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# Reductions of order in difference equations defined as products of exponential and power functions

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Using the method of semiconjugate factorization we obtain reductions in orders for difference equations that are defined as products of complex exponential and power functions. As an application of this type of reduction in order, we explain the behaviours of positive solutions in special cases of such equations with real parameters.

**Keywords:** order reduction; semiconjugate; factorization; power functions; exponential functions

## 1. Introduction

Consider the following nonlinear difference equation of order $k+1$

$$x_{n+1} = x_n^{a_0} x_{n-1}^{a_1} \cdots x_{n-k}^{a_k} \mathrm{e}^{\alpha_n - b_0 x_n - \cdots - b_k x_{n-k}}, \tag{1}$$

where the parameters $a_j, b_j$ are complex numbers for $j = 0, 1, \ldots, k$ and $\{\alpha_n\}$ is a given sequence of complex numbers. A solution of (1) is a sequence $\{x_n\}$ of complex numbers that satisfies (1) for all $n \geq 1$ if a set of initial values $x_0, x_{-1}, \ldots, x_{-k}$ is specified.

Equation (1) may also be written in the equivalent, more stylized form as a product of power and exponential functions:

$$x_{n+1} = \gamma_n x_n^{a_0} x_{n-1}^{a_1} \cdots x_{n-k}^{a_k} \beta_0^{x_n} \beta_1^{x_{n-1}} \cdots \beta_k^{x_{n-k}},$$

with constants $\beta_j = \mathrm{e}^{-b_j}$, $j = 0, 1, \ldots, k$ and $\gamma_n = \mathrm{e}^{\alpha_n}$ for all $n \geq 0$. We may refer to equation (1) as an *expow* difference equation for short.

Special cases of (1) with real parameters appear in the literature. For example, the following *expow* difference equation of order two is derived from a discrete time population model in [3]:

$$x_{n+1} = x_{n-1} \mathrm{e}^{\alpha - x_n - x_{n-1}}, \tag{2}$$

with $\alpha$ a fixed real number. In particular, it is shown in [3] that if $0 < \alpha \leq 1$ then every positive solution of (2) converges to a solution with period two. Numerical simulations indicate that this statement is not true for $\alpha > 1$. In this case, it is shown in [6] (also see Example 12 below) that equation (2) exhibits a large variety of stable solutions depending on the choice of (positive) initial values $x_0, x_{-1}$. These positive solutions include periodic

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solutions of all possible even periods as well as bounded, non-periodic solutions. The approach in [6] is based on reducing equation (2) to an equation of order one whose properties are known:

$$y_{n+1} = t_{n+1} y_n \mathrm{e}^{-y_n} \tag{3}$$

where,

$$t_{n+1} = \frac{\mathrm{e}^{\alpha}}{t_n}.$$

We note that the second equation above for $t_n$ is independent of the equation (3) and an explicit formula for its 2-periodic solution is easy to calculate. With this in mind, (3) may be considered a reduction of order of equation (2) from two to one.

Equation (3) together with its associated equation (for the parameter $t_{n+1}$) constitute a *semiconjugate factorization* of (2) over $(0, \infty)$. The method of semiconjugate factorization is discussed in detail in [6]. This method applies to any algebraic group so its potential applicability extends to areas where discrete modelling naturally occurs (e.g. dynamics on networks; see for instance, [4]).

In this paper, we use the concept of semiconjugacy to obtain reductions of order for equation (1) over the set of complex numbers $\mathbb{C}$. We first generalize a result stated in [6] from the multiplicative group $(0, \infty)$ of all positive real numbers to all multiplicative subgroups of $\mathbb{C}$. Then we use this extended result (Lemma 1 below) to obtain our main results in this paper on the reduction of order of equation (1). The proofs given here are technically self-contained and do not require the PDE-based approach in [6]. However, since the results in this paper belong to the category of semiconjugate factorization a brief review of [6] offers additional insights into the nature of our results here.

As applications of our results we study the positive solutions of some special cases of (1) with real parameters. These special cases are the ones seen in scientific models. The exponential and power functions are then single-valued and one to one. Positive solutions are guaranteed to exist uniquely for a given set of initial values $x_0, x_{-1}, \dots, x_{-k}$, if

$$a_j, b_j, \alpha_n \in \mathbb{R} \quad \text{for all} \quad 0 \le j \le k, \quad n \ge 0, \qquad x_0, x_{-1}, \dots, x_{-k} > 0. \tag{4}$$

## 2. Semiconjugacy and reduction of order

Let $\mathbb{C}$ be the set of all complex numbers. The non-autonomous difference equation (1) of order $k+1$ unfolds to a sequence of self maps of $\mathbb{C}^{k+1}$ as

$$F_n(z_0, z_1, \dots, z_k) = (z_0^{a_0} z_1^{a_1} \dots z_k^{a_k} \mathrm{e}^{\alpha_n - b_0 z_0 - \dots - b_k z_k}, z_0, \dots, z_{k-1}),$$

for every point $(z_0, z_1, \dots, z_k) \in \mathbb{C}^{k+1}$ at which the expression on the right is defined. The unfolding functions $F_n$ are also referred to as the associated vector maps of equation (1); see, e.g. [5]. It follows that if $\{x_n\}$ is a solution of (1) then for all $n$,

$$\begin{aligned} F_n(x_n, \dots, x_{n-k}) &= (x_n^{a_0} \dots x_{n-k}^{a_k} \mathrm{e}^{\alpha_n - b_0 x_n - \dots - b_k x_{n-k}}, x_n, x_{n-1}, \dots, x_{n-k+1}) \\ &= (x_{n+1}, x_n, x_{n-1}, \dots, x_{n-k+1}). \end{aligned}$$

Let $\mathbb{C}_0 = \mathbb{C} \backslash \{0\}$, a group under the ordinary multiplication of complex numbers. Our primary purpose is to use the ideas in [6] to obtain a sequence of self maps $\phi_n : \mathbb{C}_0 \to \mathbb{C}_0$

and a map $H : \mathbb{C}_0^{k+1} \rightarrow \mathbb{C}_0$ such that

$$H \circ F_n = \phi_n \circ H \quad \text{for all} \;\; n \geq 0. \tag{5}$$

If (5) holds then we say that the difference equation (1) is *semiconjugate* to the first order difference equation

$$t_{n+1} = \phi_n(t_n), \tag{6}$$

on $\mathbb{C}_0$. Equation (6) is the *factor* equation and the function $H(z_0, z_1, \ldots, z_k)$ defines a *form symmetry*. In the autonomous case (no explicit dependence on the independent variable $n$) this is the same type of semiconjugacy as that introduced in [5] for maps of $\mathbb{R}^N$ (where maps are not limited to unfoldings of higher order equations).

In [6] a form symmetry is obtained for the following equation that generalizes (1)

$$x_{n+1} = \beta_n \psi_0(x_n)\psi_1(x_{n-1}) \cdots \psi_k(x_{n-k}), \tag{7}$$

where $\beta_n > 0$ for all $n$ and each $\psi_j$ is a self map of the subgroup $(0, \infty)$ of $\mathbb{C}_0$. The method in [6] is based on the fact that equation (7) can be transformed into additive form using exponential and logarithmic functions. Since the latter functions are homeomorphisms for real numbers and functions, the solutions of (7) can be easily related to those of its additive version.

The following basic lemma uses a *new proof* to extend the result in [6] to arbitrary subgroups of $\mathbb{C}_0$ *without* using the complex exponential and logarithmic functions. This lemma is essential for the derivation of our results on the *expow* equations. For a deeper understanding of the origins of this result we refer to [6].

LEMMA 1. *Let G be a non-trivial subgroup of* $\mathbb{C}_0$ *under ordinary multiplication and assume that*

$$\beta_n, x_{-j} \in G, \quad \psi_j : G \rightarrow G, \quad j = 0, \ldots k, \quad n \geq 1.$$

*If there is* $c \in \mathbb{C}_0$ *such that the following equality holds for all* $z \in G$,

$$\psi_0(z)^{c^k}\psi_1(z)^{c^{k-1}} \ldots, \psi_k(z) = z^{c^{k+1}}, \tag{8}$$

*then* (5) *holds and equation* (7) *has a form symmetry*

$$H(z_0, z_1, \ldots, z_k) = z_0 h_1(z_1) \cdots h_k(z_k), \quad z_0, z_1, \ldots, z_k \in G, \tag{9}$$

*with the functions* $h_j : G \rightarrow \mathbb{C}_0$ *defined as*

$$h_j(z) = z^{c^j}\psi_0(z)^{-c^{j-1}} \cdots \psi_{j-1}(z)^{-1}, \quad j = 1, \ldots k. \tag{10}$$

*With the form symmetry defined by* (9) *and* (10), *the following pair of lower order equations* (*called a semiconjugate factorization*) *is equivalent to* (7):

$$t_{n+1} = \beta_n t_n^c, \quad t_0 = x_0 h_1(x_{-1}) \cdots h_k(x_{-k}). \tag{11}$$

$$y_{n+1} = \frac{t_{n+1}}{h_1(y_n) \ldots h_k(y_{n-k+1})}, \quad y_{-j} = x_{-j}, \; j = 0, 1, \ldots, k-1 \tag{12}$$

.

*Proof*. The unfoldings or associated vector functions $F_n$ of equation (7) are

$$F_n(z_0, z_1, \ldots, z_k) = (\beta_n \psi_0(z_0)\psi_1(z_1)\ldots\psi_k(z_k), \ \ z_0, z_1, \ldots, z_{k-1}).$$

where $(z_0, z_1, \ldots, z_k) \in G^{k+1}$. Therefore, if $H$ is defined by (9) and (10) then

$$\begin{aligned} H(F_n(z_0, z_1, \cdots, z_k)) &= \beta_n\psi_0(z_0)\cdots\psi_k(z_k)h_1(z_0)h_2(z_1)\cdots h_k(z_{k-1}) \\ &= \beta_n\psi_0(z_0)\psi_1(z_1)\cdots\psi_{k-1}(z_{k-1})\psi_k(z_k)\left[z_0^c\psi_0(z_0)^{-1}\right] \\ &\quad \left[z_1^{c^2}\psi_0(z_1)^{-c}\psi_1(z_1)^{-1}\right]\cdots\left[z_{k-1}^{c^k}\psi_0(z_{k-1})^{-c^{k-1}}\cdots\psi_{k-1}(z_{k-1})^{-1}\right] \\ &= \beta_n\psi_k(z_k)z_0^c z_1^{c^2}\psi_0(z_1)^{-c}\cdots z_{k-1}^{c^k}\psi_0(z_{k-1})^{-c^{k-1}}\cdots\psi_{k-1}(z_{k-1})^{-c}. \end{aligned}$$

On the other hand, if we define $\phi_n(z) = \beta_n z^c$ as in (11) for all $z \in G$ and all integers $n \geq 0$ then

$$\begin{aligned} \phi_n(H(z_0, z_1, \cdots, z_k)) &= \beta_n[z_0 h_1(z_1)h_2(z_2)\cdots h_k(z_k)]^c \\ &= \beta_n z_0^c\left[z_1^c\psi_0(z_1)^{-1}\right]^c\cdots\left[z_{k-1}^{c^{k-1}}\psi_0(z_{k-1})^{-c^{k-2}}\cdots\psi_{k-1}(z_{k-1})^{-1}\right]^c[h_k(z_k)]^c \\ &= \beta_n z_0^c z_1^{c^2}\psi_0(z_1)^{-c}\cdots z_{k-1}^{c^k}\psi_0(z_{k-1})^{-c^{k-1}}\cdots\psi_{k-1}(z_{k-1})^{-c}\psi_k(z_k), \end{aligned}$$

where the last equality follows by (8) because

$$[h_k(z_k)]^c = z_k^{c^{k+1}}\psi_0(z_k)^{-c^k}\cdots\psi_{k-1}(z_k)^{-c} = z_k^{c^{k+1}}z_k^{-c^{k+1}}\psi_k(z_k) = \psi_k(z_k).$$

Hence equality (5) holds for each $n$.

To establish the equivalence of (7) to the system of equations (11) and (12) we show that every solution $\{x_n\}$ of (7) corresponds uniquely to a solution $\{(t_n, y_n)\}$ of the system and vice versa. First, let $\{x_n\}$ be the unique solution of (7) generated by a given set of initial values $x_0, x_{-1}, \ldots, x_{-k}$ in $G$ and define the sequence

$$t_n = H(x_n, \ldots, x_{n-k}).$$

Then the semiconjugate relation (5) implies that

$$\begin{aligned} t_{n+1} &= H(x_{n+1}, x_n, \ldots, x_{n-k+1}) \\ &= H(\beta_n\psi_0(x_n)\ldots\psi_k(x_{n-k}), x_n, x_{n-1}, \ldots, x_{n-k+1}) \\ &= H(F_n(x_n, \ldots, x_{n-k})) \\ &= \phi_n(H(x_n, \ldots, x_{n-k})) \\ &= \phi_n(t_n). \end{aligned}$$

Therefore, $\{t_n\}$ is the solution of equation (11) that is uniquely defined by the initial value

$$t_0 = x_0 h_1(x_{-1}) \ \cdots \ h_k(x_{-k}).$$

Further, notice that

$$x_{n+1}h_1(x_n)\cdots h_k(x_{n-k+1}) = H(x_{n+1}, x_n, \cdots, x_{n-k+1}) = t_{n+1},$$

so that (12) holds with $y_n = x_n$ for all $n \geq -k$.

Conversely, let $\{(t_n, y_n)\}$ be the unique solution of the system of equations (11) and (12) with a given set of initial values

$$t_0, y_{-1}, y_{-2}, \ldots, y_{-k} \in G. \tag{13}$$

We note that $t_0$ generates the sequence $\{t_n\}$ which satisfies (11) independently of equation (12). These values $t_n$ then contribute to the calculation of the sequence $\{y_n\}$ which satisfies (12). By the latter equation,

$$t_{n+1} = y_{n+1}h_1(y_n)\cdots h_k(y_{n-k+1}) = H(y_{n+1}, \cdots, y_{n-k+1}).$$

It follows that for all $n \geq 0$, $t_n = H(y_n, \ldots, y_{n-k})$. Now by equations (5), (11), (12) and the definition of $H$,

$$\begin{aligned}
y_{n+1} &= \frac{\phi_n(t_n)}{h_1(y_n)\cdots h_k(y_{n-k+1})} \\
&= \frac{\phi_n(H(y_n, \ldots, y_{n-k}))}{h_1(y_n)\cdots h_k(y_{n-k+1})} \\
&= \frac{H(F_n(y_n, \ldots, y_{n-k}))}{h_1(y_n)\cdots h_k(y_{n-k+1})} \\
&= \frac{H(\beta_n\psi_0(y_n)\cdots\psi_k(y_{n-k}), y_n, y_{n-1}, \ldots, y_{n-k+1})}{h_1(y_n)\cdots h_k(y_{n-k+1})} \\
&= \beta_n\psi_0(y_n)\cdots\psi_k(y_{n-k}).
\end{aligned}$$

It follows that $\{y_n\}$ is a solution of (7) with initial values (13). □

*Remark*. (*Triangular systems*). We refer to equation (12) in Lemma 1 as the *cofactor* equation for (7). The pair of equations (11) and (12) form a *triangular system* that is uncoupled in the sense that equation (11) is independent of the variable $y_n$. Triangular systems have general properties that simplify studying their solutions; for instance, the general structure of the periodic solutions of triangular systems is determined in [1]. Also see [7] for some background on these systems.

## 3. Reductions of order in *expow* equations

The difference equation (12) has order $k$, one less than (7) and its solutions $\{y_n\}$ subject to the conditions of Lemma 1 are identical with the solutions $\{x_n\}$ of (7). In this sense, equation (12) represents a *reduction of order* for (7). The variable coefficient $t_{n+1}$ is calculated independently from the first order equation (11) by induction as stated in the next lemma whose straightforward proof is omitted.

LEMMA 2. *The general solution of equation* (11) *is*

$$t_{n+1} = t_0^{c^{n+1}} \prod_{j=0}^{n} \beta_{n-j}^{c^j}. \tag{14}$$

*If $\beta_n = \beta$ is constant for all $n$ (i.e. (7) is an autonomous equation) then*

$$t_{n+1} = \begin{cases} \rho(t_0/\rho)^{c^{n+1}} & \text{with } \rho = \beta^{1/(1-c)}, \quad \text{if } c \neq 1, \\ t_0\beta^{n+1}, & \text{if } \quad c = 1. \end{cases}$$

To find sufficient conditions on the parameters $a_j, b_j$ that allow a reduction of order of (1) via Lemmas 1 and 2, we require that the functions

$$\psi_j(z) = z^{a_j}\, e^{-b_j z}, \quad j = 0, 1, \ldots, k,$$

satisfy identity (8), i.e. for some $c \in \mathbb{C}_0$ the following must hold for all $z \in G$

$$z^{a_0 c^k} \mathrm{e}^{-b_0 c^k z} z^{a_1 c^{k-1}} \mathrm{e}^{-b_1 c^{k-1} z} \ldots z^{a_k} \mathrm{e}^{-b_k z} = z^{c^{k+1}}. \tag{15}$$

After some rearranging of terms, we see that identity (15) holds for all $z$ if both of the following equations hold:

$$c^{k+1} - a_0 c^k - a_1 c^{k-1} - \cdots - a_{k-1} c - a_k = 0,$$
$$b_0 c^k + b_1 c^{k-1} + \cdots + b_{k-1} c + b_k = 0.$$

For $c \in \mathbb{C}_0$ satisfying both of the above polynomial equations the form symmetry and semiconjugate factorization are determined using the functions

$$h_j(z) = z^{c^j} \psi_0(z)^{-c^{j-1}} \cdots \psi_{j-1}(z)^{-1} = z^{c^j - a_0 c^{j-1} - \cdots - a_{j-1}} \mathrm{e}^{(b_0 c^{j-1} + b_1 c^{j-2} + \cdots + b_{j-1}) z}, \tag{16}$$

as shown in the following result.

Theorem 3. *Assume that the following polynomials*

$$P_0(z) = z^{k+1} - a_0 z^k - a_1 z^{k-1} - \cdots - a_{k-1} z - a_k;$$

$$Q_0(z) = b_0 z^k + b_1 z^{k-1} + \cdots + b_{k-1} z + b_k,$$

*have a common root $c_0 \in \mathbb{C}_0$. Then the following statements are true:*

(*a*) *Equation* (1) *has a reduction of order to the* expow *equation*

$$y_{n+1} = t_{n+1} y_n^{-p_{0,0}} y_{n-1}^{-p_{0,1}} \ldots y_{n-k+1}^{-p_{0,k-1}} \mathrm{e}^{-q_{0,0} y_n - q_{0,1} y_{n-1} - \cdots - q_{0,k-1} y_{n-k+1}}, \tag{17}$$

*where for $j = 0, 1, \ldots, k-1$,*

$$p_{0,j} = c_0^{j+1} - a_0 c_0^j - a_1 c_0^{j-1} - \cdots - a_{j-1} c_0 - a_j, \tag{18}$$

$$q_{0,j} = b_0 c_0^j + b_1 c_0^{j-1} + \cdots + b_{j-1} c_0 + b_j, \tag{19}$$

*and*

$$t_{n+1} = t_0^{c_0^{n+1}} \mathrm{e}^{\sigma_n}, \quad \sigma_n = \sum_{j=0}^{n} \alpha_{n-j} c_0^j, \quad t_0 = x_0 h_1(x_{-1}) \cdots h_k(x_{-k}).$$

(*b*) *Let* $c_1 \in \mathbb{C}_0$ *be also a common root of both* $\mathrm{P}_0$ *and* $\mathrm{Q}_0$. *Then* $\mathrm{c}_1$ *is a root of both of the following polynomials*

$$P_1(z) = z^k + p_{0,0} z^{k-1} + p_{0,1} z^{k-2} + \cdots + p_{0,k-1},$$
$$Q_1(z) = q_{0,0} z^{k-1} + q_{0,1} z^{k-2} + \cdots + q_{0,k-1},$$

*so by Part* (*a*) *the* expow *equation* (17) *has a reduction of order to*

$$z_{n+1} = r_{n+1} z_n^{-p_{1,0}} z_{n-1}^{-p_{1,1}} \ldots z_{n-k+2}^{-p_{1,k-2}} \mathrm{e}^{-q_{1,0} z_n - q_{1,1} z_{n-1} - \cdots - q_{1,k-2} z_{n-k+2}}, \tag{20}$$

*where for* $i = 0, 1, \ldots, k-2$,

$$p_{1,i} = c_1^{i+1} + p_{0,0} c_1^i + p_{0,1} c_1^{i-1} + \cdots + p_{0,i-1} c_1 + p_{0,i},$$
$$q_{1,i} = q_{0,0} c_1^i + q_{0,1} c_1^{i-1} + \cdots + q_{0,i-1} c_1 + q_{0,i},$$

*and*

$$r_{n+1} = r_0^{c_1^{n+1}} \prod_{j=0}^{n} t_{n+1-j}^{c_1^j}.$$

*Proof*. (a) By Lemma 1 the cofactor equation is

$$y_{n+1} = t_{n+1} h_1(y_n)^{-1} \cdots h_k(y_{n-k+1})^{-1}.$$

The proof of this part can now be concluded by induction using (16) and combining various exponents then using Lemma 2 (with $\beta_n = \mathrm{e}^{\alpha_n}$) for the numbers $t_{n+1}$.

(b) By assumption,

$$P_0(c_1) = Q_0(c_1) = 0. \tag{21}$$

Now

$$(z - c_0) P_1(z) = (z - c_0)\left(z^k + \sum_{j=0}^{k-1} p_{0,j} z^{k-j-1}\right)$$
$$= z^{k+1} + \sum_{j=0}^{k-1} [p_{0,j} - c_0 p_{0,j-1}] z^{k-j} - c_0 p_{0,k-1}.$$

where we define $p_{0,-1} = 1$. From (18) for each $j = 0, 1, \ldots, k-1$ we obtain

$$p_{0,j} - c_0 p_{0,j-1} = -a_j,$$

and further, since $P_0(c_0) = 0$,

$$\begin{aligned} c_0 p_{0,k-1} &= c_0\left(c_0^k - a_0 c_0^{k-1} - \cdots - a_{k-1}\right) \\ &= P_0(c_0) + a_k \\ &= a_k. \end{aligned}$$

Thus

$$(z - c_0)P_1(z) = P_0(z),$$

and if $c_1 \neq c_0$ then $P_1(c_1) = 0$ by (21). If $c_1 = c_0$ then $c_0$ is a double root of both $P_0$ and $Q_0$ so that their derivatives are zeros, i.e.,

$$P_0'(c_0) = Q_0'(c_0) = 0. \tag{22}$$

In this case, using (18) we obtain

$$\begin{aligned} P_1(c_0) &= c_0^k + \sum_{j=0}^{k-1} (c_0^{j+1} - a_0 c_0^j - \cdots - a_{j-1}c_0 - a_j)c_0^{k-j-1} \\ &= (k+1)c_0^k - \sum_{j=0}^{k-1} (k-j)a_j c_0^{k-j-1} = P_0'(c_0). \end{aligned}$$

Therefore, if $c_1 = c_0$ then $P_1(c_1) = 0$ by (22). Similar calculations show that $Q_1(c_1) = 0$; thus by Part (a) equation (17) has a semiconjugate factorization.

The factorization is obtained as in Part (a). For equation (20) we need only to change $a_j$ to $-p_{0,j}$ and $b_j$ to $q_{0,j}$ in our hypotheses for each $j = 0, 1, \ldots, k-1$ to obtain the new numbers $p_{1,i}$ and $q_{1,i}$ for (20) as stated in the statement of the theorem. □

The next result applies Theorem 3 to second order *expow* equations.

*Corollary* 4. *Let $a_0, a_1, b_0, b_1, \alpha_n \in \mathbb{C}$ for $n \geq 1$ such that*

$$b_1^2 + a_0 b_0 b_1 - a_1 b_0^2 = 0, \quad b_1 \neq 0. \tag{23}$$

*Then the second order* expow *equation*

$$x_{n+1} = x_n^{a_0} x_{n-1}^{a_1} \mathrm{e}^{\alpha_n - b_0 x_n - b_1 x_{n-1}}, \tag{24}$$

*has a reduction of order to the first order equation*

$$\begin{aligned} y_{n+1} &= t_0^{c^{n+1}} y_n^{a_0 - c} \mathrm{e}^{\sigma_n - b_0 y_n}, \quad c = -b_1/b_0, \quad y_0 = x_0, \\ \sigma_n &= \sum_{j=0}^{n} \alpha_{n-j} c^j, \quad t_0 = x_0 x_{-1}^{c-a_0} e^{b_0 x_{-1}}. \end{aligned} \tag{25}$$

*In particular, if $a_0, a_1, b_0, b_1, \alpha_n \in \mathbb{R}$ for $n \geq 1$ then $c \in \mathbb{R}$ and reduction of order is defined on $\mathbb{R}$.* □

*Proof*. First we note that conditions (23) imply that $b_0$ and at least one of $a_0, a_1$ are nonzero. Now $c = -b_1/b_0$ is the unique nonzero root of $Q_0$ and by the equality in (23),

$$c^2 - a_0 c - a_1 = \frac{b_1^2}{b_0^2} + a_0 \frac{b_1}{b_0} - a_1 = 0.$$

So the above value of $c$ is also a root of $P_0$. Now the proof is completed by applying Theorem 3. The last assertion is obvious. □

The next corollary gives reduction of order for a generalization of equation (2) that includes an arbitrary delay in the exponential term and an arbitrary sequence $\{\alpha_n\}$. In this case, like Corollary 4, the reductions of order are defined over the real numbers if the parameters are real.

COROLLARY 5. *Let $k \geq 1$, $b_0, b_k \in \mathbb{C}$ with $b_k \neq 0$ and $\alpha_n \in \mathbb{C}$ for $n \geq 0$. Consider the* expow *delay equation:*

$$x_{n+1} = x_{n-1} \mathrm{e}^{\alpha_n - b_0 x_n - b_k x_{n-k}}. \tag{26}$$

(*a*) *Equation* (26) *has two possible reductions of order:* (*i*) *If $b_k = -b_0$ then*

$$\begin{aligned} y_{n+1} &= \frac{t_0}{y_n} \mathrm{e}^{\sigma_n - b_0 y_n - b_0 y_{n-1} - \cdots - b_0 y_{n-k+1}}, \\ \sigma_n &= \sum_{j=0}^{n} \alpha_j, \quad t_0 = x_0 x_{-1} \mathrm{e}^{b_0 (x_{-1} + x_{-2} + \cdots + x_{-k+1})}. \end{aligned} \tag{27}$$

(*ii*) *If* k *is odd and $b_k = b_0$ then*

$$\begin{aligned} y_{n+1} &= t_{n+1} y_n \mathrm{e}^{-b_0 y_n + b_0 y_{n-1} - \cdots + (-1)^{k-1} b_0 y_{n-k+1}}, \\ t_{n+1} &= t_0^{(-1)^{n+1}} \mathrm{e}^{\sigma_n}, \quad \sigma_n = \sum_{j=0}^{n} (-1)^j \alpha_{n-j}, \quad t_0 = \frac{x_0}{x_{-1}} \mathrm{e}^{b_0 (x_{-1} - x_{-2} + \cdots + (-1)^{k-1} x_{-k+1})}. \end{aligned} \tag{28}$$

(*b*) *If k is even then the* expow *equation* (27) *has a further reduction of order to*

$$\begin{aligned} z_{n+1} &= r_{n+1} \mathrm{e}^{-b_0 z_n + b_0 z_{n-2} - b_0 z_{n-4} + \cdots + (-1)^{k-1} b_0 z_{n-k+2}}, \\ r_{n+1} &= r_0^{(-1)^{n+1}} \prod_{j=0}^{n} t_{n+1-j}^{(-1)^j}, \quad r_0 = x_0 \mathrm{e}^{b_0 (x_0 - x_{-2} + \cdots + (-1)^k x_{-k+2})}. \end{aligned}$$

(*c*) *If $b_0, b_k \in \mathbb{R}$ with $b_k \neq 0$ and $\alpha_n \in \mathbb{R}$ for all n then the reductions of order in Parts* (*a*) *and* (*b*) *are defined on* $\mathbb{R}$.

*Proof*. (a) Since $a_1 = 1$ and $a_j = 0$ for $j \neq 1$, we have

$$P_0(c) = c^{k+1} - c^{k-1} = 0 \Rightarrow c^{k-1}(c^2 - 1) = 0 \Rightarrow c = \pm 1.$$

If $c = 1$ then $Q_0(c) = b_0 + b_k = 0$ or $b_k = -b_0$. Applying Theorem 3 with $c = 1$ yields the claimed reduction of order. The other order reduction for odd $k$ is obtained similarly with $c = -1$.

(b) In equation (27) which has order $k$ the new parameter values are $a_0 = -1$ and $a_j = 0$ for $j > 0$. Now $P_0(c) = c^k + c^{k-1} = 0$ yields a nonzero value $c = -1$. This must also be a root of $Q_0$; since the new values of the parameters $b_j$ are $b_j = b_0$ for $j = 0, \ldots, k-1$ we must have

$$(-1)^{k-1}b_0 + (-1)^{k-2}b_0 + \cdots + (-1)b_0 + b_0 = 0. \tag{29}$$

With $b_0 \neq 0$ (29) holds if and only if $k$ is even, in which case applying Theorem 3 with $c = -1$ yields the stated reduction of order for (27).

(c) This is clear, since $c = \pm 1$ is real in Parts (a) and (b). □

The following result involves a more general delay pattern than that in Part (a)(i) of Corollary 5, i.e., when the parameters $b_0, b_k$ have equal magnitudes but opposite signs.

Corollary 6. *For $k, m \geq 1$, $\beta \in \mathbb{C}_0$ and $\alpha_n \in \mathbb{C}$ for $n \geq 0$ the* expow *delay equation*

$$x_{n+1} = x_{n-m}\mathrm{e}^{\alpha_n + \beta x_n - \beta x_{n-k}}, \tag{30}$$

*has a reduction of order to*

$$y_{n+1} = \frac{t_0 \mathrm{e}^{\sigma_n - \beta(y_n + y_{n-1} + \cdots + y_{n-k+1})}}{y_n y_{n-1} \cdots y_{n-m+1}}, \quad \text{where :}$$

$$\sigma_n = \sum_{j=0}^{n} \alpha_j, \quad t_0 = x_0 x_{-1} \cdots x_{-m+1} \mathrm{e}^{\beta(x_{-1} + x_{-2} + \cdots + x_{-k+1})}.$$

*For $m = 0$ equation* (30) *has a reduction of order to*

$$y_{n+1} = t_0 \mathrm{e}^{\sigma_n - \beta(y_n + y_{n-1} + \cdots + y_{n-k+1})}.$$

*Proof*. First assume that $k \geq m \geq 1$. Then

$$P_0(c) = c^{k+1} - a_m c^{k-m} = c^{k-m}(c^{m+1} - 1), \quad Q_0(c) = -\beta c^k + \beta.$$

Clearly $c = 1$ is a common nonzero root of $P_0$ and $Q_0$ so Theorem 3 can be applied to obtain the order reduction by calculating

$$p_{0,j} = \begin{cases} c^{j+1} = 1, & \text{if } j < m \\ c^{j+1} - a_m c^{j-m} = 0 & \text{if } j \geq m, \end{cases}$$

and

$$q_{0,j} = \begin{cases} b_0 c^j = -\beta, & \text{if } j < k \\ b_0 c^j + b_k = -\beta + \beta = 0, & \text{if } j = k. \end{cases}$$

Next, if $m > k \geq 1$ then

$$P_0(c) = c^{m+1} - a_m = c^{m+1} - 1, \quad Q_0(c) = -\beta c^m + \beta c^{m-k} = -\beta c^{m-k}(c^k - 1).$$

Again $c = 1$ is a common nonzero root of $P_0$ and $Q_0$, so Theorem 3 can be applied similarly to the preceding case.

In the case $m = 0$ we have

$$P_0(c) = c^{k+1} - a_0 c^k = c^k(c-1), \quad Q_0(c) = -\beta c^k + \beta.$$

Thus $c = 1$ is a common nonzero root and the proof is completed by applying Theorem 3. □

We note that equation (30) is not a generalization of equation (2). The following result generalizes the delay pattern in both equation (2) and in Part (a)(ii) of Corollary 5, i.e., when the parameters $b_0, b_k$ are equal. The proof follows the same argument as in the preceding corollary by showing that $c = -1$ is a common root of $P_0$ and $Q_0$ We omit the details of this proof.

Corollary 7. *For* $k, m \geq 1$, $\beta \in \mathbb{C}_0$ *and* $\alpha_n \in \mathbb{C}$ *for* $n \geq 0$ *the* expow *delay equation*

$$x_{n+1} = x_{n-2m+1} e^{\alpha_n - \beta x_n - \beta x_{n-2k+1}}, \tag{31}$$

*has a reduction of order to*

$$y_{n+1} = \frac{t_0^{(-1)^{n+1}} y_n y_{n-2} \cdots y_{n-2m+2}}{y_{n-1} y_{n-3} \cdots y_{n-2m+3}} e^{\sigma_n - \beta[y_n - y_{n-1} + \cdots - y_{n-2k+2}]}, \quad \textit{where}:$$

$$\sigma_n = \sum_{j=0}^{n} (-1)^j \alpha_{n-j}, \quad t_0 = \frac{x_0 x_{-2} \cdots x_{-2m+2}}{x_{-1} x_{-3} \cdots x_{-2m+3}} e^{\beta(x_{-1} - x_{-2} + \cdots + x_{-2k+1})}.$$

Example 12 below uses the order reduction in Corollary 7 to study the positive solutions of a special case of equation (31).

## 4. Repeated reductions of order

It is of interest that equation (17) in Theorem 3 is again an *expow* equation similar to (1). This suggests that the methods of the preceding section can be applied to (17). Indeed, under the conditions in Part (b) of Theorem 3 a further reductions of order was obtained by repeating the semiconjugate factorization process. A similar situation was encountered in Part (b) of Corollary 5.

The main impediment to repeatedly using Theorem 3 in this way is the requirement that the polynomials $P_0$ and $Q_0$ have *common* nonzero roots. The next result illustrates a special case in which $P_0$ and $Q_0$ reduce to a single polynomial and thus, the factorization process continues until a triangular system of first order equations is obtained. In this way the original equation of order $k+1$ is reduced to an equation of order one.

Corollary 8. *The* expow *equation*

$$x_{n+1} = x_n^{a_0} x_{n-1}^{a_1}, \ldots, x_{n-k+1}^{a_{k-1}} e^{\alpha_n + b x_n - a_0 b x_{n-1} - \cdots - a_{k-1} b x_{n-k}}. \tag{32}$$

*where for* $j = 0, 1, \ldots, k-1$ *and all* $n \geq 0$*,*

$$a_j, \alpha_n, b \in \mathbb{C}, a_{k-1} \neq 0.$$

*has a complete semiconjugate factorization into a triangular system of first order equations over* $\mathbb{C}_0$.

*Proof*. First assume that $b \neq 0$. For equation (23) setting $P_0(c) = Q_0(c) = 0$ gives

$$c^{k+1} - a_0 c^k - a_1 c^{k-1} - \cdots - a_{k-1} c = 0, \quad bc^k - ba_0 c^{k-1} - \cdots - ba_{k-1} = 0. \quad (33)$$

Since $a_{k-1} \neq 0$ the nonzero roots of the above polynomials are identical to the zeros of the following polynomial

$$P(c) = c^k - a_0 c^{k-1} - \cdots - a_{k-1}.$$

Note that *every* root of $P_{(c)}$ is nonzero and a root of *both* $P_0$ and $Q_0$. Therefore, by Theorem 3, not only equation (32) has a semiconjugate factorization, but also if $k > 1$ then the factorization process continues until the order of the cofactor equation is reduced to one.

Finally, if $b = 0$ (no exponential functions) then equation (33) reduces to a trivial identity so once again only one polynomial $P(c)$ remains.

We point out that by denoting $\mathrm{e}^{\alpha_n} = \gamma_n$ and $e^{-b} = \beta$, equation (32) can be written in the following more symmetric form

$$x_{n+1} = \gamma_n x_n^{a_0} x_{n-1}^{a_1} \cdots x_{n-k+1}^{a_{k-1}} \beta^{-x_n} \beta^{a_0 x_{n-1}} \cdots \beta^{a_{k-1} x_{n-k}}.$$

□

*Example* 9. For each positive integer $k$, the *expow* delay equation

$$x_{n+1} = x_{n-k+1}^{a} \mathrm{e}^{\alpha_n + bx_n - abx_{n-k}}, \quad a, b, \alpha_n \in \mathbb{R}, a, b \neq 0, \quad (34)$$

is clearly of type (32), so by Corollary 8 it has a complete semiconjugate factorization into a triangular system of first order equations. The polynomial $P(c) = c^k - a$ in this case so the common nonzero roots of $P_0$ and $Q_0$ are just the $k$-th roots of $a$.

In particular, if $k = 2$ then the polynomial $P(c) = c^2 - a$ has roots $\pm\sqrt{a}$. Using Theorem 3 we obtain the following system of first order equations

$$t_{n+1} = \mathrm{e}^{\alpha_n} t_n^{\sqrt{a}}, \quad r_{n+1} = t_{n+1} r_n^{-\sqrt{a}}, \quad y_{n+1} = r_{n+1} \mathrm{e}^{by_n}.$$

Note that if $a < 0$ then the above system is defined over the complex numbers even though all parameters in equation (34) are real. This situation is analogous to the case of a linear difference equation having complex eigenvalues; see [6] for full semiconjugate factorizations of linear equations into triangular systems of first order equations.

## 5. Positive solutions

This section applies some of the results obtained in previous sections to study the positive solutions of certain *expow* equations subject to conditions (4). We discuss the asymptotic behaviours of solutions in two examples below, where direct analysis (without order reduction) seems to be more difficult.

The first example examines a slightly modified version of equation (2) that nevertheless shows a markedly different behaviour. The semiconjugate factorization makes transparent the root causes of this substantial change in behaviour that might be difficult to explain otherwise.

*Example* 10. By way of comparison with equation (2) consider the following *expow* equation

$$x_{n+1} = x_{n-1}\mathrm{e}^{a+bx_n-bx_{n-1}}, \quad a \in \mathbb{R}, \ b > 0, \tag{35}$$

which is similar to (2) except for a sign change in the exponent. By Corollary 5, equation (35) reduces to the first order equation:

$$y_{n+1} = \frac{t_0}{y_n}\mathrm{e}^{(n+1)a+by_n}, \quad t_0 = x_0x_{-1}\mathrm{e}^{-bx_{-1}}. \tag{36}$$

Since $t_0$ and $y_0 = x_0$ are positive, if $a > 0$ then evidently each solution of (36) is unbounded. If $a = 0$ then because the function $\mathrm{e}^{bu}/u$ is unimodal with a single minimum, the equation

$$y_{n+1} = \frac{t_0}{y_n}\mathrm{e}^{by_n} = \frac{x_0x_{-1}\mathrm{e}^{by_n-bx_{-1}}}{y_n}, \tag{37}$$

has two, one or no fixed points depending on how large the initial value $t_0$ is; in particular, for sufficiently small values of $t_0$ there are two fixed points.

This bifurcation is not so transparent in a direct examination of the second order equation (35) but it has a significant effect on the asymptotic behaviours of the solutions of that equation because the appearance of the fixed point in (37) yields bounded solutions for equation (35) when $a = 0$. In this case, using straightforward analysis it is possible to determine the regions of initial points $(x_{-1}, x_0)$ in the Euclidean plane that imply the occurrence of a particular asymptotic behaviour. In all cases, solutions are either unbounded or they converge to a positive fixed point. Finally, if $a < 0$ then we may in addition have convergence to zero. In no case are the complex behaviours exhibited by the solutions of equation (2) observed.

Next we consider a more general, non-autonomous version of (2) with $\alpha$ replaced by a periodic sequence $\alpha_n$. We first present a lemma to facilitate the discussion of the non-autonomous equation below.

LEMMA 11. *Let $\{\sigma_n\}$ be a periodic sequence of positive numbers with minimal period $p \geq 1$ and let $\{t_n\}$ be a solution of the difference equation*

$$t_{n+1} = \frac{\sigma_n}{t_n}, \tag{38}$$

*for a given initial value $t_0 > 0$. Then the following are true:*

(*a*) *If p is odd then $\{t_n\}$ is periodic with period 2p* (*not necessarily minimal*).
(*b*) *If p is even and*

$$\sigma_0\sigma_2\ldots\sigma_{p-2} = \sigma_1\sigma_3\ldots\sigma_{p-1}, \tag{39}$$

*then $\{t_n\}$ is periodic with period $p$.*
(*c*) *If $p$ is even but* (39) *does not hold then $\{t_n\}$ has a subsequence that decreases to zero and another subsequence that increases to infinity.*

*Proof*. (a) Using straightforward induction we find that

$$t_n = \frac{t_0\sigma_1\sigma_3\cdots\sigma_{n-3}\sigma_{n-1}}{\sigma_0\sigma_2\cdots\sigma_{n-4}\sigma_{n-2}}, \quad \text{if } n \text{ is even and;} \tag{40}$$

$$t_n = \frac{\sigma_0\sigma_2\cdots\sigma_{n-3}\sigma_{n-1}}{t_0\sigma_1\sigma_3\cdots\sigma_{n-4}\sigma_{n-2}}, \quad \text{if } n \text{ is odd.} \tag{41}$$

If $p = 2q + 1$ is odd then

$$\sigma_{2q+1+j} = \sigma_j \quad \text{for } j = 0, 1, \ldots, 2q. \tag{42}$$

Further, $t_{2p} = t_{4q+2}$ has even index so by (40)

$$\begin{aligned} t_{2p} &= \frac{t_0\sigma_1\sigma_3\cdots\sigma_{2q-3}\sigma_{2q-1}\sigma_{2q+1}\sigma_{2q+3}\cdots\sigma_{4q-1}\sigma_{4q+1}}{\sigma_0\sigma_2\cdots\sigma_{2q-2}\sigma_{2q}\sigma_{2q+2}\sigma_{2q+4}\cdots\sigma_{4q-2}\sigma_{4q}} \\ &= t_0\left(\frac{\sigma_1\sigma_3\cdots\sigma_{2q-3}\sigma_{2q-1}}{\sigma_{2q+2}\sigma_{2q+4}\cdots\sigma_{4q-2}\sigma_{4q}}\right)\left(\frac{\sigma_{2q+1}\sigma_{2q+3}\cdots\sigma_{4q-1}\sigma_{4q+1}}{\sigma_0\sigma_2\cdots\sigma_{2q-2}\sigma_{2q}}\right) = t_0, \end{aligned} \tag{43}$$

where the last equality holds because each of the two ratios in (43) equals 1 by (42). It follows that $\{t_n\}$ has period $2p$. This may not be a prime period; for example, if $\sigma_j = \sigma$ for $j = 0, 1, \ldots, p-2$ where $\sigma > 0$ and $\sigma \neq 1$ and if also $\sigma_{p-1} = t_0 = 1$ then by (41)

$$t_p = \frac{\sigma_0\sigma_2\cdots\sigma_{p-3}\sigma_{p-1}}{t_0\sigma_1\sigma_3\cdots\sigma_{p-4}\sigma_{p-2}} = \frac{\sigma^{(p-1)/2}}{\sigma^{(p-1)/2}} = 1 = t_0,$$

i.e., $\{t_n\}$ has period $p$.

(b) Suppose that $p$ is even. Then again by (40)

$$t_p = t_0\frac{\sigma_1\sigma_3\cdots\sigma_{p-3}\sigma_{p-1}}{\sigma_0\sigma_2\cdots\sigma_{p-4}\sigma_{p-2}} = t_0,$$

where the last equality is true by (39).

(c) If $p$ is even then as in Part (b)

$$t_p = t_0\frac{\sigma_1\sigma_3\cdots\sigma_{p-3}\sigma_{p-1}}{\sigma_0\sigma_2\cdots\sigma_{p-4}\sigma_{p-2}}. \tag{44}$$

Since $\sigma$ has even period $p$, by (40)

$$t_{2p} = t_0\left(\frac{\sigma_1\sigma_3\cdots\sigma_{p-3}\sigma_{p-1}}{\sigma_0\sigma_2\cdots\sigma_{p-4}\sigma_{p-2}}\right)\left(\frac{\sigma_{p+1}\sigma_{p+3}\ldots\sigma_{2p-3}\sigma_{2p-1}}{\sigma_p\sigma_{p+2}\ldots\sigma_{2p-4}\sigma_{2p-2}}\right) = t_0\left(\frac{\sigma_1\sigma_3\cdots\sigma_{p-3}\sigma_{p-1}}{\sigma_0\sigma_2\cdots\sigma_{p-4}\sigma_{p-2}}\right)^2.$$

Inductively, we see that

$$t_{mp} = t_0 \left( \frac{\sigma_1 \sigma_3 \cdots \sigma_{p-3} \sigma_{p-1}}{\sigma_0 \sigma_2 \cdots \sigma_{p-4} \sigma_{p-2}} \right)^m . \tag{45}$$

If (39) does not hold then the ratio inside the parentheses in (45) is either greater than one or less than one. If greater than one then $t_{mp} \longrightarrow \infty$ monotonically as $m \longrightarrow \infty$ and

$$t_{mp+1} = \frac{\sigma_{mp}}{t_{mp}} = \frac{\sigma_0}{t_{mp}} \longrightarrow 0.$$

Similarly, if the ratio in (45) is less than one then $t_{mp} \longrightarrow 0$ and $t_{mp+1} \longrightarrow \infty$ monotonically as $m \longrightarrow \infty$. Thus the proof of (c) is complete. □

*Example* 12. Consider the *expow* equation

$$x_{n+1} = x_{n-1} \mathrm{e}^{\alpha_n - bx_n - bx_{n-1}}, \quad x_0, x_{-1}, \ b > 0, \tag{46}$$

where $\alpha_n$ is a periodic sequence of real numbers with minimal or prime period $p \geq 1$. By Corollary 7 with $k = m = 1$ this equation reduces in order to

$$y_{n+1} = t_{n+1} y_n \mathrm{e}^{-by_n}, \quad \text{where} \ \ t_{n+1} = \frac{\mathrm{e}^{\alpha_n}}{t_n}.$$

If $p$ is either odd or it is even and satisfies (39) then by Lemma 11 the solution $\{t_n\}$ of the factor equation above is periodic with period $p$ or $2p$. A full cycle of $\{t_n\}$ consists of $q$ distinct points where $q \leq 2p$. It follows that each orbit of (46) is confined to $q$ distinct curves of type

$$\xi_i(u) = t_i u \mathrm{e}^{-bu},$$

in the plane where the $t_i$ are the distinct values of $t_n$. For sufficiently large values of $\alpha_n$ some of the mappings $\xi_i$ are chaotic (due to their having a period 3 solution or a snap-back repeller; see e.g. [2,6]). Thus the collection of points $(y_{n-1}, y_n)$ on each curve $\xi_i$ tends to be densely distributed on a segment of $\xi_i$. These dense patches are visibly highlighted in a numerically generated plot of the orbit of (46) in its state space (or 'phase plane'). Figure 1 depicts this situation for (46) with $b = 1$ and

$$\alpha_n = 5 + 0.6 \sin \frac{\pi n}{3},$$

which has period 6. Thus $\mathrm{e}^{\alpha_n}$ also has period 6 and further, it satisfies (39) with

$$\mathrm{e}^{\alpha_1} \mathrm{e}^{\alpha_3} \mathrm{e}^{\alpha_5} = \mathrm{e}^{\alpha_2} \mathrm{e}^{\alpha_4} \mathrm{e}^{\alpha_6} = 3269017.37.$$

The sequence $\{t_n\}$ has only 4 distinct points per cycle:

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $t_n$ | 8.16 | 30.6 | 8.16 | 18.2 | 4.85 | 18.2 |

so that we observe patches on only four distinct curves in Figure 1. We note that these patches are not continuous curves, but made of 20, 000 tightly packed points. Clearly this

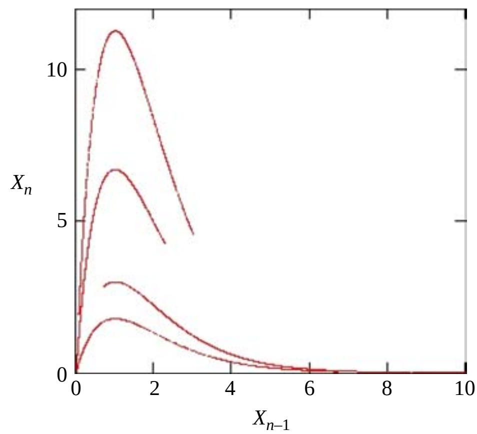


Figure 1. A non-periodic phase plane orbit of equation (46) with periodic $\alpha_n$.

is not a periodic orbit for equation (46). For smaller values of $\alpha_n$ for which the mappings $\xi_i$ have periodic points, numerical simulations show that the solutions of (46) are also periodic; these periods must be integer multiples of $p$ by the preceding argument.

If $p$ is even but (39) is not satisfied then the unbounded subsequences of $\{t_n\}$ cause a spread of points in the orbit of (46) because now there are an infinite number of curves like the $\xi_i$ above. Figure 2 shows a numerically generated orbit of this type where the sequence

$$\alpha_n = \ln\left(120 + 20\sin\frac{\pi n}{4}\right),$$

has period 8 with

$$e^{\alpha_1}e^{\alpha_3}e^{\alpha_5}e^{\alpha_7} = 201640000, \quad e^{\alpha_2}e^{\alpha_4}e^{\alpha_6}e^{\alpha_8} = 201600000.$$

The planar orbit in Figure 2 is a plot of 60000 points ($b = 1$). As more points are generated numerically and plotted, the peak of the cone will rise without bound since the sequence $\{t_n\}$ is unbounded.

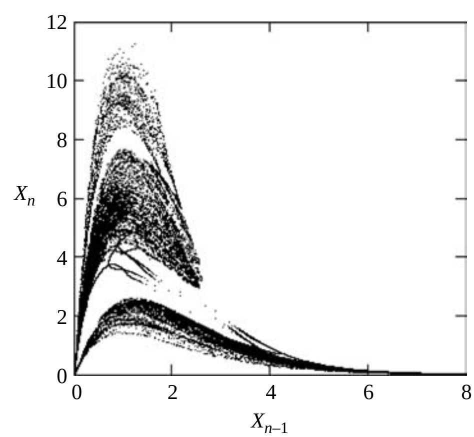


Figure 2. A phase plane orbit of equation (46) with $\alpha_n$ still periodic.

## 6. Conclusion and future directions

Our study of *expow* equations above points to a large variety of equations with semiconjugate factorizations and thus, reducible in order. We have also shown that such equations are capable of generating rich dynamic behaviours. Further, the behaviours of positive solutions in the last two examples above might be difficult to explain without the reduction of order that results from the semiconjugate factorization. On the other hand, the interested reader will have noticed that many types of *expow* equations that are amenable to analysis using Lemma 1 and Theorem 3 have not been considered in this first study. Some of those equations and the behaviours of their solutions on various subgroups of $\mathbb{C}_0$ (e.g., $(0, \infty)$ or the circle group $\mathbb{T}$) can provide significant challenges and rewards in the future studies of *expow* equations.

Finally, many *expow* equations do not satisfy identity (8) and, therefore, the above discussion is not applicable to such equations. Whether the latter types of *expow* equations possess form symmetries and semiconjugate factorizations of a different kind remains an open question.

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