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# Periodic and non-periodic solutions in a Ricker-type second-order equation with periodic parameters

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**ABSTRACT**

We study the dynamics of the positive solutions of the exponential difference equation

$$x_{n+1} = x_{n-1}e^{a_n - x_{n-1} - x_n}$$

where the sequence $\{a_n\}$ is periodic. We find that qualitatively different dynamics occurs depending on whether the period $p$ of $\{a_n\}$ is odd or even. If $p$ is odd then periodic and non-periodic solutions coexist (with different initial values) if the amplitudes of the terms $a_n$ are allowed to vary over a sufficiently large range. But if $p$ is even then all solutions converge to an asymptotically stable limit cycle of period $p$ if either all the odd-indexed or all the even-indexed terms of $\{a_n\}$ are less than 2, and the sum of the even terms of $\{a_n\}$ does not equal the sum of its odd terms. The key idea in our analysis that explains this behavioural dichotomy is a semiconjugate factorization of the above equation into a triangular system of two first-order equations.

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## 1. Introduction

We study the behaviour of solutions of the second-order difference equation

$$x_{n+1} = x_{n-1}e^{a_n - x_{n-1} - x_n} \tag{1}$$

where the parameter $\{a_n\}$ is a periodic sequence of real numbers. This equation is a special case of a stage-structured population model with a Ricker-type recruitment function; see [3]. For more information and additional Ricker-type models see, e.g. [5–7,11].

Equation (1) has a rich variety of periodic and non-periodic solutions. It exhibits coexisting periodic solutions if the range of variation, or amplitude of $a_n$ is limited. We also show that an expanded range or greater amplitudes for $a_n$ leads to the occurrence of coexisting non-periodic solutions (including chaotic solutions). The period $p$ of the sequence $\{a_n\}$ is the natural parameter to consider in this study.

In addition, there is an unexpected dichotomy between the behaviour of solutions when the period $p$ of $\{a_n\}$ is odd and when $p$ is even. When $p$ is odd different periodic and non-periodic stable solutions are generated from different pairs of initial values. But if $p$ is even

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and the amplitude of $a_n$ is less than 2 then asymptotically stable $p$-cycles occur in a generic fashion, i.e. independently of initial values.

Such differences in behaviour among the solutions of (1) are readily explained by a semiconjugate factorization of (1) into a triangular system of two first-order difference equations. The latter pair of equations determine the full structure of (1) and allow us to explain the aforementioned dichotomy as follows: One of the two first-order equations has periodic, hence bounded solutions when $p$ is odd; however, if $p$ is even then the solutions of the same first-order equation are unbounded (except for a boundary case). Thus the corresponding *bounded* solutions of (1) 'forget' the initial values and approach a single asymptotically stable solution.

In the next section we list the main results with some elaboration. The proofs of these results and the supporting material are provided in the following section. We finally conclude with a summary and some possible future directions.

## 2. The main results

The solutions of (1) exhibit qualitatively different behaviour depending on whether the period of the parameter sequence $\{a_n\}$ is odd or even. Accordingly, this section is divided into two subsections where we list the main results for each case. The proofs and other relevant material are presented in the next section.

### 2.1. The odd period case

There are two main results for the case where the sequence $\{a_n\}$ has an odd minimal (or prime) period. The first shows that if each $a_n$ is bounded by 2 then we can expect solutions of (1) to converge to solutions of period $2p$ or, in exceptional cases, period $p$. The second main result of this section shows that if $a_n > 2$ for some $n$ then (1) has periodic solutions of periods other than $p$ or $2p$ and if the range of variation of $a_n$ is large enough then nonperiodic and chaotic solutions also exist.

The following result on the existence of periodic solutions is a consequence of Theorem 12 below.

**Theorem 1:** *Let $\{a_n\}$ be periodic with minimal odd period $p$ and assume that $0 < a_i < 2$ for $i = 0, \ldots, p-1$.*

(a) *Each solution of (1) converges to a cycle with length $2p$ that depends on the initial values $x_{-1}, x_0 > 0$;*
(b) *If $x_0 = x_{-1}e^{-\sigma/2 - x_{-1}}$ where*

$$\sigma = -a_0 + a_1 - a_2 + \ldots - a_{p-1} \tag{2}$$

*then the solutions of (1) converge to a cycle of length $p$.*

It is important to note that the periodic solutions in Theorem 1 may be distinct if the initial values are distinct, so these cycles are not locally stable (i.e. they are not ordinary limit cycles). Thus we see that (1) exhibits multistability in this case. Figures 1 and 2 illustrate this situation for period $p = 3$ with

$$a_0 = 1, \quad a_1 = 1.9, \quad a_2 = 0.8$$

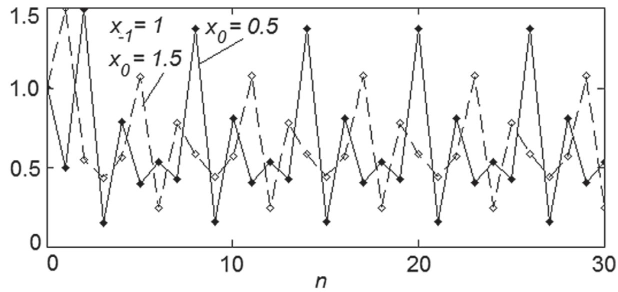


**Figure 1.** Coexisting period 6 solutions with parameter period $p = 3$.

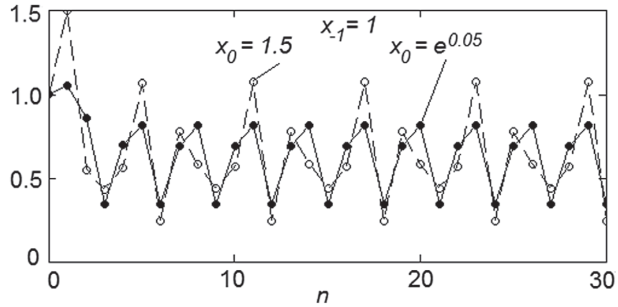


**Figure 2.** Coexisting period 6 and period 3 (exceptional) solutions, $p = 3$.

This multistable behaviour resembles neutral stability seen in nonhyperbolic systems although it is clearly not neutral stability due to the fact that each periodic solution attracts a certain number of solutions and is thus a limit cycle of some type.

A similar situation exists when $a_n$ exceeds 2 and is discussed in the next result where the range of $\{a_n\}$ is not restricted.

**Theorem 2:** *Suppose that $\{a_n\}$ is periodic with minimal odd period $p \geq 1$ and let $f$ be the interval map in Lemma 13 below and assume that $t_0 > 0$ is a fixed real number.*

(a) *If $s$ is a periodic point of $f$ with period $\omega$ then all solutions of (1) with initial values $x_{-1} = s$ and $x_0 = t_0 s e^{-s}$ have period $2p\omega$.*
(b) *If $t_0 = e^{-\sigma/2}$ and $s$ is a periodic point of $f$ with period $\omega$, then all solutions of (1) with initial values $x_{-1} = s$ and $x_0 = s e^{-\sigma/2-s}$ have period $p\omega$.*
(c) *If the map $f$ has a non-periodic point, then (1) has a non-periodic solution.*
(d) *If $f$ has a period-three point then (1) has periodic solutions of period $2pn$ for all positive integers $n$ as well as chaotic solutions in the sense of Li–Yorke* ([1,4]).

In the case $p = 1$, i.e. when (1) is autonomous with $a_n = a$ for all $n$, the conditions stated in Theorem 2 were examined in [3] where it was also verified that if $a \geq 3.13$ then (1) has chaotic solutions from certain initial conditions. The multistable nature of solutions of (1) was also discussed in detail to distinguish them from both locally stable and neutrally stable solutions.

For odd $p \geq 3$ the associated interval map $f$ is a composition of $2p$ functions and therefore, analytically less tractable than the autonomous case. We use numerical simulations to

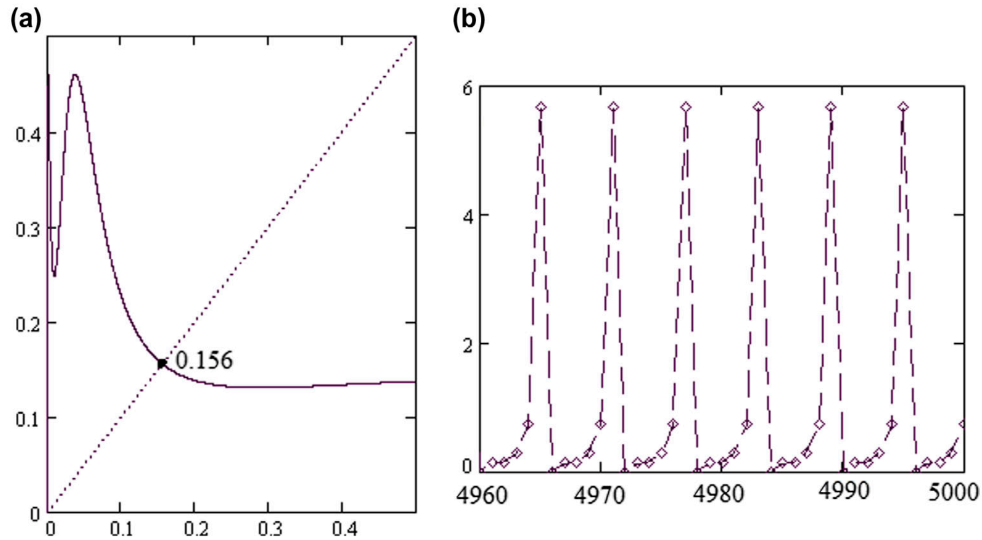


**Figure 3.** (a) Graph of $f$; (b) Corresponding 6-cycle.

highlight the rich variety of coexisting solutions that Theorem 2 indicates. As noted above, and explained in greater detail in [3], these solutions are attracting (though not locally stable) so they are observable and may be recorded numerically.

In the next four figures, $p = 3$ with

$$a_0 = 1, \quad a_1 = 2, \quad a_2 = 4.$$

The initial values that generate the 6-cycle in Figure 3 are $x_{-1} = 1$ and $x_0 = 0.8$. Panel (a) shows the map $f$ (a composition of six exponential maps) together with a single stable positive fixed point that corresponds to the solution of (1) shown in Panel (b).

In Figure 4 the initial values are $x_{-1} = x_0 = 1$. In Panel (a) the graphs of $f$ and $f^2$ are shown that indicate the presence of a stable 2-cycle (the fixed point of $f$ is unstable in this case). Panel (b) shows the corresponding 12-cycle for (1), as required by Theorem 2 with $\omega = 2$. We emphasize that this solution coexists stably, in the sense of [3], with the 6-cycle in Figure 3. We further note that 12-cycles do not exist under the hypotheses of Theorem 1 which imply the stability of the fixed point of $f$.

In Figure 5 the initial values are $x_{-1} = 1$ and $x_0 = 3.8$. In Panel (a) the graphs of $f$ and $f^3$ are shown where a stable 3-cycle is indicated that corresponds to the solution of (1) that is shown in Panel (b). As stated in Theorem 2 this is an 18-cycle since now $\omega = 3$. This solution coexists stably, in the sense of [3], with the 6-cycle and the 12-cycle above.

Finally, in Figure 6 the initial values are $x_{-1} = 1$ and $x_0 = 6$. Panel (a) shows the graphs of $f$ and $f^3$ where we can identify a pair of unstable 3-cycles where the graph of $f^3$ crosses the identity line (in addition to the unstable fixed point of $f$). The map $f$ then exhibits Li–Yorke type chaos. A portion of the plot of the corresponding solution of (1) is shown in Panel (b). This nonperiodic solution coexists stably, in the sense of [3], with the periodic solutions mentioned above. However, nonperiodic solutions do not exist under the hypotheses of Theorem 1.

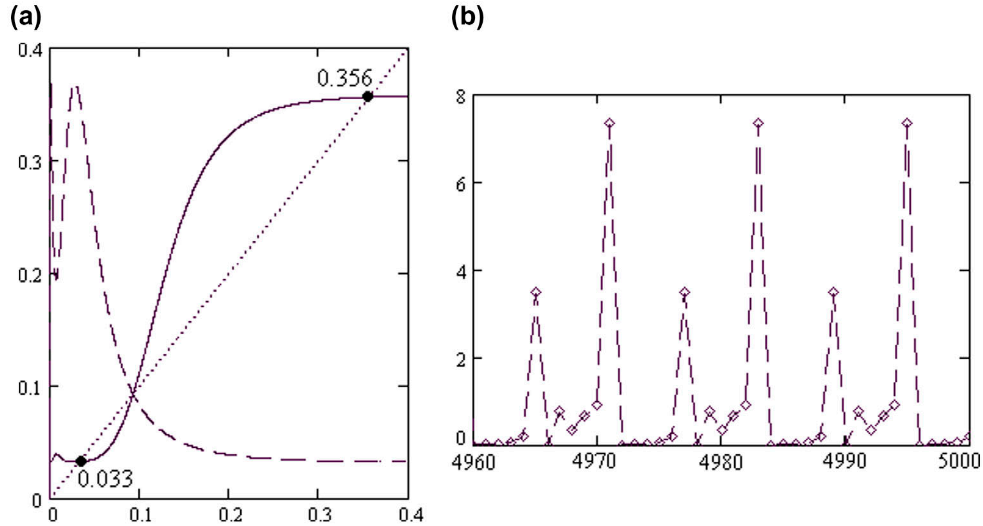


**Figure 4.** (a) Graphs of $f$ and $f^2$; (b) Corresponding 12-cycle.

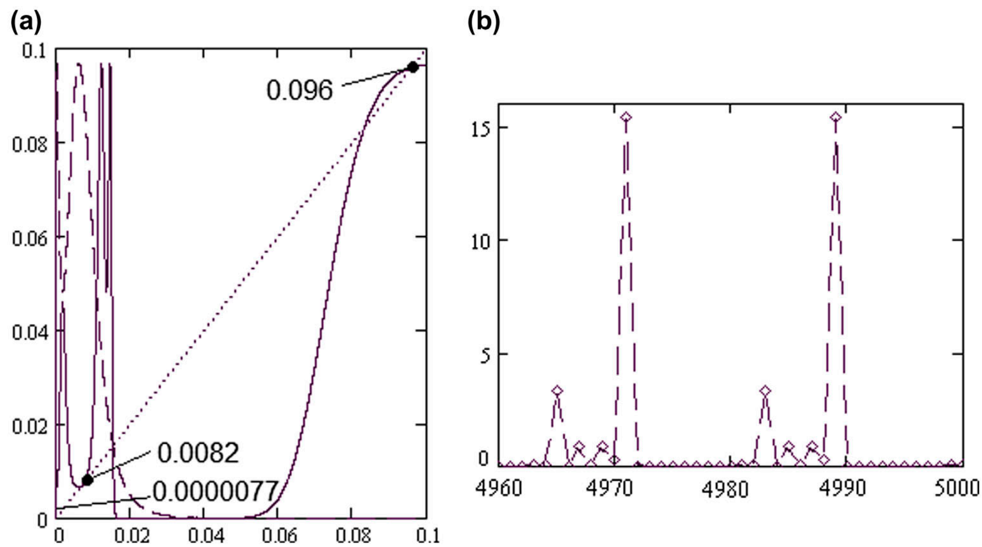


**Figure 5.** (a) Graphs of $f$ and $f^3$; (b) Corresponding 18-cycle.

### *2.2. The even period case*

The solutions of (1) behave quite differently when the parameter sequence $\{a_n\}$ has an even minimal period (the reason for this difference is a change in the dynamics of an associated first-order equation introduced in the next section). The main theorem of this section is the following, which is the even-period analog of Theorem 1. Note that multistability is replaced by ordinary stability, in the sense that solutions converge to just one limit cycle that is independent of the initial values.

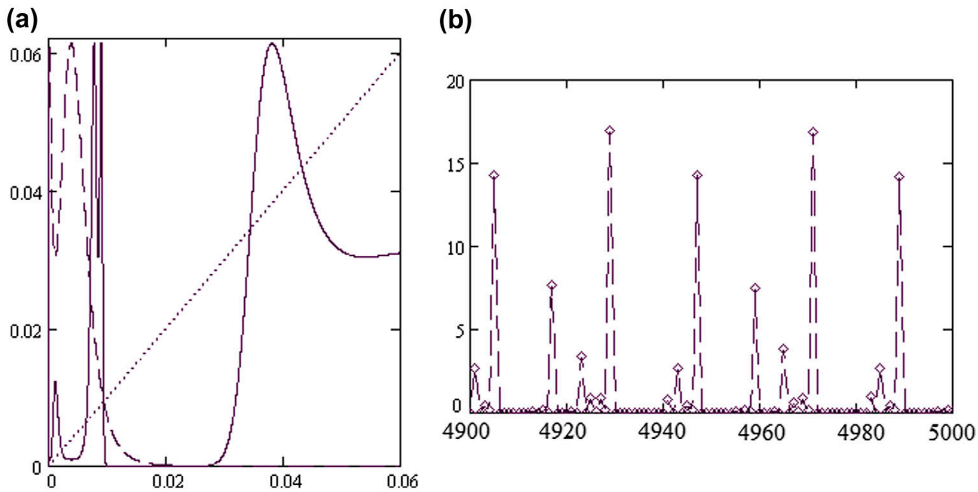


**Figure 6.** (a) Graphs of $f$ and $f^3$; (b) Corresponding nonperiodic solution.

**Theorem 3:** *Let $\{a_n\}$ be periodic with minimal even period $p \geq 2$ and let $\sigma$ be the characteristic sum defined in (2).*

(a) *If $\sigma > 0$ and $0 < a_{2k-1} < 2$ for $k = 1, 2, \ldots p/2$ then solutions of (1) converge, regardless of the initial values $x_{-1}, x_0$, to a solution $\{\bar{x}_n\}$ with period $p$ such that $\bar{x}_{2n-1} = 0$ and $\bar{x}_{2n}$ is a sequence of period $p/2$ satisfying the equality*

$$\sum_{i=1}^{p/2} \bar{x}_{2i-2} = \sum_{i=1}^{p/2} a_{2i-1}.$$

(b) *If $\sigma < 0$ and $0 < a_{2k-2} < 2$ for $k = 1, 2, \ldots p/2$ then solutions of (1) converge, regardless of the initial values $x_{-1}, x_0$, to a solution $\{\bar{x}_n\}$ with period $p$ such that $\bar{x}_{2n} = 0$ and $\bar{x}_{2n-1}$ is a sequence of period $p/2$ satisfying the equality*

$$\sum_{i=1}^{p/2} \bar{x}_{2i-1} = \sum_{i=1}^{p/2} a_{2i-2}.$$

Figure 7 illustrates Theorem 3 with $p = 4$ and

$$a_0 = 1.4, \quad a_1 = 1.8, \quad a_2 = 1.6, \quad a_3 = 0.3$$

Theorem 3 leaves out the boundary case $\sigma = 0$. In this special case multistability returns and the dynamics of (1) resembles that of the odd period case (an associated first-order equation has periodic solutions in this case, just as in the odd period case; see the next section for details). The next result is analogous to Theorem 2 and is proved using the same lemmas. Figures 3–6 also provide an illustration of the behaviour of solutions in this case, although of course, now periods do not double.

**Theorem 4:** *Suppose that $\{a_n\}$ is periodic with minimal even period $p$ and $\sigma = 0$. Let $f$ be the interval map in Lemma 13 and $t_0 > 0$ is a fixed real number.*

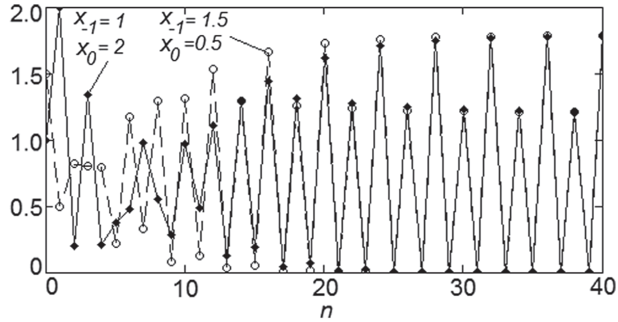


**Figure 7.** Solutions converging to a single 4-cycle with parameter period $p = 4$.

(a) *If $s$ is a periodic point of $f$ with period $\omega$ then all solutions of* (1) *with initial values $x_{-1} = s$ and $x_0 = t_0 s e^{-s}$ have period $p\omega$.*
(b) *If the map $f$ has a non-periodic point, then* (1) *has a non-periodic solution.*
(c) *If $f$ has a period-three point then* (1) *has periodic solutions of period $pn$ for all positive integers $n$ as well as chaotic solutions in the sense of Li–Yorke* [4].

Theorem 4 shows that much (though not all) of the types of behaviour seen when $\{a_n\}$ has odd period is also seen when it has even period with characteristic sum $\sigma = 0$. This suggests another possible classification of solutions based only on the even period case, distinguishing between the cases $\sigma = 0$ and $\sigma \neq 0$. In the particular case $p = 2$ the condition $\sigma = 0$ in Theorem 4 does yield a minimal *odd* period since $a_0 = a_1$ and the sequence $\{a_n\}$ is constant, i.e. it has minimal period 1 (which is odd). Thus we expect to see multistable behaviour and indeed, the dynamics described in Theorem 4 is that which is observed with a constant parameter; see [3] for more information on the nature of the stability of the solutions mentioned above.

For even $p \geq 4$ the condition $\sigma = 0$ does not imply that $\{a_n\}$ has odd period. For instance, if $p = 4$ then $\sigma = 0$ requires only that $a_0 + a_2 = a_1 + a_3$. The parameter sequence

$$\{1, 1/2, 1, 3/2, 1, 1/2, 1, 3/2 \ldots\}$$

has even period 4 with $\sigma = 0$ so in this case, Theorem 4 applies rather than Theorem 3. On the other hand, the parameter sequence

$$\{1, 1, 1/2, 3/2, 1, 1, 1/2, 3/2 \ldots\}$$

yields $\sigma = 1$ so the dynamics of (1) is determined by Theorem 3 in this case. Thus if the period of the parameter sequence is even then multistable behaviour is a borderline case that, while not prevalent, must be considered to complete the dynamics picture.

Finally, we note that compared with Theorem 3 the range of variation (or amplitude) of $a_n$ is unrestricted in Theorem 4. We expect that Theorem 3 can be extended in future studies to possibly include asymptotically stable periodic solutions whose period is greater than $p$ and even asymptotically stable nonperiodic solutions.

## 3. Explanation of the main results

This section contains not only the proofs of the main results but also explanation of why there is such a marked difference in the qualitative behaviour of solutions of (1).

We begin with the observation that if the initial values $x_{-1}, x_0$ are positive then $x_n > 0$ for all $n \geq 1$ and

$$x_{n+1} < x_{n-1}e^{a_n - x_{n-1}} \leq e^{a_n}\frac{1}{e} = e^{a_n - 1}$$

Thus the following result is obvious.

**Lemma 5:** *Let $\{a_n\}$ be a sequence of real numbers that is bounded from above with $\sup_n a_n = a$. If $x_{-1}, x_0 > 0$ then the corresponding solution $\{x_n\}$ of (1) is bounded and for all $n$*

$$0 < x_n < e^{a-1}. \tag{3}$$

### 3.1. Order reduction

The first step in understanding the dichotomy between the odd and even period cases is to obtain and examine the semiconjugate factorization of Equation (1). Following [9], we define

$$t_n = \frac{x_n}{x_{n-1}e^{-x_{n-1}}}$$

for each $n \geq 1$ and note that

$$t_{n+1}t_n = \frac{x_{n+1}}{x_n e^{-x_n}} \frac{x_n}{x_{n-1}e^{-x_{n-1}}} = \frac{x_{n+1}}{x_{n-1}e^{-x_{n-1}-x_n}} = e^{a_n}$$

or equivalently,

$$t_{n+1} = \frac{e^{a_n}}{t_n}. \tag{4}$$

Now

$$x_{n+1} = e^{a_n}x_{n-1}e^{-x_{n-1}}e^{-x_n} = e^{a_n}\frac{x_n}{t_n}e^{-x_n} = \frac{e^{a_n}}{t_n}x_n e^{-x_n} = t_{n+1}x_n e^{-x_n} \tag{5}$$

The pair of Equations (4) and (5) constitute the semiconjugate factorization of (1):

$$t_{n+1} = \frac{e^{a_n}}{t_n}, \quad t_0 = \frac{x_0}{x_{-1}e^{-x_{-1}}} \tag{6}$$

$$x_{n+1} = t_{n+1}x_n e^{-x_n} \tag{7}$$

Every solution $\{x_n\}$ of (1) is generated by a solution of the system (6)–(7). Using the initial values $x_{-1}, x_0$ we obtain a solution $\{t_n\}$ of the first-order Equation (6), called the factor equation. This solution is then used to obtain a solution of the cofactor Equation (7) and thus also of (1). The system (6)–(7) is said to be triangular basically because one equation (i.e. the factor equation) is independent of the other; see [10] for more information on triangular systems.

For an arbitrary sequence $\{a_n\}$ and a given $t_0 \neq 0$ by iterating (6) we obtain

$$t_1 = \frac{e^{a_0}}{t_0}, \quad t_2 = \frac{e^{a_1}}{t_1} = t_0 e^{-a_0+a_1}, \quad t_3 = \frac{e^{a_2}}{t_2} = \frac{1}{t_0} e^{a_0-a_1+a_2},$$
$$t_4 = \frac{e^{a_3}}{t_3} = t_0 e^{-a_0+a_1-a_2+a_3}, \ldots$$

This pattern of development implies the following result.

**Lemma 6:** *Let $\{a_n\}$ be an arbitrary sequence of real numbers and $t_0 \neq 0$.*

(a) *The general solution of (6) is given by*

$$t_n = t_0^{(-1)^n} e^{(-1)^n s_n}, \quad n = 1, 2, \ldots \tag{8}$$

*where*

$$s_n = \sum_{j=1}^{n} (-1)^j a_{j-1} \tag{9}$$

(b) *For all $n$,*

$$x_n \le \frac{1}{e} t_n.$$

***Proof:*** (a) For $n = 1$, (8) yields

$$t_1 = t_0^{-1} e^{-s_1} = \frac{1}{t_0} e^{-(-a_0)} = \frac{e^{a_0}}{t_0}$$

which is true. Suppose that (8) is true for $n \le k$. Then by (8) and (9)

$$t_0^{(-1)^{k+1}} e^{(-1)^{k+1} s_{k+1}} = \frac{1}{t_0^{(-1)^k} e^{(-1)^k s_k}} e^{(-1)^{2k+2} a_k} = \frac{e^{a_k}}{t_k} = t_{k+1}$$

which is again true and the proof is now complete by induction.

(b) This is an immediate consequence of (7) and the fact that $xe^{-x} \le 1/e$.

□

In the sequel, whenever the sequence $\{a_n\}$ has period $p$ the following quantity plays an essential role:

$$\sigma = s_p = \sum_{j=1}^{p} (-1)^j a_{j-1} = -a_0 + a_1 - a_2 + \cdots - a_{p-1} \tag{10}$$

The following special-case result will be useful later on.

**Lemma 7:** *Assume that $\{a_n\}$ is periodic with minimal period $p$. If $\sigma = 0$ and $t_0 = 1$, then $\{t_n\}$ is periodic with period $p$.*

***Proof:*** If $\sigma = 0$, then by (8) and (9) in Lemma 6 we have:

$$t_p = t_0^{(-1)^p} e^{(-1)^p s_p} = e^{(-1)^p \sigma} = 1 = t_0$$

and

$$t_{n+p} = t_0^{(-1)^{n+p}} e^{(-1)^{n+p} s_{n+p}} = e^{(-1)^{n+p} s_{n+p}}.$$

Now since $\sigma = 0$,

$$s_{n+p} = \sum_{j=1}^{n+p} (-1)^j a_{j-1} = \sum_{j=1}^{p} (-1)^j a_{j-1} + \sum_{j=p+1}^{p+n} (-1)^j a_{j-1} = \sum_{j=p+1}^{p+n} (-1)^j a_{j-1}$$

If $p$ is even, then

$$\sum_{j=p+1}^{p+n} (-1)^j a_{j-1} = -a_p + a_{p+1} + \cdots + (-1)^{n+p} a_{n+p-1} = -a_0 + a_1 + \cdots + (-1)^n a_{n-1} = s_n$$

so

$$t_{n+p} = e^{(-1)^{n+p} s_{n+p}} = e^{(-1)^n s_n} = t_n.$$

If $p$ is odd, then

$$\sum_{j=p+1}^{p+n} (-1)^j a_{j-1} = a_p - a_{p+1} + \cdots + (-1)^{n+p} a_{n+p-1} = a_0 - a_1 + \cdots - (-1)^n a_{n-1} = -s_n$$

so

$$t_{n+p} = e^{(-1)^{n+p} s_{n+p}} = e^{-(-1)^n (-s_n)} = e^{(-1)^n s_n} = t_n.$$

and the proof is complete. □

Note that the solution $\{t_n\}$ of (6) in Lemma 6 need not be bounded even if $\{a_n\}$ is a bounded sequence. The next result expresses a useful fact for this case.

**Lemma 8:** *Assume that $\{a_n\}$ is bounded from above and $x_0, x_{-1} > 0$. If the sequence $\{t_n\}$ from $t_0$ given in (6) is unbounded then some subsequence of the corresponding solution $\{x_n\}$ of (1) converges to 0.*

***Proof:*** By the hypotheses, $\sup_n a_n = a < \infty$ and there is a subsequence $\{t_{n_k}\}$ such that $\lim_{k\to\infty} t_{n_k} = \infty$. By (6) and Lemma 6(b)

$$x_{n_k+1} \leq \frac{1}{e} t_{n_k+1} \leq \frac{e^{a-1}}{t_{n_k}}$$

Therefore,

$$\lim_{k\to\infty} x_{n_k+1} = \lim_{k\to\infty} \frac{e^{a-1}}{t_{n_k}} = 0.$$

□

### 3.2. *The odd period case*

The dynamics of (1) depends critically on whether the period of the parameter sequence $\{a_n\}$ is odd or even. In this section we consider the odd case and the nature of solutions of (4) in this case. An important result in this section is Theorem 12, from which Theorem 1 readily follows.

**Lemma 9:** *Suppose that $\{a_n\}$ is sequence of real numbers with minimal odd period $p \geq 1$ and let $\{t_n\}$ be a solution of (4).*

(a) *$\{t_n\}$ has period $2p$ with a complete cycle $\{t_0, t_1, \ldots, t_{2p-1}\}$ where $t_k$ is given by (8) with*

$$s_k = \begin{cases} \sum_{j=1}^{k} (-1)^j a_{j-1}, & \text{if } 1 \leq k \leq p \\ \sum_{j=k}^{2p-1} (-1)^j a_{j-p}, & \text{if } p+1 \leq k \leq 2p-1 \end{cases} \tag{11}$$

(b) *If $t_0 = e^{-\sigma/2}$ then $\{t_n\}$ is periodic with period $p$.*

***Proof:*** (a) Let $\{a_0, a_1, \ldots, a_{p-1}\}$ be a full cycle of $a_n$ and define $\sigma$ as in (10), i.e.

$$\sigma = -a_0 + a_1 - a_2 + \ldots - a_{p-1}.$$

Since a full cycle of $a_n$ has an odd number of terms, expanding $s_n$ in (9) yields a sequence with alternating signs in terms of $\sigma$

$$s_n = \sigma - \sigma + \cdots + (-1)^{m-1}\sigma + (-1)^m \sum_{j=1}^{i} (-1)^j a_{j-1}$$

for integers $i, m$ such that $n = pm + i$, $m \geq 0$ and $1 \leq i \leq p$. If $m$ is even then for $i = 1, 2, \ldots, p$

$$s_n = \sum_{j=1}^{i} (-1)^j a_{j-1} = \begin{cases} -a_0 & n = pm+1 \quad \text{(odd)} \\ -a_0 + a_1 & n = pm+2 \quad \text{(even)} \\ \vdots & \vdots \\ -a_0 + a_1 \ldots - a_{p-1} & n = pm+p \quad \text{(odd)} \end{cases}$$

Similarly, if $m$ is odd then for $i = 1, 2, \ldots, p$

$$s_n = \sigma - \sum_{j=0}^{i} (-1)^j a_j = \begin{cases} \sigma + a_0 & n = pm+1 \quad \text{(even)} \\ \sigma + a_0 - a_1 & n = pm+2 \quad \text{(odd)} \\ \vdots & \vdots \\ \sigma + a_0 - a_1 + \ldots - a_{p-1} & n = pm+p \quad \text{(even)} \end{cases}$$

The above list repeats for every consecutive pair of values of $m$ and yields a complete cycle for $\{s_n\}$. In particular, for $m = 0$ we obtain for $i = 1, 2, \ldots, p$

$$s_n = \sum_{j=1}^{i} (-1)^j a_{j-1} = \begin{cases} -a_0 & n = 1 \\ -a_0 + a_1 & n = 2 \\ \vdots & \vdots \\ -a_0 + a_1 \ldots - a_{p-1} & n = p \end{cases}$$

and for $m = 1$ we obtain for $i = 1, 2, \ldots, p-1$

$$s_n = \sigma - \sum_{j=0}^{i} (-1)^j a_j = \begin{cases} a_1 - a_2 + \ldots - a_{p-1} & n = p+1 \\ -a_2 + \ldots - a_{p-1} & n = p+2 \\ \vdots & \vdots \\ -a_{p-1} & n = 2p-1 \end{cases}$$

$$= \sum_{j=p+1}^{2p-1} (-1)^j a_{j-p}$$

This proves the validity of (11) and shows that the sequence $\{s_n\}$ has period $2p$. Now (8) implies that $\{t_n\}$ also has period $2p$ as claimed.

(b) If $\sigma = 0$ then the statement follows immediately from Lemma 7. If $\sigma \neq 0$ and $p$ is odd then

$$t_p = t_0^{(-1)^p} e^{(-1)^p s_p} = e^{\sigma/2 - \sigma} = e^{-\sigma/2} = t_0$$

In the proof of Lemma 7 it was shown that $s_{n+p} = \sigma - s_n$. Thus

$$\begin{aligned} t_{n+p} &= t_0^{(-1)^{n+p}} e^{(-1)^{n+p} s_{n+p}} \\ &= e^{(-1)^n \sigma/2} e^{-(-1)^n (\sigma - s_n)} \\ &= e^{-(-1)^n \sigma/2 + (-1)^n s_n} = t_0^{(-1)^n} e^{(-1)^n s_n} = t_n \end{aligned}$$

and the proof is complete. □

For $p = 1$, Lemma 9 implies that $\{t_n\}$ is the two-cycle

$$\left\{ t_0, \frac{e^a}{t_0} \right\}$$

where $a$ is the constant value of the sequence $\{a_n\}$. For $p = 3$, $\{t_n\}$ is the six-cycle

$$\left\{ t_0, \frac{e^{a_0}}{t_0}, t_0 e^{a_1 - a_0}, \frac{e^{a_2 - a_1 + a_0}}{t_0}, t_0 e^{a_1 - a_2}, \frac{e^{a_2}}{t_0} \right\}.$$

From the cofactor Equation (7) we obtain

$$\begin{aligned} x_{2n+2} &= t_{2n+2} x_{2n+1} e^{-x_{2n+1}} = t_{2n+2} t_{2n+1} x_{2n} \exp\left(-x_{2n} - t_{2n+1} x_{2n} e^{-x_{2n}}\right) \\ x_{2n+1} &= t_{2n+1} x_{2n} e^{-x_{2n}} = t_{2n+1} t_{2n} x_{2n-1} \exp\left(-x_{2n-1} - t_{2n} x_{2n-1} e^{-x_{2n-1}}\right) \end{aligned}$$

For every solution $\{t_n\}$ of (6), $t_{n+1} t_n = e^{a_n}$ for all $n$, so the even terms of the sequence $\{x_n\}$ satisfy

$$x_{2n+2} = x_{2n} \exp\left(a_{2n+1} - x_{2n} - t_{2n+1} x_{2n} e^{-x_{2n}}\right) \tag{12}$$

and the odd terms satisfy

$$x_{2n+1} = x_{2n-1} \exp\left(a_{2n} - x_{2n-1} - t_{2n} x_{2n-1} e^{-x_{2n-1}}\right) \tag{13}$$

To reduce the notational clutter, let

$$y_n = x_{2n} \quad \rho_n = a_{2n+1} \quad \mu_n = t_{2n+1} \tag{14}$$

for $n \geq 0$ and also

$$z_n = x_{2n-1} \quad \zeta_n = a_{2n} \quad \eta_n = t_{2n}. \tag{15}$$

Then we can write (12) and (13) as

$$y_{n+1} = y_n e^{\rho_n - y_n - \mu_n y_n e^{-y_n}} \tag{16}$$

$$z_{n+1} = z_n e^{\zeta_n - z_n - \eta_n z_n e^{-z_n}} \tag{17}$$

The next result establishes the existence of an attracting, invariant interval for (12) and (13), or equivalently, (16) and (17).

**Lemma 10:** *Let $\{a_n\}$ be a bounded sequence where $\inf_{n\geq 0} a_n \in (0,2)$. Let $x_0, x_{-1} > 0$ and $t_0$ be given as in (6). Assume that the sequence $\{t_{2n+1}\}$ (respectively, $\{t_{2n}\}$) is bounded and let $\{x_n\}$ be the corresponding solution of (1).*

(a) *There exists an interval $[\alpha, \beta]$ with $\alpha > 0$ such that if $x_{-1}, x_0 \in [\alpha, \beta]$ then $x_{2n} \in [\alpha, \beta]$ (respectively, $x_{2n+1} \in [\alpha, \beta]$) for $n \geq 1$.*
(b) *For all $x_0, x_{-1} > 0$ there exists an integer $N > 0$ such that $x_{2n} \in [\alpha, \beta]$ (respectively, $x_{2n+1} \in [\alpha, \beta]$) for all $n \geq N$.*

***Proof:*** (a) First, note that if $x_0, x_{-1} > 0$ then $x_n > 0$ for all $n$ and by Lemma 5 $x_n \leq e^{a-1}$ for $n \geq 1$ where

$$a = \sup_{n\geq 0} a_n.$$

Thus if

$$\beta = e^{a-1}$$

then $x_n \leq \beta$ for all $n$. Next, let

$$\rho = \inf_{n\geq 0} a_n \in (0,2)$$

and consider the map

$$f(x) = xe^{\rho - x - \gamma x e^{-x}}$$

where $\gamma > 0$ is fixed. Now $x^*$ is a fixed point of $f$ if and only if

$$x^* = f(x^*) = x^* e^{\rho - x^* - \gamma x^* e^{-x^*}}$$

which is true if and only if $\rho - x^* - \gamma x^* e^{-x^*} = h(x^*) = 0$. Since $h(0) = \rho > 0$ and $h(\rho) = -\gamma\rho e^{-\rho} < 0$, there is $x^* \in (0, \rho)$ such that $h(x^*) = 0$. Thus $f$ has a fixed point $x^* \in (0, \rho)$. Further $f(x) > x$ for $x \in (0, x^*)$ and $f(x) < x$ for $x \in (x^*, \beta)$. If

$$\alpha = \min\{x^*, f(\beta), f(1)\}.$$

then we now show that $[\alpha, \beta]$ is invariant under $f$, i.e. $f(x) \in [\alpha, \beta]$ for all $x \in [\alpha, \beta]$. There are two possible cases:

*Case 1*: $\gamma \leq e$. In this case, $f(x)$ has one critical point at $x = 1$ and it is increasing in $(0, 1)$ and decreasing on $(1, \infty)$. Thus $f(1)$ is a global maximum and thus $\alpha \neq f(1)$. First, consider the case where $x^* < f(\beta) < \beta$ and let $x \in [x^*, \beta]$. If $x < 1$, then $f(x) > f(x^*) = x^* \geq \alpha$ because $f$ is increasing on $(0, 1)$. If $x > 1$, then $f(x) > f(\beta) > x^* \geq \alpha$, because $f$ is decreasing on $(1, \beta)$. In either case $f(x) \in [\alpha, \beta]$.

Next, consider the case where $f(\beta) < x^* < \beta$ and let $x \in [f(\beta), \beta]$. Then $f(x) > x > f(\beta) = \alpha$ for $f(\beta) < x < x^*$. On the other hand, if $x^* < x < 1$, then $f(x) > f(x^*) > f(\beta) = \alpha$ and if $x^* < 1 < x < \beta$, then $f(x) > f(\beta) = \alpha$. It follows that $f(x) \in [\alpha, \beta]$ if $\gamma \leq e$.

*Case 2*: $\gamma > e$. In this case, $f(x)$ has three critical points $x', 1$ and $x''$ with $x' < 1 < x''$, where local maxima occur at $x'$ and $x''$and a local minimum at 1. There are three possibilities:

(i) $\alpha = x^*$. In this case, for $x^* \leq x \leq x', f(x) \geq f(x^*) = x^* = \alpha$, since $f$ is increasing on $(0, x')$. If $x \in (x', x'')$, then $f(x) \geq f(1) \geq \alpha$. If $\beta \geq x''$ and $x'' \leq x \leq \beta$ then $f(x) \geq f(\beta) \geq \alpha$, since $f$ is decreasing on $(x'', \infty)$.

(ii) $\alpha = f(\beta)$. In this case, for $x \in [f(\beta), x^*), f(x) > x \geq f(\beta) = \alpha$. If $x^* \leq x'$ and $x \in [x^*, x']$ then $f(x) \geq f(x^*) = x^* \geq \alpha$ since $f$ is increasing. If $x \in (x', x'')$ then $f(x) \geq f(1) \geq \alpha$. If $\beta \geq x''$ and $x'' \leq x \leq \beta$ then $f(x) \geq f(\beta) = \alpha$ since $f$ is decreasing.

(iii) $\alpha = f(1)$. In this case, if $x \in [f(1), x^*)$ then $f(x) > x > f(1) = \alpha$. If $x^* < x'$ and $x \in [x^*, 1]$ then $f(x) \geq f(x^*) = x^* \geq \alpha$ for $x \in [x^*, x')$ and $f(x) \geq f(1) = \alpha$ for $x \in [x', 1]$. On the other hand, if $x^* \geq x'$ then $f(x) \geq f(1) = \alpha$ for $x \in [x^*, 1]$. Finally, if $\beta > 1$ and $x \in (1, \beta]$ then $f(x) > f(1) = \alpha$ for $x \in (1, x'')$ since $f$ is increasing on $(1, x'')$, and $f(x) \geq f(\beta) \geq \alpha$ for $x \in (x'', \beta]$ if $\beta > x''$.

The above three cases exhaust all possibilities so $f(x) \in [\alpha, \beta]$ if $\gamma > e$.

Next, assume that $\{t_{2n+1}\}$ is bounded and let $\{y_n\}$ be as defined by (16). If

$$\gamma = \sup\{t_{2n+1}\} + 1 < \infty$$

and $y_n \in [\alpha, \beta]$ then

$$y_{n+1} = y_n e^{a_{2n+1} - y_n - t_{2n+1} y_n e^{-y_n}} > y_n e^{\rho - y_n - \gamma y_n e^{-y_n}} = f(y_n) \geq \alpha$$

Similarly, if $\{t_{2n}\}$ is bounded and $\gamma = \sup\{t_{2n}\} + 1$ then

$$z_{n+1} = z_n e^{a_{2n+2} - z_n - t_{2n+2} z_n e^{-z_n}} > z_n e^{\rho - z_n - \gamma z_n e^{-z_n}} = f(z_n) \geq \alpha$$

which proves (a).

(b) It suffices to consider the case where $z_n, y_n < \alpha$. We will do this for $z_n$, since the case for $y_n$ can be done similarly. Let

$$\tau = \sup\{t_{2n}\} + \frac{1}{2} = \gamma - \frac{1}{2} > 0$$

so for $x < x^*$ and $n \geq 0$

$$e^{a_{2n} - x - t_{2n} x e^{-x}} > e^{\rho - x - \tau x e^{-x}} > e^{\rho - x^* - \tau x^* e^{-x^*}} > e^{\rho - x^* - \gamma x^* e^{-x^*}} = 1$$

Define

$$k = e^{\rho - x^* - \tau x^* e^{-x^*}} > 1.$$

If $z_n < \alpha \leq x^*$, then

$$z_{n+1} = z_n e^{a_{2n+2} - z_n - t_{2n+2} z_n e^{-z_n}} > z_n e^{\rho - x^* - \tau x^* e^{-x^*}} = k z_n$$

If $z_{n+1} > \alpha$ then we're done; otherwise,

$$z_{n+2} = z_{n+1} e^{a_{2n+4} - z_{n+1} - t_{2n+4} z_{n+1} e^{-z_{n+1}}} > z_{n+1} e^{\rho - x^* - t x^* e^{-x^*}} = k z_{n+1} = k^2 z_n$$

and we continue in this way inductively. Since $k > 1$ it follows that $z_{n+N} > z_n k^N > \alpha$ for sufficiently large $N$. □

**Lemma 11:** *Let $\{a_n\}$ be periodic of period $p$ and $0 < a_n < 2$. If as noted above, $\{y_n\}$ and $\{z_n\}$ are the even and odd indexed terms of the solution $\{x_n\}$ of (1) with initial values $x_0, x_{-1} \in [\alpha, \beta]$ then there are constants $K > 0$ and $\delta \in (0, 1)$ such that*

$$\left| \prod_{i=0}^{n-1} (1 - y_i)(1 - z_i) \right| \leq K\delta^n \tag{18}$$

***Proof:*** Recall that if $g$ is a continuous function on the compact interval $[\alpha, \beta]$ with $|g(x)| < 1$ for all $x \in [\alpha, \beta]$ then by the extreme value theorem there is a point $\tilde{x} \in [\alpha, \beta]$ such that $|g(x)| \leq |g(\tilde{x})| < 1$ for $x \in [\alpha, \beta]$. Thus if $\delta = |g(\tilde{x})| \in (0, 1)$ then $|g(x)| \leq \delta$ for all $x \in [\alpha, \beta]$.

Now we establish the inequality in (18). First, if $a = \max_{0 \leq i \leq p-1}\{a_i\} < 1 + \ln 2$ then $\beta = e^{a-1} < 2$. Thus if $u_i$ denotes either $y_i$ or $z_i$ then $u_i \in (0, 2) \supset [\alpha, \beta]$, i.e. $|1 - u_i| < 1$ and there exists $\delta_1 \in (0, 1)$ so that $|1 - u_i| < \delta_1$ for $u_i \in [\alpha, \beta]$.

Next, suppose that $a \geq 1 + \ln 2$ and let

$$2 \leq u_i \leq e.$$

Consider the preimage $u_{i-1}$ of $u_i$. There are two possible cases: Either $u_{i-1} \leq 1$ or $u_{i-1} \geq 1$.

*Case 1*: If $u_{i-1} \leq 1$ then

$$\begin{aligned} |1 - u_{i-1}||1 - u_i| &= (1 - u_{i-1})(u_i - 1) \\ &\leq (1 - u_{i-1})(u_{i-1} e^{a - u_{i-1} - \tau_{i-1} u_{i-1} e^{-u_{i-1}}} - 1) \\ &< (1 - u_{i-1})(u_{i-1} e^{2 - u_{i-1}} - 1) \end{aligned}$$

where $\tau_i = \mu_i$ or $\eta_i$ depending on the case ($y_n$ or $z_n$ respectively). Note that

$$(1 - x)(x e^{2-x} - 1) < 1 \tag{19}$$

for $x \in (0, 1]$ because (19) can be written as $x(1 - x) < (2 - x)e^{x-2}$ and this inequality is true since its left hand side has a maximum of 1/4 on (0,1] whereas its right hand side has a minimum of $2e^{-2} > 1/4$ on (0,1]. In particular, (19) holds for $x \in [\alpha, 1]$ so there exists $\delta_2 \in (0, 1)$ such that

$$|1 - u_{i-1}||1 - u_i| < \delta_2.$$

*Case 2*: If $u_{i-1} \geq 1$ then

$$\begin{aligned}|1-u_{i-1}||1-u_i| &= (u_{i-1}-1)(u_i-1)\\ &\leq (u_{i-1}-1)(u_{i-1}e^{a-u_{i-1}-\tau_{i-1}u_{i-1}e^{-u_{i-1}}}-1)\\ &< (u_{i-1}-1)(u_{i-1}e^{a-u_{i-1}}-1).\end{aligned}$$

If $\phi(x) = (x-1)(xe^{2-x}-1)$ then

$$\phi'(x) = [x-(x-1)^2]e^{2-x}-1, \quad \phi''(x) = (x-1)(x-4)e^{2-x}.$$

Since $\phi$ is smooth with $\phi'(2) = 0$ and $\phi''(x) < 0$ for $x \in (1,4)$ it follows that $\phi$ is maximized on [1,4] at 2 and $\phi(2) = 1$. In particular, for $u_{i-1} \in [1,\beta] \subset [1,4]$,

$$(u_{i-1}-1)(u_{i-1}e^{a-u_{i-1}}-1) < \phi(u_{i-1}) \leq 1$$

and it follows that there is $\delta_3 \in (0,1)$ such that

$$|1-u_{i-1}||1-u_i| < \delta_3.$$

Finally, there are at most $m$ pairings $|1-u_{i-1}||1-u_i|$ where $m = [n/2]$ (i.e. $m$ is $n/2$ rounded down to the nearest integer). If $n$ is even, then $m = n/2$, if $n$ is odd, $m = (n-1)/2$ and we have one last unpaired term left, namely, $|1-u_0| < (e-1)$. Choosing $\delta = \max\{\delta_1,\delta_2,\delta_3\}$, we get

$$\left|\prod_{i=0}^{n-1}(1-u_i)\right| < (e-1)\delta^m.$$

Therefore,

$$\left|\prod_{i=0}^{n-1}(1-y_i)(1-z_i)\right| < (e-1)^2\delta^{2m} \leq \frac{\beta}{\alpha}K\delta^{nm}$$

where $K = (e-1)^2/\delta$ and the proof is complete. □

The next result generalizes similar results in [2] and [3].

**Theorem 12:** *Let $\{a_n\}$ be a periodic sequence with $0 < a_n < 2, x_0, x_{-1} > 0$ and the sequence $\{t_n\}$ with $t_0 = x_0/x_{-1}e^{x_{-1}}$ be periodic with period q. Then each solution of (1) from the initial values $x_0, x_{-1}$ converges to a periodic solution (dependent on the choice of initial values) with period q.*

***Proof:*** Let $q$ be the period of the sequence $\{t_n\}$ from initial value $t_0 = x_0/x_{-1}e^{x_{-1}}$. For each $i = 1, 2, \ldots, q$, define the map

$$g_i(x) = t_i x e^{-x}$$

and let

$$\phi = g_q \circ g_{q-1} \circ \cdots \circ g_1$$

Then by the cofactor Equation (6), $\phi$ generates the orbit of (1) from initial values $x_0, x_{-1}$. Also note that $\phi$ is an autonomous interval map, and by Lemma 10, there exist real numbers

$\alpha, \beta > 0$ and a positive integer $N$ so that $\phi : [\alpha, \beta] \to [\alpha, \beta]$ and $\phi^n(x) \in [\alpha, \beta]$ for all $n \geq N$. Hence, by Brouwer's fixed point theorem, there exists a $x^* \in [\alpha, \beta]$ so that $\phi(x^*) = x^*$. Now, let $x_0 \in [\alpha, \beta]$ be given.

Since

$$g_i'(x) = t_i e^{-x}(1-x) = \frac{g_i(x)}{x}(1-x)$$

then

$$\begin{aligned}\phi'(x_0) &= \prod_{i=1}^{q} g_i'(x_{i-1}) = \prod_{i=1}^{q} \frac{g_i(x_{i-1})}{x_{i-1}}(1-x_{i-1}) \\ &= \frac{g_1(x_0)}{x_0}\frac{g_2(x_1)}{x_1}\cdots\frac{g_q(x_{q-1})}{x_{q-1}}\prod_{i=1}^{q}(1-x_{i-1})\end{aligned}$$

Noting that $g_i(x_{i-1}) = x_i$, we get

$$\phi'(x_0) = \frac{x_q}{x_0}\prod_{i=1}^{q}(1-x_{i-1})$$

Similarly,

$$(\phi^2)'(x_0) = (\phi \circ \phi)'(x_0) = \frac{x_{2q}}{x_0}\prod_{i=1}^{2q}(1-x_{i-1})$$

and in general,

$$(\phi^n)'(x_0) = \frac{x_{nq}}{x_0}\prod_{i=1}^{nq}(1-x_{i-1})$$

Now, let $m = [nq/2]$. If $nq$ is even, then $m = nq/2$ and by Lemma (11)

$$\left|\prod_{i=1}^{nq}(1-x_{i-1})\right| = \left|\prod_{i=0}^{m-1}(1-y_i)(1-z_i)\right| \leq K\delta^m$$

for some $K > 0$, $\delta \in (0, 1)$, where $y_i$ and $z_i$ are the even and odd indexed terms of the $\{x_n\}$ as noted above. If $nq$ is odd, then $m = (nq-1)/2$, so

$$\left|\prod_{i=1}^{nq}(1-x_{i-1})\right| = \left|(1-x_{nq})\prod_{i=0}^{m-1}(1-y_i)(1-z_i)\right| \leq (e-1)K\delta^m$$

hence,

$$\left|(\phi^n)'(x_0)\right| \leq \frac{\alpha}{\beta}(e-1)K\delta^{(nq-1)/2}.$$

Finally,

$$\begin{aligned}|\phi^n(x_0) - x^*| &= |\phi^n(x_0) - \phi^n(x^*)| = |\left(\phi^n\right)'(w)||x_0 - x^*| \\ &\leq \frac{\beta}{\alpha}K(e-1)\delta^{(nq-1)/2}|x_0 - x^*| \to 0\end{aligned}$$

as $n \to \infty$ and the proof is complete. □

Theorem 1 is now an immediate consequence of the preceding result.

***Proof:*** (of Theorem 1)

(a) By Lemma 9, $\{t_n\}$ is periodic with period $2p$, so $\{a_n\}$ and $\{t_n\}$ have a common period $2p$. The rest follows from Theorem 12.

(b) If $x_0 = x_{-1}e^{-\sigma/2-x_{-1}}$, i.e. $t_0 = e^{-\sigma/2}$, then by Lemma 9 $t_n$ is periodic with period $p$. Therefore, $\{a_n\}$ and $\{t_n\}$ have a common period $p$ and the rest follows from Theorem 12.

□

Next, we consider a wider range of values for $a_n$ and the existence of non-periodic solutions for (1). The next two results are needed for proving Theorem 2.

**Lemma 13:** *Let $\{x_n\}$ be a solution of (1) with initial values $x_{-1}, x_0 > 0$ and assume that $\{a_n\}$ is periodic with minimal period $p \geq 1$ and $\{t_n\}$ is periodic with period $q \geq 1$. Define*

$$g_k(x) = t_k x e^{-x}, \; k = 0, 1, \ldots, q-1$$

*where $t_0 = x_0/(x_{-1}e^{-x_{-1}})$ and $t_k$ is given by (8) and (9). Also define*

$$\begin{aligned} h_k &= g_k \circ g_{k-1} \circ \cdots \circ g_0, \quad k = 0, 1, \ldots, q-1 \\ f &= h_{q-1} = g_{q-1} \circ g_{q-2} \circ \cdots \circ g_1 \circ g_0 \end{aligned}$$

*Then $\{x_n\}$ is determined by the $q$ sequences*

$$x_{qm+k} = h_k \circ f^{\,m}(x_{-1}), \quad k = 0, 1, \ldots, q-1 \tag{20}$$

*that are obtained by iterations of one-dimensional maps of the interval $(0, \infty)$, with $f^{\,0}$ being the identity map.*

***Proof:*** Given the initial values $x_{-1}, x_0 > 0$ the definition of $t_0$ and (7) imply that

$$\begin{aligned} x_0 &= t_0 x_{-1} e^{-x_{-1}} = g_0(x_{-1}) = h_0(x_{-1}) \\ x_1 &= t_1 x_0 e^{-x_0} = g_1(x_0) = g_1 \circ g_0(x_{-1}) = h_1(x_{-1}) \end{aligned}$$

and so on:

$$x_k = h_k(x_{-1}), \quad k = 0, 1, \ldots, q-2$$

Thus (20) holds for $m = 0$. Further, $x_{q-1} = h_{q-1}(x_{-1}) = f(x_{-1})$. Inductively, we suppose that (20) holds for some $m \geq 0$ and note that for $k = 0, 1, \ldots, q-2$

$$h_{k+1} = g_{k+1} \circ g_k \circ \cdots \circ g_0 = g_{k+1} \circ h_k$$

Now by (7)

$$\begin{aligned} x_{q(m+1)-1} &= t_{qm+q-1} x_{qm+q-2} e^{-x_{qm+q-2}} \\ &= t_{q-1} h_{q-2} \circ f^{\,m}(x_{-1}) e^{-h_{q-2} \circ f^{m}(x_{-1})} \\ &= g_{q-1} \circ h_{q-2} \circ f^{\,m}(x_{-1}) \end{aligned}$$

$$\begin{aligned}&= h_{q-1} \circ f^{m}(x_{-1}) \\ &= f^{m+1}(x_{-1})\end{aligned}$$

So (20) holds for $k = q - 1$ by induction. Further, again by (7) and the preceding equality

$$\begin{aligned}x_{q(m+1)} &= t_{qm+q} x_{qm+q-1} q^{-x_{qm+q-1}} \\ &= t_0 f^{m+1}(x_{-1}) e^{-f^{m+1}(x_{-1})} \\ &= g_0 \circ f^{m+1}(x_{-1}) \\ &= h_0 \circ f^{m+1}(x_{-1})\end{aligned}$$

Similarly,

$$\begin{aligned}x_{q(m+1)+1} &= t_{q(m+1)+1} x_{q(m+1)} e^{-x_{q(m+1)}} \\ &= t_1 h_0 \circ f^{m+1}(x_{-1}) e^{-h_0 \circ f^{m+1}(x_{-1})} \\ &= g_1 \circ h_0 \circ f^{m+1}(x_{-1}) \\ &= h_1 \circ f^{m+1}(x_{-1})\end{aligned}$$

Repeating this calculation $q - 2$ times establishes (20) and completes the induction step and the proof. □

**Lemma 14:** *Suppose that $\{a_n\}$ and $\{t_n\}$ are periodic and $\{t_n\}$ has minimal period $q \geq 1$.*

(a) *If the map $f$ in Lemma 13 has a (positive) periodic point of minimal period $\omega$ then there is a solution of (1) with period $\omega q$.*

(b) *If the map $f$ in Lemma 13 has a non-periodic point then (1) has a non-periodic solution.*

***Proof:*** (a) By hypothesis, there is a number $s \in (0, \infty)$ such that $f^{n+\omega}(s) = f^n(s)$ for all $n \geq 0$. Let $x_{-1} = s$ and define $x_0 = h_0(s)$. By Lemma 13 the solution $x_n$ corresponding to these initial values follows the track shown below:

$$\begin{array}{rcll}x_{-1} = s \to & x_0 = h_0(s) \to \cdots \to & & x_{q-2} = h_{q-2}(s) \to \\ \to x_{q-1} = h_{q-1}(s) = f(s) \to & x_q = h_0(f(s)) \to \cdots \to & & x_{2q-2} = h_{q-2}(f(s)) \to \\ \to x_{2q-1} = h_{q-1}(f(s)) = f^2(s) \to & x_{2q} = h_0(f^2(s)) \to \cdots \to & & x_{3q-2} = h_{3q-2}(f^2(s)) \to \\ \vdots & \vdots & \vdots & \vdots \\ x_{\omega q-1} = h_{q-1}(f^{\omega-1}(s)) = f^{\omega}(s) = s \to & x_{q\omega} = h_0(s) \to \cdots \to & x_{(\omega+1)q-2} = h_{q-2}(s) \to \cdots & \end{array}$$

The pattern in this list evidently repeats after $\omega q$ entries. So $x_{\omega q+n} = x_n$ for $n \geq 0$ and it follows that the solution $\{x_n\}$ of (1) has period $\omega q$.

(b) Suppose that $\left\{f^n(x_{-1})\right\}$ is a non-periodic sequence for some $x_{-1} > 0$. Then by Lemma 13 the solution $\{x_n\}$ of (1) with initial values $x_{-1}$ and $x_0 = g_0(x_{-1})$ has the non-periodic subsequence

$$x_{qn-1} = f^n(x_{-1})$$

It follows that $\{x_n\}$ is non-periodic. □

Now Theorem 2 readily follows.

***Proof:*** (of Theorem 2)

Parts (a) and (c) are immediate consequences of Lemma 14 with $q = 2p$ because of Lemma 9(a). Part (b) is true by Lemma 9(b).

(d) As is well-known from [4], if $f$ has a period three point then $f$ has periodic points of every period $n \geq 1$, as well as aperiodic, chaotic solutions in the sense of Li and Yorke. Therefore, by parts (a) and (b), (1) also has periodic solutions of period $2pn$, as well as chaotic solutions. □

### 3.3. The even period case

When $\{a_n\}$ is periodic with minimal even period $p$ the next result shows that the sequence $\{t_n\}$ is *not* periodic with the exception of a boundary case. This causes a fundamental change in the dynamics of (1). Once again, the quantity $\sigma$ is defined by (10), i.e.

$$\sigma = -a_0 + a_1 - a_2 + \ldots + a_{p-1}.$$

**Lemma 15:** *Suppose that $\{a_n\}$ is a sequence of real numbers with minimal even period $p \geq 2$ and let $\{t_n\}$ be a solution of (4). Then*

$$t_n = \left(t_0 e^{d_n\sigma+\gamma_n}\right)^{(-1)^n} \tag{21}$$

*where the integer divisor $d_n = [n - n(\bmod p)]/p$ is uniquely defined by each $n$ and*

$$\gamma_n = \begin{cases} \sum_{j=1}^{n(\bmod p)} (-1)^j a_{j-1} & \text{if } n(\bmod p) \neq 0 \\ 0 & \text{if } n(\bmod p) = 0 \end{cases} \tag{22}$$

*The sequence $\{t_n\}$ is periodic with period $p$ iff $\sigma = 0$, i.e.*

$$a_0 + a_2 + \cdots a_{p-2} = a_1 + a_3 + \cdots + a_{p-1}. \tag{23}$$

***Proof:*** Let $\{a_0, a_1, \ldots, a_{p-1}\}$ be a full cycle of $a_n$ with an even number of terms. Since $n = pd_n + n(\bmod p)$ for $n \geq 1$, expand $s_n$ in (9) to obtain

$$s_n = d_n\sigma + \sum_{j=1}^{n(\bmod p)} (-1)^j a_{j-1}$$

if $n(\bmod p) \neq 0$. If $p$ divides $n$ so that $n(\bmod p) = 0$ then we assume that the sum is 0 and $s_n = d_n\sigma$. Thus $s_n = d_n\sigma + \gamma_n$ where $\gamma_n$ is as defined in (22).

The $\sigma$ terms have uniform signs in this case since there are an even number of terms in each full cycle of $a_n$. Now (8) yields

$$t_n = t_0^{(-1)^n} e^{(-1)^n s_n} = t_0^{(-1)^n} e^{(-1)^n(d_n\sigma+\gamma_n)}$$

which is the same as (21).

Next, if $\sigma \neq 0$ then $d_n\sigma$ is unbounded as $n$ increases without bound so $\{t_n\}$ is not periodic. But if $\sigma = 0$ then (21) reduces to

$$t_n = \left(t_0 e^{\gamma_n}\right)^{(-1)^n} \tag{24}$$

Since the sequence $\gamma_n$ has period $p$, the expression on the right hand side of (24) has period $p$ with a full cycle

$$t_1 = \frac{e^{a_0}}{t_0}, t_2 = t_0 e^{-a_0+a_1}, t_3 = \frac{e^{a_0-a_1+a_2}}{t_0}, \ldots, t_p = t_0 e^{-a_0+a_1+\cdots+(-1)^p a_{p-1}} = t_0.$$

□

By the preceding result,

$$t_{2m} = t_0 e^{\gamma_{2m}} e^{d_{2m}\sigma} \qquad \text{if } n = 2m \text{ is even}$$
$$t_{2m+1} = \frac{1}{t_0} e^{-\gamma_{2m+1}} e^{-d_{2m+1}\sigma} \quad \text{if } n = 2m+1 \text{ is odd}$$

Suppose that $\sigma \neq 0$. If $\sigma > 0$ then since $\lim_{n\to\infty} d_n = \infty$ it follows that $t_{2m}$ is unbounded but $t_{2m+1}$ converges to 0, and the reverse is true if $\sigma < 0$. Therefore,

$$\lim_{m\to\infty} t_{2m} = \infty, \quad \lim_{m\to\infty} t_{2m+1} = 0, \quad \text{if } \sigma > 0, \tag{25}$$
$$\lim_{m\to\infty} t_{2m} = 0, \quad \lim_{m\to\infty} t_{2m+1} = \infty, \quad \text{if } \sigma < 0. \tag{26}$$

**Lemma 16:** *Suppose that $\{a_n\}$ is a sequence of real numbers with minimal even period $p \geq 2$ and let $\{x_n\}$ be a solution of (1) with initial values $x_{-1}, x_0 > 0$. Then $\lim_{n\to\infty} x_{2n+1} = 0$ if $\sigma > 0$ and $\lim_{n\to\infty} x_{2n} = 0$ if $\sigma < 0$.*

***Proof:*** Assume first that $\sigma > 0$. Then by (25) $\lim_{n\to\infty} t_{2n} = \infty$ so as in the proof of Lemma 8 $\lim_{n\to\infty} x_{2n+1} = 0$. If $\sigma < 0$ then a similar argument using (26) yields $\lim_{n\to\infty} x_{2n} = 0$ to complete the proof. □

Lemma 16 indicates that half of the terms of every solution $\{x_n\}$ of (1) converge to 0 in the even period case if $\sigma \neq 0$. We now consider what happens to the *other* half.

**Lemma 17:** *Let $\{u_n\}$ be the solution of*

$$u_{n+1} = u_n e^{a_{2n+1} - u_n} \tag{27}$$

*and $\{w_n\}$ be the solution of*

$$w_{n+1} = w_n e^{a_{2n+2} - w_n}. \tag{28}$$

(a) *The sequence $\{x_n\}$ with $x_{2n} = u_n$ and $x_{2n+1} = 0$ is a solution of (1).*
(b) *The sequence $\{x_n\}$ with $x_{2n} = 0$ and $x_{2n+1} = w_n$ is a solution of (1).*

***Proof:*** (a) Let $\{u_n\}$ be a solution to (27) from initial value $u_0 > 0$. If $x_0 = u_0$ and $x_1 = 0$, then

$$x_2 = x_0 e^{a_1 - x_0 - x_1} = u_0 e^{a_1 - u_0} = u_1$$

and

$$x_3 = x_1 e^{a_2 - x_2 - x_1} = 0$$

Inductively, if $x_{2k} = u_k$ and $x_{2k+1} = 0$ for some $k \geq 1$ then

$$x_{2k+2} = x_{2k}e^{a_{2k+1}-x_{2k}-x_{2k+1}} = u_k e^{a_{2k+1}-u_k} = u_{k+1}$$

and

$$x_{2k+3} = x_{2k+1}e^{a_{2k+2}-x_{2k+1}-a_{2k+2}} = 0$$

which proves (a).

(b) Let $\{w_n\}$ be a solution to (28) from initial value $w_0 > 0$. If $x_0 = 0$ and $x_1 = w_0$, then

$$x_2 = x_0 e^{a_1 - x_0 - x_1} = 0$$

and

$$x_3 = x_1 e^{a_2 - x_2 - x_1} = w_0 e^{a_2 - w_0} = w_1$$

Inductively, if $x_{2k} = 0$ and $x_{2k+1} = w_k$ for some $k \geq 1$ then

$$x_{2k+2} = x_{2k}e^{a_{2k+1}-x_{2k}-x_{2k+1}} = 0$$

and

$$x_{2k+3} = x_{2k+1}e^{a_{2k+2}-x_{2k+1}-a_{2k+2}} = w_k e^{a_{2k+2}-w_k} = w_{k+1}$$

which proves (b). □

The next result is proved in [8].

**Lemma 18:** *Consider the first-order difference equation*

$$y_{n+1} = y_n e^{\alpha_n - y_n} \tag{29}$$

*where $\alpha_n$ is a sequence of real numbers with period $q$. If $0 < \alpha_n < 2$ then (29) has a globally asymptotically stable solution $\{y_n^*\}$ with period $q$ such that*

$$\sum_{i=1}^{q} y_i^* = \sum_{i=1}^{q} \alpha_i .$$

We now prove Theorem 3.

***Proof:*** (of Theorem 3)

We prove part (a) and part (b) is demonstrated similarly. By Lemma 18 the equation in (27) has a periodic solution of period $p/2$ given by $\{u_i^*\}$ with $0 \leq i \leq p/2 - 1$. By Lemma 17, the sequence $\{u_0^*, 0, u_1^*, 0, \ldots, u_{p/2-1}^*, 0\}$ is a $p$ periodic solution of (1) This means that $\bar{x}_{2n-1} = 0$ and $\bar{x}_{2n} = u_i^*$ with

$$\sum_{i=1}^{p/2} \bar{x}_{2i-2} = \sum_{i=1}^{p/2} u_i^* = \sum_{i=1}^{p/2} a_{2i-1} .$$

Let the even indexed terms of the solution $\{x_n\}$ be defined as in (16) and for each $n \geq 0$, define

$$F_n(x) = xe^{\rho_n - x - \mu_n x e^{-x}}$$

Then $F_n(u_n^*) = u_{n+1}^*$. Now observe that with $\xi_n = F_n \circ F_{n-1} \circ \cdots \circ F_0$

$$F_n(y_n) = F_n(F_{n-1}(y_{n-1})) = F_n(F_{n-1}(\cdots F_0(y_0))\cdots) = \xi_n(y_0)$$

Also note that

$$|\xi_n'(y_0)| = \left|\prod_{i=0}^{n} e^{\rho_i - y_i - \mu_i y_i e^{-y_i}}(1-\mu_i y_i e^{-y_i})(1-y_i)\right|$$

Since $\mu_n \to 0$, for sufficiently large $N$, $0 < (1-\mu_n y_n e^{-y_n}) \leq 1$ for $n \geq N$. Then there exists a constant $M > 0$ so that

$$\left|\prod_{i=0}^{n}(1-\mu_i y_i e^{-y_i})\right| \leq \left|\prod_{i=0}^{N}(1-z_{i+1})\right| \leq M$$

Proceeding now as in the proof of Lemma 11, if we let $m = [n/2]$, we can find constants $K > 0$ and $\delta \in (0,1)$ so that

$$\left|\prod_{i=0}^{n}(1-y_i)\right| \leq K\delta^m$$

Therefore,

$$|\xi_n'(y_0)| = \frac{y_{n+1}}{y_0}\left|\prod_{i=0}^{n}(1-\mu_i y_i e^{-y_i})(1-y_i)\right| \leq \frac{\beta}{\alpha}KM\delta^m$$

Finally,

$$\begin{aligned}|y_{n+1} - u_{n+1}^*| &= |F_n(y_n) - F_n(u_n^*)| = |\xi_n(y_0) - \xi_n(u_0^*)| = |\xi'(w)||y_0 - u_0^*| \\ &\leq \frac{b}{\alpha}KM\delta^m|y_0 - u_0^*| \to 0\end{aligned}$$

as $n \to \infty$ which completes the proof. □

**Remark 19:**

(1) In Theorem 3(a) the even-indexed terms $a_{2k}$ are not restricted to (0,2) as long as $\sigma > 0$, i.e.

$$a_1 + a_3 + \cdots + a_{p-1} > a_0 + a_2 + \cdots + a_{p-2}$$

This imposes an upper bound $a_{2k} < 2(p/2) = p$ for each $k$ but clearly some $a_{2k}$ may exceed 2. Similarly, in (b) the odd-indexed terms are not restricted to (0,2) as long as $\sigma < 0$.

(2) Note that $2p$ is not a minimal period for $\{\bar{x}_n\}$. For example, if $p = 4$ with $a_1 = a_3$ and $2a_1 > a_0 + a_2$ (so that $\sigma > 0$) then $\bar{x}_{2n-1}$ satisfies (29) with constant $\rho_n$. In this case, Lemma 18 yields a globally asymptotically stable fixed point for (29), and thus a globally attracting period two solution for (1).

In the boundary special case $\sigma = 0$, the solutions of (1) have entirely different dynamics that resemble the odd period case, as stated in Theorem 4, which follows readily from Lemma 14

***Proof:*** (of Theorem 4)

By Lemma 15 $\{t_n\}$ has period $p$ so the application of Lemma 14 completes the proof. □

## 4. Summary and future directions

We used a semiconjugate factorization of (1) to investigate its dynamics. Semiconjugate factorizations for difference equations of exponential type are not generally known (unlike linear equations) but fortunately we have one for (1). As we see above, the decomposition of (1) into the triangular system (6)–(7) of first-order equations makes it clear why the solutions of (1) behave differently in a fundamental way depending on whether the period of $\{a_n\}$ is odd or even: in the former case the sequence $\{t_n\}$ is periodic, hence bounded while in the latter case $\{t_n\}$ is unbounded when $\sigma \neq 0$.

The main results of this paper are Theorems 1 and 2 for when the period $p$ of the parameter sequence $\{a_n\}$ is odd, and Theorems 3 and 4 for when $p$ is even. Theorems 1, 2 and 4 show that (1) has multistable coexisting solutions, including non-periodic and chaotic solutions if the amplitude of the parameter sequence $a_n$ is unrestricted. Theorem 3 indicates a completely different dynamics where globally stable limit cycles occur when $a_n$ is restricted to the interval (0, 2). Another of our main results is Theorem 12 that extends previous special cases in [2] and [3]. Further, Theorem 1 is an immediate consequence of Theorem 12.

An extension of Theorem 3 that includes non-periodic solutions when $a_n$ exceeds 2 for some indices $n$ is expected and may be of future interest. Such an extension may also yield asymptotically stable non-periodic solutions, including chaotic solutions for (1) when $\sigma \neq 0$.

A natural extension of the above results is not obvious for higher order versions of (1) such as

$$x_{n+1} = x_{n-1} e^{a_n - x_n - x_{n-k}} \tag{30}$$

For instance, (30) may have unbounded solutions if $k \geq 2$ and thus different dynamics than (1) are exhibited. Further, known semiconjugate factorizations for (30) yield a factor equation with order at least 2 if $k \geq 2$; see [9]. Such an equation is less tractable than the first-order case studied above. A detailed study of difference equations such as (30) and similar with periodic $\{a_n\}$ may yield interesting and possibly unexpected results.


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