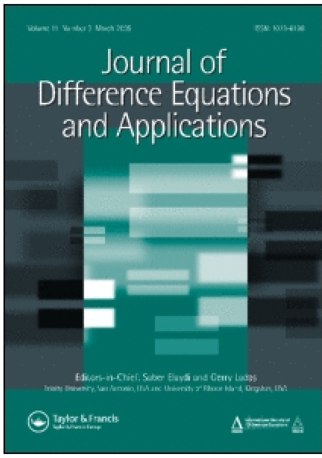

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Open Problems and Conjectures

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Open Problems and Conjectures

Edited by Gerry Ladas
In this section, we present some open problems and conjectures about some interesting types of difference equations. Please submit your problems and conjectures with all relevant information to G. Ladas

On third-order rational difference equations with quadratic terms

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Rational difference equations are natural extensions of linear ones that present us with deep new insights into dynamics of non-linear equations. Rational equations not only exhibit a rich variety of dynamic behaviours, they also present us with interesting challenges through a large number of conjectures and open problems. In this note, we consider third-order rational difference equations of type

$$x_{n+1} = x_{n-k}\left(\frac{\alpha x_n + \beta x_{n-1} + \gamma x_{n-2}}{Ax_n + Bx_{n-1} + Cx_{n-2}}\right), \tag{1}$$

where

$$k \in \{0, 1, 2\} \quad \text{and} \tag{2a}$$

$$\alpha, \beta, \gamma, A, B, C \geq 0, \quad \text{with} \quad \alpha + \beta + \gamma, \;\; A + B + C > 0. \tag{2b}$$

The seven parameters in equation (1) result in 192 separate difference equations if the three values of k in (2a) are counted along with the 2^6 possible cases in which the six coefficients in (2b) can be zeros. For easy reference, ordered 7-tuples can be used to denote various cases of (1); thus, e.g. $(2; \alpha, 0, 0, A, B, 0)$ represents the following special case of (1): $x_{n+1} = \alpha x_n x_{n-2}/(Ax_n + Bx_{n-1})$. As a space-saving measure, we drop zeros as well (except $k = 0$) so the preceding example can be represented as $(2; \alpha, A, B)$. In this notation, it is assumed that $\alpha, A, B \neq 0$.

Some of these cases are quite simple while others present significant challenges. Without intending to give an exhaustive list here, we indicate what is known about some non-trivial special cases of (1) and state some open problems and conjectures about this equation. Our choice of non-negative coefficients helps keep the number of special cases limited, but certainly an unstated open problem here would be the extensions of all of our

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results to real coefficients. Such extensions give us a larger collection of qualitatively different behaviours and further improve our understanding of rational equations.

Equation (1) is a natural extension of the third-order rational equation

$$x_{n+1} = \frac{\alpha x_n + \beta x_{n-1} + \gamma x_{n-2}}{A x_n + B x_{n-1} + C x_{n-2}}, \tag{3}$$

that has been studied in Ref. [3]. The term x_{n-k} that is missing in equation (3) has an important consequence for equations of type (1). With that term (1) is a homogeneous equation of degree 1 (or HD1); see Refs. [9,10]. Although, we do not deal with the HD1 property explicitly in this note, the following facts are worth mentioning: The HD1 property implies that (1) has no isolated fixed points, so standard methods such as linearization or semicycle analysis do not apply directly. On the other hand, the homogeneous property lets us reduce (1) to a second-order rational equation via the substitution

$$r_n = \frac{x_n}{x_{n-1}}. \tag{4}$$

In addition to having a lower order than the original equation, the factor equation often possesses isolated fixed points and is amenable to standard analytical methods. Equation (4) can be written as

$$x_n = r_n x_{n-1}. \tag{5}$$

Using this linear non-autonomous equation, we can obtain information about (1) from a solution $\{r_n\}$ of the rational factor equation. Note that the explicit solution of (5) is

$$x_n = x_0 r_1 r_2 \ldots r_n. \tag{6}$$

1. Second-order cases

With $k \in \{0, 1\}$ and $\gamma = C = 0$, equation (1) reduces to one of the following second-order equations:

$$\text{(a)}\;\; x_{n+1} = x_n \left(\frac{\alpha x_n + \beta x_{n-1}}{A x_n + B x_{n-1}} \right), \quad \text{(b)}\;\; x_{n+1} = x_{n-1} \left(\frac{\alpha x_n + \beta x_{n-1}}{A x_n + B x_{n-1}} \right). \tag{7}$$

The positive solutions of equation (7) are studied in Ref. [5]. For a detailed study of related equations of type

$$x_{n+1} = \frac{\alpha x_n + \beta x_{n-1}}{A x_n + B x_{n-1}},$$

we refer to Ref. [7] and references therein. The following propositions summarize the known results from Ref. [5] for equation (7).

PROPOSITION 1. *Let $\alpha + B, \alpha + \beta, A > 0$ in equation (a) in (7). Then, the following are true:*

(a) *Equation $(7a)$ has no positive periodic solutions.*

(b) *Every positive solution of* (7*a*) *converges to* 0 *eventually monotonically if either of the following conditions holds:*
  (i) $\alpha + \beta < A + B$ *with* $\alpha > B$ *if* $\beta = 0$*;*
  (ii) $\beta = 0$ *and* $\alpha \leq B$.
(c) *Every positive solution of* (7*a*) *converges to* ∞ *eventually monotonically if* $A + B < \alpha + \beta$.
(d) *Every positive solution of* (7*a*) *converges to a finite limit if* $A + B = \alpha + \beta$, $\beta > 0$ *and* $A < 2\alpha + \beta$.

The results in Proposition 1 also address Open Problem 6.10.1(b) in Ref. [7] for non-negative coefficients.

PROPOSITION 2. *Let* $A > 0$ *and* $\alpha B > \beta A$ *in equation* (*b*) *in* (7). *Then, the following are true:*

(a) *Every positive solution of* (7*b*) *converges to* 0 *eventually monotonically if* $A + B > \alpha + \beta$.
(b) *Every positive solution of* (7*b*) *converges to* ∞ *eventually monotonically if* $A + B < \alpha + \beta$.
(c) *Every positive solution of* (7*b*) *converges to a finite limit if* $A + B = \alpha + \beta$ *and* $\alpha > A$.

The solutions of equation (7b) exhibit a greater variety of qualitatively different behaviours if the inequality in Proposition 2 is reversed. In particular, we state the following conjectures:

CONJECTURES 1–3.

1. Let $\alpha < A$ and $\beta > B$ in equation (b) in (7). Then for each positive solution $\{x_n\}$ of (7b) one of the subsequences $\{x_{2n}\}$, $\{x_{2n-1}\}$ converges to zero and the other to infinity. (this excludes trivial or constant solutions starting from equal initial values when α and β add up to $A + B$).
2. Let $\alpha = A$ and $\beta > B$ in equation (b) in (7). Then for each positive solution $\{x_n\}$ of (7b) one of the subsequences $\{x_{2n}\}$, $\{x_{2n-1}\}$ converges to infinity and the other to a positive number that can be arbitrarily large depending on initial values.
3. Let $\alpha < A$ and $\beta = B$ in equation (b) in (7). Then for each positive solution $\{x_n\}$ of (7b) one of the subsequences $\{x_{2n}\}$, $\{x_{2n-1}\}$ converges to zero and the other to a non-negative number.

In Conjecture 3 note that each positive solution converges to a 2-cycle $\{\ldots, 0, \psi, 0, \psi, \ldots\}$, where $\psi \geq 0$. The next conjecture suggests that the value $\psi = 0$ is possible with the existence of monotonically decreasing solutions, a situation that is similar to what occurs for the equation $x_{n+1} = x_{n-1}/(1 + x_n)$; see Ref. [1]. Thus, initial values play a role in determining the asymptotic behaviour in this case as well as for Conjecture 2.

CONJECTURES 4 AND 5.

4. If $\alpha < A$ and $\beta = B$ in equation (b) in (7) then there are positive initial values for which the corresponding solution $\{x_n\}$ decreases monotonically to zero.
5. If $\alpha = A$ and $\beta > B$ in equation (b) in (7) then there are positive initial values for which the corresponding solution $\{x_n\}$ increases monotonically to infinity.

If considering *all* solutions, i.e. $x_{-1}, x_0 \in \mathbb{R}$ then we must determine the forbidden sets (see Ref. [7]) of each equation in (7), i.e. the sets of singular initial points (x_0, x_{-1}) that after a finite number of iterations lead to division by zero and thus an undefinable value for x_{n+1}. A first observation about the forbidden or singularity sets of both equations in (7) is that these sets do not contain any points from the first or the third quadrants of the plane. This is self-evident for the first quadrant; for the third, we observe that for each positive solution $\{x_n\}$ of equation (7a) or (b), $\{-x_n\}$ is also a solution and conversely.

The forbidden set of equation (7a) is relatively easy to find, at least recursively. Clearly, if n is the least non-negative integer such that

$$Ax_n + Bx_{n-1} = 0, \tag{8}$$

then clearly x_{n+1} is undefined. Also if n is the least non-negative integer such that

$$\alpha x_n + \beta x_{n-1} = 0, \tag{9}$$

then $x_{n+1} = 0$. Therefore, $x_{n+2} = 0$ which leads to a singularity in (7a) for x_{n+3}. Let us assume that $A, B, \alpha, \beta \neq 0$. Then, using the notation in (4) with $x_{n-1} \neq 0$, the equalities (8) and (9) are equivalent to the two equalities

$$r_n = -\frac{B}{A}, \quad r_n = -\frac{\beta}{\alpha}.$$

The substitution (4) transforms equation (7a) into a first-order equation as follows:

$$r_{n+1} = \frac{x_{n+1}}{x_n} = \frac{(\alpha x_n/x_{n-1}) + \beta}{(Ax_n/x_{n-1}) + B} = \frac{\alpha r_n + \beta}{Ar_n + B}. \tag{10}$$

The mapping $g(r) = (\alpha r + \beta)/(Ar + B)$ is invertible and its inverse is easily calculated as

$$g^{-1}(t) = -\frac{Bt - \beta}{At - \alpha}.$$

Now, if $t_0 = -B/A$ and we define $t_{n+1} = g^{-1}(t_n)$ then the sequence $\{t_n\}$ is a backward orbit of (10) starting from its forbidden. Similarly, we can generate a backward orbit from $t_0 = -\beta/\alpha$. These observations lead to the following conclusion, which also resolves Open Problem 6.10.1(a) in Ref. [7] for non-negative coefficients.

PROPOSITION 3. *Assume that $A, B, \alpha, \beta > 0$ and let $\{\delta_n\}$ and $\{\nu_n\}$ be orbits of the equation $t_{n+1} = g^{-1}(t_n)$ with initial values of $\delta_0 = -B/A$ and $\nu_0 = -\beta/\alpha$, respectively. Then, the forbidden set of (7a) consists of the sequence of lines through the origin given by*

$$\bigcup_{n=0}^{\infty} \{(u,v) : u = \delta_n v\} \cup \bigcup_{n=0}^{\infty} \{(u,v) : u = \nu_n v\} \cup \{(u,v) : u = 0\}.$$

These lines are contained in the second and fourth quadrants of the plane, and if $\bar{t}_-$ denotes the unique negative fixed point of g^{-1} then $\bar{t}_-$ attracts all negative orbits of g^{-1} so that

$$\lim_{n\to\infty} \delta_n = \lim_{n\to\infty} \nu_n = \bar{t}_- = \frac{\alpha - B - \sqrt{(\alpha - B)^2 + 4\beta A}}{2A}.$$

We can use a similar procedure for equation (7b). In this case, the analogue of (10) is calculated as follows:

$$r_{n+1}r_n = \frac{x_{n+1}}{x_n}\frac{x_n}{x_{n-1}} = \frac{(\alpha x_n/x_{n-1}) + \beta}{(Ax_n/x_{n-1)} + B},$$

so that

$$r_{n+1} = \frac{\alpha r_n + \beta}{r_n(Ar_n + B)}. \tag{11}$$

Unlike the function g of (10), the mapping on the right-hand side of (11) is not invertible so a little more complication is expected in calculating the forbidden set of equation (7b). On the other hand, we have a slight simplification because the observation involving the numerator quantity in (9) does not apply to (7b); this equation does have nontrivial solutions that are zeros every other term if precisely one of the initial values is 0. In fact, if $x_0x_{-1} = 0$ with $\sigma = x_0 + x_{-1} \neq 0$ then these alternating solutions are

$$\{\ldots 0, (\beta/B)^n\sigma, 0, (\beta/B)^{n+1}\sigma \ldots\}.$$

We close this section with two open problems concerning equation (7b).

Open Problem 1. Determine the forbidden set of equation (7b) and extend Proposition 2 to all solutions of equation (7b).

Open Problem 2. Give a complete characterization of the behaviours of the solutions of equation (7b) if $\alpha B < \beta A$.

2. Third-order cases

In this section, we examine a few nontrivial special cases of (1) that have order 3.

Cases. $(0; \beta, A, C)$ and $(1; \alpha, A, C)$.

Either of these cases yields the third-order rational equation

$$x_{n+1} = \frac{x_n x_{n-1}}{ax_n + bx_{n-2}}, \quad a = \frac{A}{\mu} > 0, \quad b = \frac{C}{\mu} > 0, \tag{12}$$

where $\mu = \alpha$ or γ depending on the specific case. The forbidden (or singularity) set and all real solutions of (12) have been studied in Ref. [11]. Using the substitution (4), equation (12) is transformed into a second-order rational equation as follows

$$r_{n+1} = \frac{x_{n+1}}{x_n} = \frac{x_{n-1}/x_{n-2}}{a(x_n/x_{n-1})(x_{n-1}/x_{n-2}) + b} = \frac{r_{n-1}}{ar_nr_{n-1} + b}. \tag{13}$$

The forbidden sets and the behaviour of all real solutions of this second-order equation have been studied in Ref. [11]; the non-negative solutions of this equation have also been discussed in Ref. [2]. Equation (13) can be transformed into a linear first-order equation via the transformation

$$r_n r_{n-1} = \frac{1}{t_n}.$$

This fact makes a detailed study of (13) possible and the results of this study then yield substantial information about (12). In particular, we have the following result.

PROPOSITION 4.

(a) *The forbidden set S of* (12) *is a sequence of planes containing the origin in* $\mathbb{R}^3$ *as follows:*

$$S = \bigcup_{n=0}^{\infty} \{(u, v, w) : u = -\gamma_n w\} \cup \{(u, v, w) : u = 0\} \cup \{(u, v, w) : v = 0\},$$

where for $n = 0, 1, 2, \ldots,$

$$\gamma_n = \begin{cases} \frac{(a-b)}{(a/b)^{n+1}-1}, & \text{if } a \neq b > \\ \frac{b}{n+1}, & \text{if } a = b \end{cases}.$$

(b) *Let* $\{x_n\}$ *be a solution of* (12) *with initial point* $(x_0, x_{-1}, x_{-2}) \notin S$.

(i) *If* $a + b > 1$ *then* $\lim_{n\to\infty} x_n = 0$.

(ii) *If* $a + b = 1$ *then* $\{x_n\}$ *converges to a cycle* $\{\zeta_0, \zeta_1\}$ *of period* 2 (*not necessarily prime*) *where* $\zeta_1 = \xi_1 \zeta_0$ *with*

$$\zeta_0 = x_0 \prod_{n=1}^{\infty} \frac{1}{1 + \theta(1 + (a/b))^{-n}},$$

where θ *is a real number depending on parameters and initial values. The quantity* ξ_1 *depends on the properties of* (13)*; in particular, if* $\xi_1 = 1$ *then* $\{x_n\}$ *converges to the single number* ζ_0.

(iii) *If* $a + b < 1$ *then each of the sequences* $\{x_{2n}\}$ *and* $\{x_{2n+1}\}$ *is unbounded.*

Open Problem 3. Discuss the occurrence of monotone and oscillatory behaviours for the solutions of (12) and (13) similarly to Propositions 1 and 2.

Cases. $(0; \beta, B, C)$ and $(1; \alpha, B, C)$.

These cases are similar to the preceding cases above. Either of these cases yields the third-order rational equation

$$x_{n+1} = \frac{x_n x_{n-1}}{a x_{n-1} + b x_{n-2}}, \quad a = \frac{B}{\mu} > 0, \quad b = \frac{C}{\mu} > 0, \tag{14}$$

where $\mu = \beta$ or α depending on the specific case. Using the substitution (4), equation (14) is transformed into a second-order rational equation as follows

$$r_{n+1} = \frac{x_{n+1}}{x_n} = \frac{x_{n-1}/x_{n-2}}{(ax_{n-1}/x_{n-2}) + b} = \frac{r_{n-1}}{ar_{n-1} + b}. \tag{15}$$

Equation (15), which is easier to deal with than (13), can be used to work out the following open problem in a manner similar to the preceding case.

Open Problem 4. Determine the forbidden sets and the behaviours of all solutions of equation (14).

Cases. $(0; \gamma, A, C)$ and $(2; \alpha, A, C)$.

These cases yield the third-order rational equation

$$x_{n+1} = \frac{x_n x_{n-2}}{ax_n + bx_{n-2}}, \quad a = \frac{A}{\mu} > 0, \quad b = \frac{C}{\mu} > 0, \tag{16}$$

where $\mu = \alpha$ or γ depending on the specific case. Equation (16) can be transformed via the reciprocal transformation

$$y_n = \frac{1}{x_n}, \tag{17}$$

into a linear third-order difference equation:

$$y_{n+1} = by_n + ay_{n-2}. \tag{18}$$

Cases. $(1; \gamma, B, C)$ and $(2; \beta, B, C)$.

These cases yield the third-order rational equation

$$x_{n+1} = \frac{x_{n-1} x_{n-2}}{ax_{n-1} + bx_{n-2}}, \quad a = \frac{B}{\mu} > 0, \quad b = \frac{C}{\mu} > 0, \tag{19}$$

where $\mu = \gamma$ or β depending on the specific case. Equation (19) can be transformed via (17) into the linear third-order equation

$$y_{n+1} = by_{n-1} + ay_{n-2}. \tag{20}$$

We note that equations (18) and (20) are distinct from the (less interesting) linear third-order equations that arise directly from equation (1) in the following cases: $(0; \alpha, \beta, \gamma, A)$, $(1; \alpha, \beta, \gamma, B)$ and $(2; \alpha, \beta, \gamma, C)$.

One way of studying the properties of rational equations (16) and (19) is through the analysis of the linear equations (18) and (20). In particular, note that after the calculation of the eigenvalues of the linear equations, one can use their explicit solutions to determine explicit formulas for the solutions of (16) and (19) via the substitution (17). Likewise, forbidden sets of the rational equations can be calculated using the explicit solutions.

Related to the preceding cases are transformations of equations (16) and (19) into second-order rational equations via (4). Specifically, (16) is transformed as

$$r_{n+1} = \frac{x_{n+1}}{x_n} = \frac{1}{a(x_n/x_{n-1})(x_{n-1}/x_{n-2}) + b} = \frac{1}{ar_n r_{n-1} + b}, \tag{21}$$

and (19) is transformed as

$$r_{n+1}r_n = \frac{x_{n+1}}{x_n}\frac{x_n}{x_{n-1}} = \frac{1}{(ax_{n-1})/(x_{n-2}) + b} = \frac{1}{ar_{n-1} + b},$$

i.e.

$$r_{n+1} = \frac{1}{r_n(ar_{n-1} + b)}. \tag{22}$$

Clearly, information about the solutions of linear equations (18) and (20), can be used to obtain information about the forbidden sets and the solutions of rational equations (21) and (22). This leads to the obvious problem to work out:

Open Problem 5. Determine the forbidden sets and the behaviours of all solutions of the four related equations (16), (19), (21) and (22).

With $a, b > 0$ it is easily verified that each of (21) and (22) has a unique positive fixed point. We claim that these fixed points attract all positive solutions of each equation.

CONJECTURE 6. All positive solutions of each of (21) and (22) converge to the unique positive fixed point for each equation.

The case. $(0; \alpha, \beta, \gamma, B, C)$.

This is the final case, that we discuss in this note. These parameter values give the third-order difference equation

$$x_{n+1} = x_n\left(\frac{\alpha x_n + \beta x_{n-1} + \gamma x_{n-2}}{Bx_{n-1} + Cx_{n-2}}\right). \tag{23}$$

Once again, this equation is transformed to a second-order rational equation using (4) as follows:

$$\begin{aligned} r_{n+1} = \frac{x_{n+1}}{x_n} &= \frac{\alpha(x_n/x_{n-1})(x_{n-1}/x_{n-2}) + (\beta x_{n-1}/x_{n-2}) + \gamma}{Bx_{n-1}/x_{n-2} + C} \\ &= \frac{\alpha r_n r_{n-1} + \beta r_{n-1} + \gamma}{Br_{n-1} + C}. \end{aligned} \tag{24}$$

The second-order equation (24) is of the type studied in Ref. [4] from which we extract the following after relabeling coefficients.

PROPOSITION 5.

(a) *Let $\alpha < B$, $\beta \geq C$ and $B\gamma \leq C\beta$. Then equation* (24) *has a unique positive fixed point*

$$\bar{r} = \frac{\beta - C + \sqrt{(\beta - C)^2 + 4\gamma(B - \alpha)}}{2(B - \alpha)},$$

that attracts all positive solutions of (24).

(b) *Let $\gamma = 0$, $\alpha \leq B$ and $\beta < C$. Then the origin is the unique non-negative fixed point of* (24) *that attracts all positive solutions.*

A quick calculation shows that $\bar{r} < 1$ (or $\bar{r} > 1$) if and only if $\gamma < B - \alpha$ (respectively, $\gamma > B - \alpha$). Thus, from Proposition 5 and equality (6), we obtain the following corollary.

Proposition 6.

(a) *Let* $\alpha < B$, $\beta \geq C$, $B\gamma \leq C\beta$ *and* $\gamma > B - \alpha$ *then every positive solution of equation* (23) *is eventually increasing to infinity.*
(b) *If either* (*i*) $\alpha < B$, $\beta \geq C$, $B\gamma \leq C\beta$ *and* $\gamma < B - \alpha$, *or* (*ii*) $\gamma = 0$, $\alpha \leq B$ *and* $\beta < C$, *then every positive solution of equation* (23) *is eventually decreasing to zero.*

When some of the conditions in Propositions 5 and 6 do not hold, the solutions of equations (23) and (24) can be more varied and complex. For instance, the special case $(0; \alpha, B, C)$, i.e. the third-order equation

$$x_{n+1} = \frac{x_n^2}{ax_{n-1} + bx_{n-2}} \quad a = \frac{B}{\alpha}, \quad b = \frac{C}{\alpha}. \tag{25}$$

is transformed to a second-order rational equation using (4) as follows:

$$r_{n+1} = \frac{x_{n+1}}{x_n} = \frac{(x_n/x_{n-1})(x_{n-1}/x_{n-2})}{(ax_{n-1}/x_{n-2}) + b} = \frac{r_n r_{n-1}}{ar_{n-1} + b}. \tag{26}$$

Using the reciprocal transformation $t_n = 1/r_n$, equation (26) takes the form

$$t_{n+1} = t_n(a + bt_{n-1}). \tag{27}$$

Equation (27) is also known as the 'extended logistic equation' and if a, b are real numbers with $b < 0 < a$ then complex behaviour is possible for (27); see Refs. [6,8]. We close with the following open problem regarding (25):

Open Problem 6. Determine the forbidden sets and the behaviours of all solutions of equation (25) for $\alpha, B, C > 0$.

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