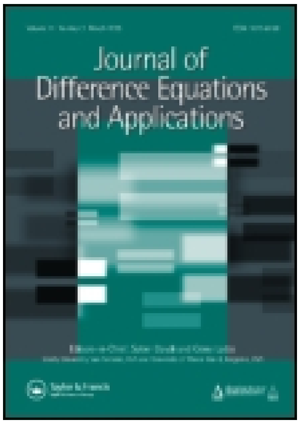

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Regarding the Equation $x_{n+1} = cx_n + f(x_n - x_{n-1})$

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Open Problems and Conjectures

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In this section we present some open problems and conjectures about some interesting types of difference equations. Please submit your problems and conjectures with all relevant information to G. Ladas.

Regarding the Equation $x_{n+1} = cx_n + f(x_n - x_{n-1})$

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The second order difference equation

$$x_{n+1} = cx_n + f(x_n - x_{n-1}), \quad \mathrm{c} \in [0, 1] \tag{1}$$

has a long history in macroeconomics that dates back to well over half a

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century. See [1–6] and [8]. Here, we present some of the few facts that are known about Eq. (1), and state a few of the many conjectures that remain unsolved.

BOUNDEDNESS AND ABSORBING INTERVALS

In the early economic literature, the "investment function" $f \in C^0(\mathbb{R})$ was assumed to be monotonically increasing, though not necessarily smooth. The following is established in Ref. [5]:

THEOREM 1 *If $c < 1$ and:*

(a) *f is nondecreasing and bounded below on $\mathbb{R}$;*

(b) *there is $t_0 > 0$ and $a \in (0,1)$ such that $f(t) \leq at$ for all $t \geq t_0$, then for every solution $\{x_n\}$ of Eq. (1), there are real numbers L, M such that $x_n \in [L, M]$ for all n exceeding some positive integer n_0. In particular, every solution of Eq. (1) is bounded.*

The interval $[L,M]$ in Theorem 1 is an *absorbing interval*. Note that L,M do not depend on the initial conditions, although the integer n_0 does. The proof of Theorem 1 utilizes properties of solutions of the first order equation

$$t_{n+1} = f(t_n), \quad t_0 \doteq x_0 - x_{-1}. \tag{2}$$

A straightforward modification shows that certain types of decreasing f that are bounded above also satisfy the conclusion of Theorem 1. On the other hand, if $c = 1$, then Eq. (1) can be written as

$$\Delta x_n = f(\Delta x_{n-1})$$

so each solution $\{x_n\}$ of Eq. (1) is a sequence of partial sums of the corresponding solution $\{t_n\}$ of Eq. (2); i.e.

$$x_n = x_0 + \sum_{k=1}^{n} t_k.$$

It follows that if $f(0) = 0$ and if $\{t_n\}$ converges at a sufficiently fast rate to zero (e.g. if $|f(t)| \leq \alpha|t|$ for some $\alpha \in (0,1)$) then $\{x_n\}$ converges to a real number (namely, the sum of the series $x_0 + \sum_{n=0}^{\infty} t_n$)which is finite but depends on initial conditions. Therefore, though every solution is bounded in this case, there are no absorbing intervals.

CONJECTURE 1 The conclusion of Theorem 1 holds if:

(a) $c < 1$ and f is nondecreasing;
(b) there is $r > 0$ and $a \in (0, 1)$ such that $|f(t)| \leq a|t|$ for all t such that $|t| \geq r$.

Conjecture 1 is true for the linear case $f(t) = \alpha t$ where $\alpha \in (0, 1)$. Thus the hypothesis "boundedness from below" in Theorem 1 is not necessary. On the other hand, if $\alpha < 0$, then the eigenvalues of the linearization of Eq. (1) are both real and the negative eigenvalue exceeds 1 in magnitude provided that $\alpha < -(1 + c)/2$. It follows that Conjecture 1 is false in the linear case if f is decreasing. Similarly, the case of linear f shows Conjecture 1 to be false if $a > 1$.

STABILITY OF THE ORIGIN

Assume that $f(0) = 0$ in Eq. (1), so that the origin is the unique fixed point of Eq. (1). Define $F(u, v) \doteq cu + f(u - v)$. If $|f(t)| \leq a|t|$for some $a > 0$, then

$$|F(u, v)| \leq c|u| + a|u - v| \leq (c + a)|u| + a|v| \leq (c + 2a)\max\{|u|, |v|\}.$$

It follows that if $c + 2a \in (0, 1)$, then the origin is globally asymptotically stable (see Ref. [7]). Note that this conclusion is valid regardless of whether f is monotonic. However, if $f(t) = at$ is linear, then for any $a \in (0, 1)$, the origin is globally asymptotically stable. Numerical simulations for nonlinear f tend to support this conclusion more generally; hence we propose:

CONJECTURE 2 If $c < 1$, f is nondecreasing and there is $a \in (0, 1)$ such that $|f(t)| \leq a|t|$ for all real t, then the origin is globally asymptotically stable.

OSCILLATIONS

The following is proved in Ref. [6].

THEOREM 2 *Suppose that* $g \in C^1(R)$, $g(0) = 0$ *and* $g'(0) > 1$, *and define* $f(t) = g(t) + \beta$ *where* $\beta > 0$. *If f satisfies the hypotheses of Theorem 1 and* β *is large enough that* $f(t) \geq 0$ *for all t, then every nontrivial solution of Eq. (1) oscillates persistently (i.e. it has two or more limit points).*

CONJECTURE 3 The conclusion of Theorem 2 holds if $f(t) = g(t) + \beta$ only satisfies the hypotheses of Conjecture 1.

Numerical simulations indicate that oscillations resulting from the conditions of Theorem 2 are always of the almost periodic type, and in certain cases, eventually periodic. This appears to be a consequence of the increasing nature of *f*. This naturally leads to an interesting problem.

OPEN PROBLEM. For $c < 1$, determine some sufficient conditions on *f* for all solutions of Eq. (1) to be eventually periodic or approach a periodic solution with period 2 or greater.

If *f* is not differentiable at 0, then solutions of Eq. (1) may exhibit complex or strange behavior. In Ref. [8] it is shown that if

(a) $f(t) = 0$ for $t \leq 0$;
(b) there is $r > 0$ such that $f(t) = bt$ for $0 < t < r$, and for $b > (1 + \sqrt{1 - c})^2$;
(c) there is $t_0 > r$ and $a \in (0, 1)$ such that $f(t) \leq at$ for all $t \geq t_0$;

then the origin is globally attracting yet unstable for all $c \in [0, 1)$. The same conclusion seems to hold if we only assume that $f(t) \geq bt$ in (b). The lower bound for *b* in statement (b) ensures that the linear segment of Eq. (1) with $f(t) = bt$ generates monotonically divergent solutions (until the interval $(0,r)$ is exited) regardless of the value of *c*.

If *f* is not monotonic, oscillatory behavior occurs in a peculiar manner. Numerical simulations indicate that the following is possibly true:

CONJECTURE 4 Let *f* be bounded and $f(0) = 0$. Assume that $f'(t)$ exists and is continuous, and that $f'(t) > 0$ for $t > 0$ and $f'(t) < 0$ for $t < 0$. If the right and left derivatives of *f* at 0 satisfy

$$f'(0^+), |f'(0^-)| > (1 + \sqrt{1 - c})^2, \quad c \in [0, 1)$$

then every solution $\{x_n\}$ of Eq. (1) satisfies the following:

$$0 \leq \liminf_{n\to\infty} x_n < \limsup_{n\to\infty} x_n < \infty.$$

Note that under the hypotheses of Conjecture 4, the origin is both the unique minimum of f and the unique fixed point of Eq. (1). Thus solutions of Eq. (1) appear to be oscillating *above*, *not about* the equilibrium point of Eq. (1).

Hypotheses like $f(0) = 0$, smoothness or boundedness are not essential; they make it easier to state the conjecture. Further, with them it is easy to show that the limit infimum is non-negative and the limit supremum is finite. A simple example of f that satisfies all of the hypotheses of Conjecture 4 is $f(t) = \arctan|bt|$ with $b > (1 + \sqrt{1-c})^2$.

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