n\cdots r_{n-k+1}) - a_1(r_{n-2}\cdots r_{n-k+1}) - \cdots - a_{k-1}r_{n-k+1} - a_k = 0. \tag{24}$$

A *constant solution (or fixed point)* $r_n = r$ of (24) must satisfy the polynomial equation

$$r^{k+1} - a_0 r^k - a_1 r^{k-1} - \cdots a_k = 0. \tag{25}$$

The right-hand side of (25) is recognizable as the characteristic polynomial of (23) whose roots are indeed the eigenvalues of the linear homogeneous equation (23).

### 4.2. Eigensequences and eigenvalues

DEFINITION 10. *The difference equation (22) in a ring R is the* characteristic equation *of the homogeneous part of (1), i.e. the linear difference equation (21). Each solution of (22) in R is an* eigensequence *of (21). An* eigenvalue *is a constant eigensequence. An eigensequence whose every term is a unit in the ring is* unitary. *An eigensequence containing a zero divisor is* improper.

To illustrate the concepts in Definition 10 and a concrete application of Theorem 9, consider the difference equation

$$x_{n+1} = x_n + x_{n-1}, \tag{26}$$

also known as the Fibonacci recurrence because with initial values $x_0 = 0$ and $x_1 = 1$ (26) generates the Fibonacci sequence $1, 1, 2, 3, 5, 8, 13, \ldots$, denoted $\{F_n\}$. The characteristic equation of (26) is

$$r_{n+1} r_n - r_n - 1 = 0. \tag{27}$$

This equation has no solutions in the ring of integers $\mathbb{Z}$, constant or otherwise. For let $r_1, r_2 \in \mathbb{Z}$ and note that $r_1 \neq 0$ because clearly $r_n = 0$ does not solve (27). Now (27) has a solution $r_2 \in \mathbb{Z}$ if and only if $r_1 = \pm 1$. Either $r_1 = 1$, $r_2 = 2$ so that $r_3 = 3/2 \notin \mathbb{Z}$ or $r_1 = -1$, $r_2 = 0$ and no value is defined for $r_3$. Hence, (26) has no eigensequences in $\mathbb{Z}$.

The eigenvalues (constant eigensequences) of this equation are roots $(1 \pm \sqrt{5})/2$ of its characteristic polynomial $r^2 - r - 1$. Thus (26) has no eigenvalues in $\mathbb{Q}$; but unlike $\mathbb{Z}$, in $\mathbb{Q}$ (27) can be stated as $r_{n+1} = 1 + 1/r_n$. Iteration starting from (say) $r_0 = 1$ yields $r_n = F_{n+1}/F_n$, a unitary eigensequence for (26). Theorem 9 then yields a SC-factorization, in $\mathbb{Q}$, of (26) consisting of the pair of equations

$$t_{n+1} = a'_{0,n} t_n, \quad a'_{0,n} = -\frac{F_n}{F_{n+1}}, \quad x_{n+1} = \frac{F_{n+2}}{F_{n+1}} x_n + t_{n+1}.$$

See Theorem 12 for a generalization of this example.

### 4.3. Non-unitary and improper eigensequences

Consider the second-order linear difference equation

$$x_{n+1} = 2x_n - 4x_{n-1}, \quad x_0, x_1 \in \mathbb{Z} \tag{28}$$

whose characteristic equation $r_{n+1} r_n - 2r_n + 4 = 0$ has no constant solutions (eigenvalues) in $\mathbb{Z}$ because the polynomial $r^2 - 2r + 4$ has complex roots $r = 1 \pm i\sqrt{3}$. But the characteristic equation does have a (non-unitary) period 3 solution in $\mathbb{Z}$ given by $\{1, -2, 4, 1, -2, 4, \ldots\}$ as may be checked by direct substitution. This is a proper eigensequence since $\mathbb{Z}$ does not contain zero divisors.

The difference equation (28) is also valid in finite rings $\mathbb{Z}_m$ of integers modulo a given positive integer $m$ and different cases occur. For instance, $\mathbb{Z}_{17}$ is a field so the above eigensequence is unitary. But in $\mathbb{Z}_{18}$ where all even numbers are zero divisors the same sequence of period 3 is an improper eigensequence. Also note that for some values of $m$ the polynomial $r^2 - 2r + 4$ has roots in $\mathbb{Z}_m$ which are eigenvalues of (28); e.g. in the field $\mathbb{Z}_7$ the roots are 3 and 6 while in the ring $\mathbb{Z}_{12}$ the roots are 4 and $10 \equiv -2 (\mathrm{mod}\, 12)$, both of which are improper.

Improper eigensequences are undesirable because ring extensions do not render them unitary and Theorem 9 cannot be applied. Nevertheless, SC-factorizations are possible with non-unitary, even improper eigensequences. Consider the following difference equation in an arbitrary non-trivial ring $R$

$$x_{n+1} = (a+b)x_n - abx_{n-1} + c_n. \tag{29}$$

The characteristic equation of the homogeneous part of (29) is $r_{n+1}r_n - (a+b)r_n + ab = 0$. The constant solutions of this equation satisfy the polynomial equation (25), which in this case is $r^2 - (a+b)r + ab = 0$. This evidently has a solution $r = b$ (if $R$ is commutative then $r = a$ is also a solution). While Theorem 9 is not applicable in the absence of an identity and unitary solutions, a SC-factorization of (29) may be directly obtained by rearranging its terms as $x_{n+1} - bx_n = a(x_n - bx_{n-1}) + c_n$ and defining a new variable $t_n = x_n - bx_{n-1}$ to obtain

$$t_{n+1} = at_n + c_n, \quad x_{n+1} = bx_n + t_{n+1}.$$

A somewhat extreme case occurs in the Boolean ring of all finite subsets of $\mathbb{Z}$ (including the empty set) with the operations $A + B = (A \backslash B) \cup (B \backslash A)$ and $AB = A \cap B$ for all finite $A, B \subset \mathbb{Z}$. Equation (29) has a SC-factorization with two improper eigenvalues $a$, $b$ in this commutative ring where the empty set is the zero element and every non-empty set is a zero divisor.

### *4.4. Unitary solutions and eigensequences*

A potential difficulty in applying Theorem 9 is finding the sequence $\{\alpha_n\}$ that satisfies (20), i.e. finding a solution of (22). In this section we discuss how to calculate $\{\alpha_n\}$ *indirectly*, by extracting it from a *unitary solution* of (21) in $R$; i.e. a solution of (21) that is contained in the unit group $G$. Let $\{x_n\}$ be such a unitary solution for a given set of initial values $x_0, x_1, \ldots, x_k \in G$. Multiplying (21) by $x_n^{-1}$ and rearranging terms gives

$$\begin{aligned} x_{n+1}x_n^{-1} &= a_{0,n} + a_{1,n}x_{n-1}x_n^{-1} + a_{2,n}x_{n-2}x_n^{-1} + \cdots + a_{k,n}x_{n-k}x_n^{-1} \\ &= a_{0,n} + a_{1,n}x_{n-1}x_n^{-1} + a_{2,n}x_{n-2}\left(x_{n-1}^{-1}x_{n-1}\right)x_n^{-1} + \cdots \\ &\quad + a_{k,n}x_{n-k}\left(x_{n-k+1}^{-1}x_{n-k+1}\right)\left(x_{n-k+2}^{-1}x_{n-k+2}\right)\cdots\left(x_{n-1}^{-1}x_{n-1}\right)x_n^{-1} \\ &= a_{0,n} + a_{1,n}\left(x_nx_{n-1}^{-1}\right)^{-1} + a_{2,n}(x_{n-1}x_{n-2})^{-1}\left(x_nx_{n-1}^{-1}\right)^{-1} + \cdots \\ &\quad + a_{k,n}\left(x_{n-k+1}x_{n-k}^{-1}\right)^{-1}\left(x_{n-k+2}x_{n-k+1}^{-1}\right)^{-1}\cdots\left(x_nx_{n-1}^{-1}\right)^{-1}. \end{aligned}$$

If $r_n = x_nx_{n-1}^{-1}$ for each $n$ then the above equation can be written as follows:

$$\begin{aligned} r_{n+1} &= a_{0,n} + a_{1,n}r_n^{-1} + a_{2,n}r_{n-1}^{-1}r_n^{-1} + \cdots + a_{k,n}r_{n-k+1}^{-1}r_{n-k+2}^{-1}\cdots r_{n-1}^{-1}r_n^{-1} \quad \text{or} \\ r_{n+1} &= a_{0,n} + a_{1,n}r_n^{-1} + a_{2,n}(r_nr_{n-1})^{-1} + \cdots + a_{k,n}(r_nr_{n-1}\ldots r_{n-k+1})^{-1}, \end{aligned} \tag{30}$$

which is precisely equation (20). Thus, the sequence $\{r_n\}$ of right ratios of $\{x_n\}$ satisfies (20). It is often easier to find a unitary solution of (21) than to look for a particular solution of (20). Once a unitary solution of (21) is identified, an eigensequence may be extracted from it using the next result that supplements and completes Theorem 9.

THEOREM 11. *Let R be a ring with identity. A (unitary) sequence in R is an eigensequence of* (21) *if and only if it is the right ratio sequence of a unitary solution of* (21).

*Proof*. Let $\{r_n\}$ be a unitary eigensequence of (21), choose $x_0 \in G$ and define $x_j = r_j x_{j-1}$ for $j = 1, \ldots, k$. Then $x_j \in G$ for each $j$ and

$$\begin{aligned} r_{j+1}x_j &= \left(a_{0,n} + a_{1,n}r_j^{-1} + a_{2,n}r_{j-1}^{-1}r_j^{-1} + \cdots + a_{k,n}r_{j-k+1}^{-1}r_{j-k+2}^{-1}\cdots r_{j-1}^{-1}r_j^{-1}\right)x_j \\ &= a_{0,n}x_j + a_{1,n}r_j^{-1}x_j + a_{2,n}r_{j-1}^{-1}r_j^{-1}x_j + \cdots + a_{k,n}r_{j-k+1}^{-1}r_{j-k+2}^{-1}\cdots r_{j-1}^{-1}r_j^{-1}x_j \\ &= a_{0,n}x_j + a_{1,n}x_{j-1} + a_{2,n}r_{j-1}^{-1}x_{j-1} + \cdots + a_{k,n}r_{j-k+1}^{-1}r_{j-k+2}^{-1}\cdots r_{j-2}^{-1}r_{j-1}^{-1}x_{j-1}, \end{aligned}$$

where we used the fact that $r_j^{-1}x_j = x_{j-1}$. Similarly, $r_{j-1}^{-1}x_{j-1} = x_{j-2}$ which yields a further reduction

$$r_{j+1}x_j = a_{0,n}x_j + a_{1,n}x_{j-1} + a_{2,n}x_{j-2} + \cdots + a_{k,n}r_{j-k+1}^{-1}r_{j-k+2}^{-1}\cdots r_{j-2}^{-1}x_{j-2}.$$

Next, $r_{j-2}^{-1}x_{j-2} = x_{j-3}$ and the above calculation may be continued to ultimately yield

$$r_{j+1}x_j = a_{0,n}x_j + a_{1,n}x_{j-1} + a_{2,n}x_{j-2} + \cdots + a_{k,n}x_{j-k}.$$

Define the right-hand side as $x_{j+1}$, then proceed to $r_{j+2}x_{j+1}$ and repeat the calculation to generate a new value

$$x_{j+2} = r_{j+2}x_{j+1} = a_{0,n}x_{j+1} + a_{1,n}x_j + a_{2,n}x_{j-1} + \cdots + a_{k,n}x_{j+1-k}.$$

The values of $x_n$ generated by the above construction satisfy the linear equation (21) for $n = j+1, j+2, \ldots$ Therefore, $\{x_n\}$ is a unitary solution of (21) whose right ratio sequence is $\{r_n\}$ (by construction). The converse is true by the definition of eigensequence and the argument preceding this theorem. □

To illustrate the use of preceding ideas, consider the following difference equation in the finite field $\mathbb{Z}_p$ where $p$ is a prime number

$$x_{n+1} = 2x_{n-1} + x_{n-2} + c_n, \quad x_0, x_1, x_2 \in \mathbb{Z}_p, \tag{31}$$

where $\{c_n\}$ is an arbitrary sequence in $\mathbb{Z}_p$. The characteristic polynomial of the homogeneous part of (31) is $r^3 - 2r - 1 = (r+1)(r^2 - r - 1)$. This has a root $-1 \in \mathbb{Z}_p$, a unit eigenvalue that yields the SC-factorization

$$t_{n+1} = a_0' t_n + a_1' t_{n-1} + c_n, \quad x_{n+1} = -x_n + t_{n+1},$$

where $t_1 = x_0 + x_1$, $t_2 = x_1 + x_2$. The constant coefficients are $a_0' = a_1' = 1$ which yield the factor equation $t_{n+1} = t_n + t_{n-1} + c_n$. The homogeneous part of this is the Fibonacci recurrence (26) whose characteristic polynomial is $r^2 - r - 1$. This polynomial has roots in

$\mathbb{Z}_p$ if and only if $p \equiv 0, 1, 4 (\mathrm{mod} 5)$; see, e.g. [5]. Such roots (eigenvalues) readily yield a SC-factorization of (26) in $\mathbb{Z}_p$. More generally, Theorem 9 is applicable via Theorem 11 if (26) has a unitary solution in $\mathbb{Z}_p$, i.e. a solution that never visits 0. $\mathbb{Z}_p$ contains (non-constant) zero-avoiding solutions of (26) for infinitely many primes of type $p \equiv 2, 3 (\mathrm{mod} 5)$; see [17]. Let $p$ be such a prime and $\{u_n\}$ a zero-avoiding solution of (26) in $\mathbb{Z}_p$. Repeating the calculations for (26) but replacing $F_n$ with $u_n$ gives the following complete SC-factorization for (31)

$$\begin{aligned} x_{n+1} &= -x_n + t_{n+1}, \\ t_{n+1} &= \frac{u_{n+1}}{u_n} t_n + s_{n+1}, \quad t_1 = x_1 + x_0, \\ s_{n+1} &= -\frac{u_{n-1}}{u_n} s_n + c_n, \quad s_2 = t_2 - \frac{u_2}{u_1} t_1. \end{aligned}$$

### 4.5. *Zero-avoiding solutions and fields of quotients*

We may take an idea in the preceding discussion one step further. If $R$ is a commutative ring with identity and no zero divisors (i.e. $R$ is an *integral domain*), then its complete ring of quotients is a field in which $R$ is embedded (see, e.g. [7], chapter 3). We denote this *field of quotients* by $Q_R$ which contains an isomorphic copy of $R$. The unit group of $Q_R$ is the set $Q_R \backslash \{0\}$ of all non-zero elements of $Q_R$ which contains $R \backslash \{0\}$. If $Q_R$ is not isomorphic to $R$ then $Q_R$ has an abundance of units that do not exist in $R$.

Let us call a sequence $\{x_n\}$ *zero-avoiding* if $x_n \neq 0$ for all $n$. In particular, if $R$ is a field then a sequence is zero-avoiding if and only if it is unitary. The next result is a consequence of Theorems 9 and 11 that reduces the search for eigensequences to a search for zero-avoiding solutions.

THEOREM 12. *Let $R$ be an integral domain with field of quotients $Q_R$ and assume that the parameters $a_{j,n}, b_n$ in (1) are in $R$ for all $n$ and all $j = 0, 1, \ldots, k$.*

(a) *If $\{x_n\}$ is a zero-avoiding solution of (21) in $R$ then $\{x_n x_{n-1}^{-1}\}$ is a (unitary) eigensequence of (21) in $Q_R$.*
(b) *If (21) has a zero-avoiding solution $\{x_n\}$ in $R$ then (1) has a SC-factorization in $Q_R$ consisting of the pair of equations (17) and (18) with parameters $\alpha_n = x_n x_{n-1}^{-1}$ and $a'_{m,n}$ all existing in $Q_R$ for all $m, n$.*

The earlier discussion of the Fibonacci recurrence (26) illustrates the use of this theorem in a familiar case where the integral domain is $\mathbb{Z}$ with field of quotients $\mathbb{Q}$.

### 4.6. *The Poincaré–Perron theorem*

The fact that eigensequences are ratio sequences of unitary solutions recalls the celebrated theorem of Poincaré and Perron; see [12,13] or section 8.2 of [4]. Let $R = \mathbb{C}$ and assume that the coefficients $a_{i,n}$ in (21) converge to constants $a_i$ as $n \to \infty$; i.e. (21) is a 'Poincaré difference equation'. In the language of eigensequences, the Poincaré–Perron theorem may be stated as follows:

Each eigenvalue of (23) is a limit of an eigensequence of (21).

For example, the following Poincaré equation in the field $\mathbb{R}$ of real numbers

$$x_{n+1} = \frac{1}{n} x_n + x_{n-1} \tag{32}$$

has limiting autonomous equation $y_{n+1} = y_{n-1}$ with eigenvalues $\pm 1$. The characteristic equation of (32) is $r_{n+1}r_n - r_n/n - 1 = 0$ which may be stated also as

$$r_{n+1} = \frac{1}{n} + \frac{1}{r_n}. \tag{33}$$

It is readily verified by induction that the solution of (33) with $r_1 = 1$ may be expressed as

$$r_{2n-1} = 1, \quad r_{2n} = \frac{2n}{2n-1}, \quad n \geq 1.$$

Thus $\lim_{n\to\infty} r_n = 1$, as expected. The eigensequence above actually yields much more information in this case; by Theorem 9 it results in a SC-factorization that readily yields a formula for the general solution of (32).

Also worth a mention is the fact that not every eigensequence of a Poincaré difference equation converges to an eigenvalue of the limiting equation. For example, equation (28) is autonomous, hence trivially of Poincaré type but as we saw previously, it has an eigensequence of period 3 that is unitary in $\mathbb{R}$.

## 5. Difference equations with periodic coefficients

In this section we study the following difference equation with periodic coefficients in a non-trivial ring $R$, i.e.

$$x_{n+1} = a_n x_n + b_n x_{n-1}, \quad a_{n+p_1} = a_n, \; b_{n+p_2} = b_n, \; n = 0, 1, 2, \ldots, \tag{34}$$

where the (minimal or prime) periods $p_1, p_2$ are positive integers with least common multiple $p = \mathrm{lcm}(p_1, p_2)$; we refer to (34) as a *difference equation of period* $p$. We study the solutions of (34) using periodic eigensequences rather than the Floquet exponents and multipliers of the standard method (see, e.g. [4]). The method that is discussed below applies in the general context of rings and is easily extended to non-homogeneous equations.

### *5.1. Periodic eigensequences*

A natural question for us is whether (34) has an eigensequence of period $p$ in $R$. If so then such an eigensequence yields a SC-factorization of the second-order equation into a pair of first-order equations. This may occur regardless of whether (34) has any periodic *solutions* since equation (34) does not generally possess any periodic solutions.

An eigensequence of period $p$ exists in $R$ if there is an initial value $r_1 \in R$ such that the characteristic equation of (34), i.e. the first-order quadratic difference equation

$$r_{n+1}r_n = a_n r_n + b_n \tag{35}$$

has a solution of period $p$ in the ring $R$. Suppose that there are $r_j \in R$ that satisfy (35) for $j = 1, 2, \ldots, p$. Then

$$r_2 r_1 = a_1 r_1 + b_1, \quad r_3 r_2 = a_2 r_2 + b_2.$$

Let $L_1 = r_1$ so that $r_2 L_1 = a_1 L_1 + b_1$. For $j = 2, \ldots, p$ define $L_{j+1} = a_j L_j + b_j L_{j-1}$. Then $r_2 r_1 = r_2 L_1 = L_2$ so that

$$r_3L_2 = (r_3r_2)r_1 = a_2r_2r_1 + b_2r_1 = a_2L_2 + b_2L_1 = L_3,$$
$$r_4L_3 = (r_4r_3)r_2r_1 = a_3r_3r_2r_1 + b_3r_2r_1 = a_3L_3 + b_3L_2 = L_4,$$
$$\vdots$$

By induction, for $j = 2, \dots, p$

$$r_{j+1}L_j = (r_{j+1}r_j)r_{j-1}\cdots r_1 = a_jr_jr_{j-1}\cdots r_1 + b_3r_{j-1}\cdots r_1 = a_jL_j + b_jL_{j-1} = L_{j+1}.$$

This process yields a solution $\{r_n\}$ of (35) with period $p$ if and only if $r_{p+1} = r_1$; thus,

$$r_1L_p = r_{p+1}L_p = a_pL_p + b_pL_{p-1} \Rightarrow (r_1 - a_p)L_p = b_pL_{p-1}. \tag{36}$$

The quantities $L_1, \dots, L_p$ that are generated above evidently depend on $r_1$ in a linear way so there are $\alpha_j, \beta_j \in R$ such that

$$L_j = \alpha_jr_1 + \beta_j$$

for $j = 1, 2, \dots, p$. Inserting this form in (36) yields

$$\begin{aligned}(r_1 - a_p)(\alpha_pr_1 + \beta_p) - b_p(\alpha_{p-1}r_1 + \beta_{p-1}) = 0,\\ r_1\alpha_pr_1 + r_1\beta_p - (a_p\alpha_p + b_p\alpha_{p-1})r_1 - (a_p\beta_p + b_p\beta_{p-1}) = 0.\end{aligned} \tag{37}$$

The definition of $L_j$ implies

$$\begin{aligned}\alpha_{j+1}r_1 + \beta_{j+1} &= a_j(\alpha_jr_1 + \beta_j) + b_j(\alpha_{j-1}r_1 + \beta_{j-1})\\ &= (a_j\alpha_j + b_j\alpha_{j-1})r_1 + a_j\beta_j + b_j\beta_{j-1}.\end{aligned}$$

Suppose that $R$ has an identity 1. By matching coefficients on the two sides of the above equality, we see that the coefficients $\alpha_j, \beta_j$ satisfy (34) for $j = 1, 2, \dots, p$ with initial values

$$\alpha_0 = 0, \quad \alpha_1 = 1; \quad \beta_0 = 1, \quad \beta_1 = 0. \tag{38}$$

Using this fact to simplify (37) we conclude that if $r_1$ is a root of the following polynomial:

$$r\alpha_pr + r\beta_p - \alpha_{p+1}r - \beta_{p+1} = 0, \tag{39}$$

then the solution $\{r_n\}$ of (35) has period $p$. These observations prove the following result.

THEOREM 13. *Let $R$ be a ring with identity 1 and for $j = 1, 2, \dots, p$, let $\alpha_j, \beta_j$ be obtained by iteration from (34) subject to (38) such that the quadratic polynomial (39) is proper, i.e. not $0 = 0$.*

(a) *If (34) has an eigensequence $\{r_n\}_{n=1}^{\infty}$ of period $p$ then $r_1$ is a root of (39) in $R$.*
(b) *If $r_1$ is a root of (39) in $R$ and there are $r_j \in R$ satisfying (35) for $j = 2, \dots, p$, then $\{r_n\}_{n=1}^{\infty}$ is an eigensequence of (34) with period $p$.*

(c) *If a root* $r_1$ *of* (39) *in* $R$ *is a unit and the recurrence*

$$r_{j+1} = a_j + b_j r_j^{-1} \tag{40}$$

*generates units* $r_2, \ldots, r_p$ *in* $R$, *then* $\{r_n\}_{n=1}^{\infty}$ *is a unitary eigensequence of* (34) *with period* $p$ *that yields the SC-factorization*

$$t_{n+1} = -b_n r_n^{-1} t_n, \quad t_1 = x_1 - r_1 x_0, \quad x_{n+1} = r_{n+1} x_n + t_{n+1}.$$

Polynomial (39) simplifies further if the coefficients $a_j, b_j$ are in the centre of $R$, i.e. they commute with all members of $R$. Then $\alpha_j, \beta_j$ are also in the centre of $R$ so (39) reduces to

$$\alpha_p r^2 + (\beta_p - \alpha_{p+1})r - \beta_{p+1} = 0. \tag{41}$$

If $a_n = a$ and $b_n = b$ are constants then $p = 1$ and the quadratic polynomial (41) reduces to $r^2 - ar - b = 0$, that is the characteristic polynomial of the autonomous linear equation of order 2.

### 5.2. *Examples of periodic difference equations*

To illustrate applications of Theorem 13, consider the following difference equation of period 3,

$$x_{n+1} = 2\cos\left(\frac{2\pi n}{3}\right) x_n + x_{n-1} \tag{42}$$

with $a_1 = a_2 = -1$ and $a_3 = 2$ and $b_n = 1$ constant. Although these coefficients are defined in any ring with identity, for convenience we assume that the underlying ring is the field $\mathbb{R}$ of real numbers. The numbers $\alpha_j, \beta_j$ are readily calculated from (42) using (38):

$$\alpha_2 = -1, \quad \alpha_3 = 2, \quad \alpha_4 = 3, \quad \beta_2 = 1, \quad \beta_3 = -1, \quad \beta_4 = -1.$$

The quadratic equation (41) $2r^2 - 4r + 1 = 0$ in this case has two zeros $(2 \pm \sqrt{2})/2$. Let $r_1 = (2 - \sqrt{2})/2$ and use (40) to calculate $r_2 = 1 + \sqrt{2}$, $r_3 = -2 + \sqrt{2}$. Since these are units in $\mathbb{R}$, by Theorem 13 a unitary eigensequence with period 3 is obtained. If $\rho = -1/(r_1 r_2 r_3)$ then the SC-factorization of (42) is readily calculated and the solution of its factor equation is found to be

$$t_{3j+1} = \rho^j t_1, \quad t_{3j+2} = -\frac{\rho^j t_1}{r_1}, \quad t_{3j+3} = \frac{\rho^j t_1}{r_1 r_2}, \quad j \geq 0, \ \ t_1 = x_1 - r_1 x_0.$$

The cofactor is $x_{n+1} = r_{n+1} x_n + t_{n+1}$. Since $\rho = 1 + \sqrt{2} > 1$ it follows that all solutions of (42) with $t_1 \neq 0$ are unbounded. However, for initial values satisfying $x_1 = r_1 x_0$ we have $t_1 = 0$; so $t_n = 0$ for all $n$ and

$$x_{3n} = \frac{(-1)^n x_0}{\rho^n}, \quad x_{3n+1} = \frac{r_1(-1)^n x_0}{\rho^n}, \quad x_{3n+2} = \frac{r_1 r_2 (-1)^n x_0}{\rho^n}, \quad n \geq 1.$$

These special solutions of (42) converge to 0 exponentially for all $x_0$.

Next, consider the following variant of (42) which is a difference equation of period 6

$$x_{n+1} = 2\cos\left(\frac{2\pi n}{3}\right)x_n + (-1)^{n-1}x_{n-1} \tag{43}$$

because the coefficients $b_n = (-1)^{n-1}$ now have period 2. The numbers $\alpha_j, \beta_j$ in this case are

$$\alpha_2 = -1,\quad \alpha_3 = 0,\quad \alpha_4 = -1,\quad \alpha_5 = 1,\quad \alpha_6 = -2,\quad \alpha_7 = -5,\quad \beta_2 = 1,$$

$$\beta_3 = -1,\quad \beta_4 = -1,\quad \beta_5 = 2,\quad \beta_6 = -3,\quad \beta_7 = -8.$$

These numbers yield the quadratic equation $r^2 - r - 4 = 0$ whose zeros are $(1 \pm \sqrt{17})/2$. Let $r_1 = (1 + \sqrt{17})/2$ and use (40) to calculate

| $r_2$ | $r_3$ | $r_4$ | $r_5$ | $r_6$ |
|---|---|---|---|---|
| $(-9+\sqrt{17})/8$ | $(1+\sqrt{17})/8$ | $(3+\sqrt{17})/2$ | $(-1-\sqrt{17})/4$ | $(-3-\sqrt{17})/4$ |

Since the above-listed numbers are units in $\mathbb{R}$ it follows that they are one cycle of an eigensequence of period 6 which leads to a SC-factorization of (43) in a manner that is similar to that discussed for (42).

If the quadratic polynomial (39) (or (41) in the commutative case) has no roots in the underlying ring, then periodic eigensequences with period $p$ do not exist in that ring. However, other periodic eigensequences may exist (e.g. recall that the autonomous equation (28) has no integer eigenvalues but has an eigensequence of period 3 in $\mathbb{Z}$).

The absence of periodic eigensequences does not imply the same for *non-periodic* eigensequences that yield SC-factorizations. In particular, for the following variant of (42)

$$x_{n+1} = 2\cos\left(\frac{2\pi n}{3}\right)x_n - x_{n-1} \tag{44}$$

we find that $\alpha_2 = -1, \alpha_3 = 0, \alpha_4 = 1, \beta_2 = -1, \beta_3 = 1, \beta_4 = 3$. With these coefficients (41) has no roots so (44) has no eigensequences of period 3. But suppose we set $r_1 = 1$. Then it can be verified by induction using the recurrence (40) that

$$r_{3j+1} = 3j+1,\quad r_{3j+2} = -\frac{3j+2}{3j+1},\quad r_{3j+3} = -\frac{1}{3j+2},\quad j \geq 0$$

is a (non-periodic) eigensequence for (44). Note that $r_n r_{n+1} r_{n+2} = 1$ for all $n$ so

$$t_{3j+1} = t_1 \prod_{i=1}^{3j} \frac{1}{r_i} = t_1 \prod_{i=0}^{j-1} \frac{1}{r_{3i+1} r_{3i+2} r_{3i+3}} = t_1 = x_1 - x_0$$

and the solution of the factor equation may be expressed as

$$t_{3j+1} = t_1,\quad t_{3j+2} = \frac{t_1}{3j+1},\quad t_{3j+3} = -\frac{t_1}{3j+2},\quad j \geq 0.$$

From the cofactor the general solution is calculated as follows:

$$x_{3j+1} = (3j+1)x_{3j} + t_1, \quad x_{3j+2} = -\frac{3j+2}{3j+1}x_{3j+1} + \frac{t_1}{3j+1} = -(3j+2)x_{3j} - t_1,$$

$$x_{3j+3} = -\frac{1}{3j+2}x_{3j+2} - \frac{t_1}{3j+2} = x_{3j}.$$

The last equation implies that $x_{3j} = x_0$ for all $j$ so the general solution of (44) is

$$x_n = \begin{cases} x_0, & \text{if } \; n = 3j, \\ x_0 n + x_1, & \text{if } \; n = 3j+1, \\ -x_0 n - x_1, & \text{if } \; n = 3j+2. \end{cases}$$

In particular, if $x_0 = 0$ then the solution $\{0, x_1, -x_1, \ldots\}$ of (44) has period 3 for all $x_1 \neq 0$.

## 6. SC-factorization in rings of functions

Difference equations in rings of functions have appeared in applied mathematics. Well-known special functions such as Bessel functions satisfy recurrence relations that are examples of difference equations on rings of real or complex-valued functions.

### *6.1. Rings of functions on a set*

For functions from a non-empty set $S$ into a non-zero ring $\mathcal{R}$ we define the operations of addition and multiplication *pointwise*, i.e. for each $s \in S$

$$(f+g)(s) = f(s) + g(s), \quad (fg)(s) = f(s)g(s).$$

Other, less common types of ring operations are possible for functions but not considered here. With these operations, the set $\mathcal{R}^S$ of all functions from $S$ into $\mathcal{R}$ is a function ring and each subring $R(S)$ of $\mathcal{R}^S$ is a ring of $\mathcal{R}$-valued functions on $S$. Note that $R(S)$ is commutative if $\mathcal{R}$ is. If $R(S)$ contains all the constant functions on $S$ then we usually think of these functions as elements of $\mathcal{R}$ and thus, think of $\mathcal{R}$ as a subring of $R(S)$. *In this section we assume that $R(S)$ contains all the constants.*

A ring of functions $R(S)$ of the above type is also a function algebra; see, e.g. [7] or [11]. An element $u$ is a unit in $R(S)$ if and only if $u(s) \neq 0$ for all $s \in S$. In this case, the inverse of $u$ is just its reciprocal $1/u$. Since $R(S)$ is closed under addition and multiplication, if the parameters and initial values $a_{j,n}, b_n, x_j : S \rightarrow \mathbb{R}$ are in $R(S)$ for all $j = 0, 1, \ldots, k$ and all $n$, then the solution $\{x_n\}$ of (1) is also contained in $R(S)$.

In the familiar ring $C[0,1]$ of all continuous, real-valued functions on the interval [0,1] the units are functions that are always either positive or negative and a zero divisor is a function whose set of zeros has a non-empty interior in [0,1]. The ring of polynomials $\mathcal{F}[x]$ with coefficients in a given field $\mathcal{F}$ is a familiar ring that may be viewed as a ring of functions on $\mathcal{F}$ (or some subset of it) by interpreting the indeterminate as a variable; see [11], chapter 4. In particular, if $S = [0,1]$ and $\mathcal{F} = \mathbb{R}$ then by the Weierstrass approximation theorem $\mathcal{F}[x]$ is a dense subring of $C[0,1]$ in the uniform topology. Various rings of differentiable functions fall in-between $\mathcal{F}[x]$ and the continuous functions and share the aforementioned properties of the continuous functions. But larger rings such as bounded functions or integrable functions have different properties.

### *6.2. SC-factorizations*

COROLLARY 14. *Let $R(S)$ be a ring of real-valued functions on a non-empty set $S$. Assume that $a_{j,n}(s) \geq 0$ for all $s \in S$, $j = 0, 1, \ldots, k$ and all $n$. If*

$$\sum_{j=0}^{k} a_{j,n}(s) > 0 \tag{45}$$

*for all $s \in S$ and all $n$ then the homogeneous part of the difference equation (1) has unitary solutions in $R(S)$. Therefore, (1) has a SC-factorization in $R(S)$ that is given by (17) and (18).*

*Proof.* Let $a_{j,n}(s) \geq 0$ for all $s \in S$ and all $n$. Choose constant initial values $u_j = 1$ for $j = 0, 1, \ldots, k$ in (21), i.e. the homogeneous part of (1). By (45), $\sum_{j=0}^{k} a_{j,n}(s) > 0$ so

$$u_{k+1}(s) = \sum_{j=0}^{k} a_{j,k}(s) > 0$$

for all $s \in S$. Thus $u_{k+1}(s)$ is a unit in $R(S)$ and

$$u_{k+2}(s) = \sum_{j=0}^{k} a_{j,k+1}(s) u_{k+1-j}(s) = a_{0,k+1}(s) u_{k+1}(s) + \sum_{j=1}^{k} a_{j,k+1}(s)$$

for all $s \in S$. If $\sum_{j=1}^{k} a_{j,k+1}(s) = 0$ for some $s$ then by (45) $a_{0,k+1}(s) \neq 0$. It follows that $u_{k+2}$ is also positive on $S$, hence a unit in $R(S)$. Proceeding in this fashion, it follows that $u_n(s) > 0$ for all $s \in S$ and all $n$. Thus $\{u_n(s)\}$ is a unitary solution of (15). By Theorem 11 the ratios sequence $\{u_n(s)/u_{n-1}(s)\}$ is a unitary eigensequence in $R(S)$ so Theorem 9 yields a SC-factorization for (1). □

*Remark* 15. Non-unitary solutions for (21) exist under the hypotheses of Corollary 14 because an initial function may not be a unit. Furthermore, none of the parameters $a_{j,n}(s), b_n(s)$ in Corollary 14 may be units. For instance, the corollary applies to the following difference equation in $C[0, 1]$

$$x_{n+1}(r) = a(1 - \sin n\pi r) x_n(r) + br^n(1 - r) x_{n-1}(r), \quad a, b > 0, \quad r \in [0, 1], \quad n \geq 1$$

in which $a_{0,n}(r) = a(1 - \sin n\pi r)$, $a_{1,n}(r) = br^n(1 - r)$ and $b_n(r) = 0$ are non-units, but for all $r$, $n$

$$a_{0,n}(r) + a_{1,n}(r) = a(1 - \sin n\pi r) + br^n(1 - r) > 0.$$

### *6.3. The recurrence for modified Bessel functions*

To illustrate an application of Corollary 14 consider the linear difference equation

$$x_{n+1}(s) = \frac{2n}{s} x_n(s) + x_{n-1}(s), \quad s \in (0, \infty). \tag{46}$$

This is the recurrence relation for the modified Bessel functions $K_n(s)$ of the second kind, so-called because they are solutions of the second-order linear differential equation known as the modified Bessel differential equation (see, e.g. [20]). In fact, the sequence of functions $\{K_n(s)\}$ is a particular solution of (46) from specified initial values $K_0(s), K_1(s)$. According to Corollary 14 a unitary solution $\{u_n(s)\}$ of (46) is generated by any pair of positive functions; e.g. $u_0(s) = u_1(s) = 1$. The first few terms are

$$u_2(s) = \frac{2}{s} + 1, \quad u_3(s) = \frac{8}{s^2} + \frac{4}{s} + 1, \quad u_4(s) = \frac{48}{s^3} + \frac{24}{s^2} + \frac{2}{s} + 1.$$

Now the ratios $u_n(s)/u_{n-1}(s)$ define an eigensequence for (46) and yield the SC-factorization

$$t_{n+1}(s) = -\frac{u_{n-1}(s)}{u_n(s)} t_n(s), \quad x_{n+1}(s) = \frac{u_{n+1}(s)}{u_n(s)} x_n(s) + t_{n+1}(s)$$

with $t_1(s) = x_1(s) - [u_1(s)/u_0(s)]x_0(s) = x_1(s) - x_0(s)$. Iteration of the factor equation yields $t_n(s) = (-1)^{n-1} t_1(s)/u_{n-1}(s)$; inserting this into the cofactor, summation yields a formula for the general solution of (46) in terms of the unitary solution $\{u_n(s)\}$ as follows:

$$x_n(s) = u_n(s)x_1(s) + \sum_{i=2}^{n-1} \frac{u_n(s)}{u_i(s)} t_i(s) = u_n(s)\left[x_0(s) + t_1(s)\sum_{i=1}^{n-1} \frac{(-1)^{i-1}}{u_i(s)u_{i-1}(s)}\right].$$

Different values of positive functions $u_0(s), u_1(s)$ yield different formulas but of course, the same quantity $x_n(s)$.

## 7. Conclusion and future directions

In this paper we studied SC-factorizations of linear difference equations in rings using the nonlinear method of SC-factorization. For the typical (recursive) linear equation this method supplements the standard methods that use modules and group actions. The advantages offered by the SC-factorization method include the following:

- Because of its nonlinear origins, SC-factorization handles non-homogeneous terms integrally, so it is not necessary to calculate a particular solution by independent methods.
- Variable coefficients naturally fit into the eigensequence framework.
- For linear equations (homogeneous or not) with constant coefficients in an abstract ring, it is often possible to find an eigensequence where eigenvalues do not exist and ring extensions are not known or not feasible.

We obtained conditions that are sufficient but not necessary for the existence of SC-factorizations. Relaxing or modifying the hypotheses in these cases should lead to broader applicability. For example, extending Corollary 14 to include negative coefficients will lead, among other things, to results for recurrences of special functions such as Bessel, Legendre, Hermite and so on that are similar to the example of modified Bessel functions above.

Many challenges at different levels of generality remain. A ring with identity may contain no unitary solutions of the homogeneous part of (1). For example, it is shown in [17] that for certain primes, e.g. $2, 3, 7, 23, \ldots$, the field $\mathbb{Z}_p$ contains no zero-avoiding (hence unitary) solutions of (26). In such cases (1) fails to have the linear form symmetry (12). However, non-existence of a particular form symmetry is not equivalent to the non-existence of a SC-factorization; see Section 3.4 on the (nonlinear) inversion form symmetry. The question of whether in the absence of (12) other types of form symmetry exist for which the factor or the cofactor equation is linear remains open.

With regard to the Poincaré–Perron theorem of Section 4.6, whether eigensequences of a Poincaré difference equation in topological rings more general than $\mathbb{R}$ or $\mathbb{C}$ (e.g. Banach algebras) converge to eigenvalues of the limiting equation is an interesting problem for future discussion.

If $R$ is not commutative then using different orders of multiplications of coefficients $a_{j,n}$ with the variables $x_{n-j}$ in (1) may result in different equations (with the same set of coefficients). Not all the results in this paper extend readily to these variants in the non-commutative cases.

Finally, linear difference equations may occur in non-recursive forms; e.g. $a_{0,n}x_n + a_{1,n}x_{n-1} + \cdots + a_{k,n}x_{n-k} = b_n$, where the leading coefficient $a_{0,n}$ is not a unit in the ring $R$ for infinitely many $n$ and the equation cannot be solved uniquely for $x_n$. For non-recursive difference equations even such basic issues as the existence and uniqueness of solutions are not assured. For a study of reduction of order of non-recursive difference equations (linear or quadratic), see [15].

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