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Monotone and oscillatory solutions of a rational difference equation containing quadratic terms

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Dedicated to Gerry Ladas on the occasion of his 70th birthday

We show that the second order rational difference equation

$$x_{n+1} = \frac{Ax_n^2 + Bx_nx_{n-1} + Cx_{n-1}^2}{\alpha x_n + \beta x_{n-1}}$$

has several qualitatively different types of positive solutions. Depending on the non-negative parameter values A,B,C,α,β, all solutions may converge to 0, or they may all be unbounded. For some parameter values both cases can occur, or coexist depending on the initial values. We find converging solutions of both monotonic and oscillatory types, as well as periodic solutions with period two. A semiconjugate relation facilitates derivations of these results by providing a link to a rational first order equation.

Keywords: rational; quadratic; monotonic; non-monotonic; period two; semiconjugate

AMS Subject Classification: 39A10; 39A11

1. Introduction

The second order difference equation

$$x_{n+1} = \frac{Ax_n^2 + Bx_nx_{n-1} + Cx_{n-1}^2}{\alpha x_n + \beta x_{n-1}} \tag{1}$$

is an example of rational equation with a quadratic numerator and a linear denominator, or a QLR equation for short. Rational difference equations having both a linear numerator and a linear denominator have been studied extensively by Gerry Ladas and colleagues; see e.g. [8–11] which include extensive lists of references. This substantial research makes a strong case for studying the behaviour of rational difference equations as well as providing a great deal of information about the behaviour of rational equations with linear terms. By comparison, rational equations containing quadratic or higher power terms in their numerators or denominators have not yet been systematically studied; a few references that contain detailed studies of such equations of order 2 are [1–5,7].

Applications of rational difference equations containing quadratic and cubic terms to biological models have been discussed in Refs. [2,3] via systems of first order rational equations

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using monotone methods. In Ref [4] we considered QLR difference equations in some detail and found that they exhibit a typically broader range of behaviours than equations having only linear terms. Therefore, understanding the nature of solutions of QLR equations should add significantly to our general understanding of the remarkable class of rational difference equations and their applications.

In this paper, we study the positive solutions of (1). Clearly, if the parameters (or coefficients) satisfy the conditions

$$\alpha > 0, A, B, C, \beta \geq 0 \quad \text{with } A + B + C > 0 \tag{2}$$

then each solution $\{x_n\}$ of (1) with initial values

$$x_0, x_{-1} \in (0, \infty) \tag{3}$$

is a positive solution. Although the weaker inequalities $\alpha + \beta > 0$ and $B > -2\sqrt{AC}$ are sufficient for positive solutions, we do not consider such cases in this paper and assume that (2) and (3) both hold without further explicit mention.

Equation (1), subject to (2) and (3), is essentially different from the QLR equations considered in Ref [4] or in most other studies because (1) has no isolated fixed points. Therefore, typical methods of analysis that utilise fixed points (e.g., linearization, semi-cycle analysis) cannot be used. By way of comparison, the linear fractional (linear–over-linear) analogue of (1), i.e., the constants-free equation

$$x_{n+1} = \frac{Ax_n + Bx_{n-1}}{\alpha x_n + \beta x_{n-1}}, \quad A, B, \alpha, \beta \geq 0, \quad A + B, \alpha + \beta > 0. \tag{4}$$

has a unique, isolated, positive fixed point $\bar{x} = (A + B)/(\alpha + \beta)$. Thus (4), is fully amenable to linearization and semi-cycle analysis; see Ref. [11] for a detailed study of the solutions of (4) using these methods.

For equation (1), we observe that it is homogeneous of degree 1 (see Ref. [14]) and thus has a semiconjugate factorization that ties it to a first order rational equation. This feature enables us to show the existence of several qualitatively different types of positive solutions for (1). We establish the occurrence of monotonic convergence, oscillatory convergence and periodic solutions with period 2. In particular, for the special case $C = 0$, we solve the Open Problem 6.10.1(b) in Ref. [11] when the coefficients are non-negative.

2. The ratio mapping

For background concepts used in this and subsequent sections we refer to the texts [6,9,12]. We may write equation (1) as

$$\frac{x_{n+1}}{x_n} = \frac{Ax_n^2 + Bx_nx_{n-1} + Cx_{n-1}^2}{\alpha x_n^2 + \beta x_n x_{n-1}} = \frac{A(x_n/x_{n-1})^2 + B(x_n/x_{n-1}) + C}{\alpha(x_n/x_{n-1})^2 + \beta(x_n/x_{n-1})}.$$

Now, if we define the ratios $r_n = x_n/x_{n-1}$ for $n \geq 0$ then the following first order, rational difference equation is satisfied by the sequence of ratios:

$$r_{n+1} = \frac{Ar_n^2 + Br_n + C}{\alpha r_n^2 + \beta r_n}. \tag{5}$$

Under conditions (2) $\{r_n\}$ is a positive solution of (5) if $r_0 = x_0/x_{-1} > 0$; this is guaranteed by (3). For such $\{r_n\}$, we note that

$$x_n = r_n x_{n-1} = r_n r_{n-1} x_{n-2} = \cdots = r_n r_{n-1} \cdots r_1 x_0. \tag{6}$$

Hence, each positive solution of (1) can be obtained from a given positive solution of (5) in the form (6). Equations (1) and (5) are semiconjugates; see Ref. [13]. Also see Ref. [14] for a generalization of the ratios idea to homogeneous difference equations on groups.

An important difference between (5) and (1) is the fact that (5) has an isolated fixed point in $(0,\infty)$.

Lemma 1. *Let $B + C > 0$ or $A > \beta$.*

(a) *Equation* (5) *has a unique fixed point $\bar{r} \in (0,\infty)$.*
(b) *$\bar{r} = 1$ if and only if $A + B + C = \alpha + \beta$.*
(c) *For $\bar{r} \neq 1$, $(\bar{r} - 1)(A + B + C - \alpha - \beta) > 0$; i.e., $\bar{r} < 1$ (respectively, $\bar{r} > 1$) if and only if $A + B + C < \alpha + \beta$ (respectively, $A + B + C > \alpha + \beta$).*

Proof. (a) Fixed points of (5) in $(0,\infty)$ are positive solutions of the equation

$$\frac{Ar^2 + Br + C}{\alpha r^2 + \beta r} = r.$$

This equation is equivalent to the polynomial equation

$$\phi(r) \doteq -\alpha r^3 + (A - \beta)r^2 + Br + C = 0.$$

First, assume that $C > 0$. Since $\alpha > 0$ and $\phi(0) = C > 0$, it is clear that ϕ has at least one positive root $\bar{r}$. Further, there is precisely one sign change in ϕ regardless of the sign of $A - \beta$, so by the Descartes rule of signs [15] ϕ has at most one positive root. It follows that $\bar{r}$ is unique.

Next, if $C = 0$ but $B > 0$ then the non-zero roots of ϕ are the same as the roots of the quadratic equation $\alpha r^2 - (A - \beta)r - B = 0$, i.e.,

$$\frac{A - \beta - \sqrt{(A - \beta)^2 + 4\alpha B}}{2\alpha} < 0 < \frac{A - \beta + \sqrt{(A - \beta)^2 + 4\alpha B}}{2\alpha}.$$

Thus, once again there is a unique positive root. Finally, if $B > C = 0$ then by hypothesis $A - \beta$ and the only non-zero root of ϕ is $(A - \beta)/\alpha$, which is positive and unique as such.

(b) $\phi(1) = A + B + C - \alpha - \beta = 0$ so $\bar{r} = 1$.

(c) To show that $\bar{r} - 1$ has the same sign as $A + B + C - \alpha - \beta$, from the equation $\phi(\bar{r}) = 0$, we obtain

$$\frac{A - \beta}{\alpha} + \frac{B}{\alpha \bar{r}} + \frac{C}{\alpha \bar{r}^2} = \bar{r}. \tag{7}$$

Suppose first that $\bar{r} < 1$. Then (7) gives

$$0 > \bar{r} - 1 \geq \frac{A - \beta}{\alpha} + \frac{B}{\alpha} + \frac{C}{\alpha} - 1 = \frac{A + B + C - \alpha - \beta}{\alpha}.$$

This chain of inequalities is valid if and only if $A + B + C < \alpha + \beta$. Similarly, if $\bar{r} > 1$ then from (7) it follows that

$$0 < \bar{r} - 1 \leq \frac{A - \beta}{\alpha} + \frac{B}{\alpha} + \frac{C}{\alpha} - 1 = \frac{A + B + C - \alpha - \beta}{\alpha},$$

which holds if and only if $A + B + C > \alpha + \beta$. The proof is now complete.

It is convenient in what follows to define the *ratio mapping*

$$g(r) = \frac{Ar^2 + Br + C}{\alpha r^2 + \beta r}$$

so that equation (5) can be stated as $r_{n+1} = g(r_n)$. Thus, Lemma 1 shows that the continuous map $g{:}(0,\infty) \rightarrow (0,\infty)$ has a unique fixed point $\bar{r}$. It is an interesting fact that although the precise value of $\bar{r}$ is not generally easy to calculate, its position relative to one is easily determined from Lemma 1. This fact is important in the study of solutions of (1).

Remark. (*The invariant ray*) Suppose that the hypotheses of Lemma 1 hold. Then the ray $\{(x, \bar{r}x) : x \in (0, \infty)\}$, or $\bar{r}x$ for short, is an invariant set of (1) in the state space $(0,\infty)^2$ since if (x_{-1},x_0) is a point on this ray so that $x_0 = \bar{r}x_{-1}$ then $r_0 = \bar{r}$ and thus $x_1 = \bar{r}x_0$; i.e. (x_0,x_1) is on $\bar{r}x$. By induction, the state-space orbit (x_{n-1},x_n) is on the invariant ray for all n. Now if $\bar{r} < 1$, then every orbit of (1) starting on $\bar{r}x$ will converge monotonically to zero on $\bar{r}x$ since by equation (6)

$$x_n = (\bar{r})^n x_0. \tag{8}$$

This inequality also shows that if $\bar{r} > 1$ then every orbit in $\bar{r}x$ goes to infinity monotonically and if $\bar{r} = 1$ then every orbit in $\bar{r}x$ is stationary (a point).

The invariant ray $\bar{r}x$ is analogous to a fixed point for (1), in the sense that by taking the quotient of $(0,\infty)^2$ modulo $\bar{r}x$, equation (1) is transformed into a topological conjugate of (5), and the ray $\bar{r}x$ into the point $\bar{r}$, on the space of rays through the origin (see Ref. [13]).

Monotonic behaviour on the invariant ray may or may not be the representative of other solutions. In most cases in Section 3, the behaviour on the invariant ray is in fact representative of all solutions but in Section 4 this is not the case.

3. Monotone solutions

Let $\{x_n\}$ be a positive solution of (1). We say that $\{x_n\}$ converges to 0 eventually monotonically if $\{x_n\}$ is a decreasing sequence for all n greater than some positive integer k and has limit 0. We also say that $\{x_n\}$ converges to ∞ eventually monotonically if $\{1/x_n\}$ converges to 0 eventually monotonically. The next result is essential for determining when all solutions of (1) are eventually monotonic. Its proof also provides information that we use in Section 4 on periodic and other non-monotonic solutions.

LEMMA 2. (*a*) *Let* $B + C > 0$. *The fixed point* $\bar{r}$ *of* g *is globally asymptotically stable on* $(0,\infty)$ *if and only if*

$$\alpha C \leq (\beta + A)B + 2A\sqrt{C(\beta + A)}. \tag{9}$$

with the inequality strict if $A = 0$.

(*b*) *If* $B = C = 0$ *and* $A > \beta$ *then the positive fixed point* $\bar{r} = (A - \beta)/\alpha$ *is globally asymptotically stable on* $(0,\infty)$.

Proof. (a) We show that the sign of the function $g^2(r) - r = (g(r)) - r$ is opposite to $r - \bar{r}$ for $r > 0$, $r \neq \bar{r}$ (Theorem 2.1.2 in Ref. [10]).

$$g^2(r) - r = \frac{A\left(\frac{Ar^2 + Br + C}{\alpha r^2 + \beta r}\right)^2 + B\left(\frac{Ar^2 + Br + C}{\alpha r^2 + \beta r}\right) + C}{\alpha\left(\frac{Ar^2 + Br + C}{\alpha r^2 + \beta r}\right)^2 + \beta\left(\frac{Ar^2 + Br + C}{\alpha r^2 + \beta r}\right)} - r$$

$$= \frac{A\left(Ar^2 + Br + C\right)^2 + B\left(Ar^2 + Br + C\right)\left(\alpha r^2 + \beta r\right) + C\left(\alpha r^2 + \beta r\right)^2}{\alpha\left(Ar^2 + Br + C\right)^2 + \beta\left(Ar^2 + Br + C\right)\left(\alpha r^2 + \beta r\right)}$$

$$- \frac{r\left[\alpha\left(Ar^2 + Br + C\right)^2 + \beta\left(Ar^2 + Br + C\right)\left(\alpha r^2 + \beta r\right)\right]}{\alpha\left(Ar^2 + Br + C\right)^2 + \beta\left(Ar^2 + Br + C\right)\left(\alpha r^2 + \beta r\right)}.$$

Combining the fractions and simplifying the numerator gives:

$$\begin{aligned} g^2(r) - r = \Big[&- \alpha A\left(A + \beta\right)r^5 + \left(A^3 + C\alpha^2 - AB\alpha - A\beta^2 - B\alpha\beta\right)r^4 \\ &+ \left(2A^2B + AB\beta + C\alpha\beta - 2AC\alpha - B\beta^2\right)r^3 \\ &+ \left(2A^2C + AB^2 + B^2\beta - BC\alpha\right)r^2 + \left(2ABC + BC\beta - C^2\alpha\right)r + AC\Big] \\ &\div \Big[\alpha\left(Ar^2 + Br + C\right)^2 + \beta\left(Ar^2 + Br + C\right)\left(\alpha r^2 + \beta r\right)\Big]. \end{aligned}$$

Let $P(r)$ be the quintic polynomial in the numerator of $g^2(r) - r$. Since, the denominator of $g^2(r) - r$ is positive for $r > 0$, the sign of $P(r)$ is the same as the sign of $g^2(r) - r$ for $r > 0$. Next, we divide $P(r)$ by the cubic polynomial $\phi(r)$ that determines the fixed points of g in Lemma 1. The polynomials are divisible and the quotient is the quadratic polynomial

$$\psi(r) = \left(A + \beta\right)Ar^2 - \left(\alpha C - AB - \beta B\right)r + AC. \tag{10}$$

Observe that if $\psi(r) > 0$ for $r > 0$ then $g^2(r) - r$ has the same sign as $\phi(r)$ on $(0,\infty)$. Under the given hypotheses Lemma 1 implies that $\phi(r)$ has a unique positive root at $\bar{r}$. So $\phi(0) \geq 0$; also $\phi(r) \to -\infty$ as $r \to \infty$, so that the sign of $\phi(r)$ is opposite of the sign of $r - \bar{r}$ on $(0,\infty)$. Thus, to complete the proof, we find conditions implying $\psi(r) > 0$.

We consider two cases: first, if $A = 0$, then $\psi(r) > 0$ for $r > 0$ if and only if

$$\alpha C < (A + \beta)B = \beta B. \tag{11}$$

Since $A = 0$ implies that $B + C > 0$, equation (9) is valid with strict inequality and reduces to (11). Now assume that $A > 0$. Then, the roots of ψ can be found explicitly as

$$\rho_{\pm} = \frac{\alpha C - AB - \beta B \pm \sqrt{(\alpha C - AB - \beta B) - 4(A+\beta)A^2 C}}{2(A+\beta)A}. \tag{12}$$

Note that $\psi(r) > 0$ for $r > 0$ if and only if ψ has no real and positive roots, i.e., if and only if $\rho_{\pm}$ are either complex or they are real and non-positive. First, $\rho_{\pm}$ are complex if and only if

$$|\alpha C - (A+\beta)B| < 2A\sqrt{C(A+\beta)}.$$

or equivalently,

$$(\beta + A)B - 2A\sqrt{C(\beta + A)} < \alpha C < (\beta + A)B + 2A\sqrt{C(\beta + A)}. \tag{13}$$

Also, since $B + C > 0$ it follows that $\rho_{\pm}$ are real and non-positive if and only if

$$|\alpha C - (A+\beta)B| \geq 2A\sqrt{C(A+\beta)} \quad \text{and} \quad \alpha C < (A+\beta)B$$

or equivalently,

$$\alpha C \leq (\beta + A)B - 2A\sqrt{C(\beta + A)} \tag{14}$$

Inequalities of (13) and (14) together, i.e., one or the other holding, are equivalent to (9) with the strict inequality. Finally, to account for possible equality, if

$$|\alpha C - (A+\beta)B| = 2A\sqrt{C(A+\beta)} \quad \text{and} \quad \alpha C > (A+\beta)B$$

we obtain

$$\alpha C - (A+\beta)B = 2A\sqrt{C(A+\beta)} \tag{15}$$

and

$$\rho_- = \rho_+ = \frac{\alpha C - (A+\beta)B}{2(A+\beta)A} = \frac{2A\sqrt{C(A+\beta)}}{2(A+\beta)A} = \sqrt{\frac{C}{A+\beta}} > 0.$$

Notice that

$$\begin{aligned}\phi\left(\sqrt{\frac{C}{A+\beta}}\right) &= \frac{-\alpha C}{A+\beta}\sqrt{\frac{C}{A+\beta}} + \frac{(A-\beta)C}{A+\beta} + B\sqrt{\frac{C}{A+\beta}} + C \\ &= \frac{-\alpha C + (A+\beta)B}{A+\beta}\sqrt{\frac{C}{A+\beta}} + \frac{2AC}{A+\beta} = 0,\end{aligned}$$

where the value 0 is obtained by using (15) again. Therefore, the unique positive root of ψ is the same as the unique root of ϕ i.e. the fixed point $\bar{r} = \sqrt{C/(A+\beta)}$. Since ψ is a quadratic polynomial this value of $\bar{r}$ gives its minimum value of 0, so $\psi(r) > 0$ if $r \neq \bar{r}$. Hence once again the sign of ϕ determines the sign of $g^2(r) - r$ and since (15) is the same as (9) with equality, the proof of (a) is complete.

(b) In this case, $\psi(r) = (A+\beta)Ar^2 > 0$ for $r > 0$. So the conclusion follows easily from the arguments in the proof of (a).

THEOREM 1. (*a*) *Assume that either* $B + C > 0$ *and* (9) *holds, or* $B = C = 0$ *and* $A > \beta$.

(*i*) *If* $A + B + C < \alpha + \beta$. *Then, every positive solution of* (1) *converges to* 0 *eventually monotonically.*

(*ii*) *If* $A + B + C > \alpha + \beta$. *Then, every positive solution of* (1) *converges to* ∞ *eventually monotonically.*

(*b*) *Let* $B = C = 0$ *and* $A \le \beta$. *Then, every positive solution of* (1) *converges to* 0 *eventually monotonically.*

Proof. (a), (i): By Lemma 1 there is a fixed point $\bar{r} \in (0,1)$ for g which is globally attracting by Lemma 2. Hence, there is $k \ge 1$ such that $r_n < 1$ for all $n > k$ and (6) implies that x_n is decreasing to zero if $n > k$.

(a), (ii): The argument is similar to that for (a), (i) except that now $\bar{r} > 1$ so that $r_n > 1$ for all sufficiently large n.

(b): In this case, $\phi(r) = -\alpha r^3 - (\beta - A)r^2 < 0$ for $r > 0$ so that $g(r) < r$ (in particular, g has no positive fixed points). Thus, $r_n \to 0$ as $n \to \infty$ and (6) implies that x_n is (eventually) decreasing to zero.

Setting $C = 0$ in Theorem 1 gives the next result concerning Open Problem 6.10.1 in the reference [11]; also see Corollaries 3 and 5 below.

COROLLARY 1. *Let* $A + \beta$, $A + B > 0$ *in the following difference equation:*

$$x_{n+1} = x_n\left(\frac{Ax_n + Bx_{n-1}}{\alpha x_n + \beta x_{n-1}}\right), \quad x_{-1}, x_0 > 0. \tag{16}$$

(a) *Every positive solution of (16) converges to 0 eventually monotonically if either of the following conditions holds:*

  (i) $A + B < \alpha + \beta$ *with* $A > \beta$ *if* $B = 0$;

  (ii) $B = 0$ *and* $A \le \beta$.

(b) Every positive solution of (16) converges to ∞ *eventually monotonically if* $A + B > \alpha + \beta$.

Setting $A = 0$ in Theorem 1 gives the next result.

COROLLARY 2. *Let* $\beta B > \alpha C$ *in the following difference equation:*

$$x_{n+1} = x_{n-1}\left(\frac{Bx_n + Cx_{n-1}}{\alpha x_n + \beta x_{n-1}}\right), \quad x_{-1}, x_0 > 0. \tag{17}$$

(a) *Every positive solution of* (17) *converges to* 0 *eventually monotonically if* $B + C < \alpha + \beta$.
(b) *Every positive solution of* (17) *converges to* ∞ *eventually monotonically if* $B + C > \alpha + \beta$.

Example. By considering a special case, we show that if the *reverse* of the inequality in Corollary 2 holds then (17) has solutions that are not eventually monotonic. Consider

$$x_{n+1} = \frac{x_{n-1}^2}{x_n}, \tag{18}$$

which is a special case of (17) with $B = \beta = 0$ and $C = \alpha$. The ratios equation for (18) is $r_{n+1} = r_n^{-2}$ whose explicit solution is obtained inductively as $r_n = r_0^{(-2)^n}$. This solution and (6) give the explicit solution of (18) as

$$x_n = r_0^{-2+(-2)^2+\cdots+(-2)^{n-1}} x_0 = x_0 r_0^{2[(-2)^n-1]/3}.$$

It is clear from this that for all $x_0, x_{-1} > 0$, one of x_{2n-1} and x_{2n} converges to 0 and the other to ∞ as $n \to \infty$; i.e. $\{x_n\}$ is unbounded but does not converge to ∞. This example also exhibits a type of extreme behaviour for the solutions of (1) that is different from the types of behaviour that we discuss in this article.

4. Non-monotonic and periodic solutions

In this section, we discuss bounded solutions, periodic solutions and solutions that converge to 0 or ∞ in a non-monotonic way depending on how $A + B + C$ compares to $\alpha + \beta$. We recall that under the hypotheses of Lemma 1, the behaviour on the invariant ray is still monotonic; thus monotonic solutions coexist with non-monotonic ones.

We start by discussing the special case $A + B + C = \alpha + \beta$. Under the conditions of Lemma 1, this equality implies the existence of a fixed point $\bar{r} = 1$ for g. When this fixed point is globally attracting, it is generally difficult to reach any conclusions about the asymptotic behaviour of solutions of (1) by using relation of (6). So we use a standard contraction result instead, which we state as a lemma without proof.

Lemma 3. *Let y_n be a given sequence of real numbers. If there exists a sequence $\{P_n\}$ of positive real numbers such that*

$$|y_{n+1} - y_n| \le p_n |y_n - y_{n-1}|, \qquad n = 0, 1, \ldots \tag{19}$$

and $\lim_{n\to\infty} p_n = p < 1$, *then y_n converges to a finite limit.*

Lemma 4. *Let $A + B + C = \alpha + \beta$ with $B + C > 0$. Then, the following conditions are equivalent:*

(i) $C < A + \beta$;
(ii) $\alpha < 2A + B$;
(iii) $\alpha C < (\beta + A)B + 2A\sqrt{(\beta + A)C}$.

Proof. (i) $\Leftrightarrow$ (ii):

$$C < A + \beta \Leftrightarrow A + B + C < 2A + B + \beta \Leftrightarrow \alpha < 2A + B.$$

(ii) $\Rightarrow$ (iii): If $C > 0$ then

$$\alpha < B + 2A \Rightarrow \alpha C < BC + 2A\left(\sqrt{C}\right)^2 < B(A + \beta) + 2A\sqrt{(A + \beta)C}.$$

If $C = 0$ then $B > 0$ so that

$$\alpha C = 0 < B(A + \beta) = B(A + \beta) + 2A\sqrt{(A + \beta)C}.$$

(iii) $\Rightarrow$ (ii): We show that the negation of (ii) implies the negation of (iii). Since the negation of (ii), i.e. $2A + B \leq \alpha$ has already been shown equivalent to the negation of (i), i.e. $A + \beta \leq C$, we have

$$B(A + \beta) + 2A\sqrt{(A + \beta)C} \leq BC + 2AC = (2A + B)C \leq \alpha C$$

which is the negation of (iii), as required.

THEOREM 2. *Let $A + B + C = \alpha + \beta$ and $B + C > 0$. If $\alpha < 2A + B$, then every positive solution of* (1) *converges to a finite limit.*

Proof. Lemma 1 implies that g has a fixed point at 1. By Lemma 4, (9) holds and thus 1 is globally attracting by Lemma 2(a). Next, use the hypotheses to write

$$x_{n+1} - x_n = \frac{(A - \alpha)x_n^2 + (B - \beta)x_n x_{n-1} + Cx_{n-1}^2}{\alpha x_n + \beta x_{n-1}} = \frac{[(\alpha - A)x_n + Cx_{n-1}](x_{n-1} - x_n)}{\alpha x_n + \beta x_{n-1}}.$$

Therefore,

$$|x_{n+1} - x_n| \leq \frac{|\alpha - A|x_n + Cx_{n-1}}{\alpha x_n + \beta x_{n-1}} |x_n - x_{n-1}|.$$

Since 1 is globally attracting. we have $\lim_{n\to\infty}(x_n/x_{n-1}) = 1$. Therefore,

$$\lim_{n\to\infty} \frac{|\alpha - A|x_n + Cx_{n-1}}{\alpha x_n + \beta x_{n-1}} = \lim_{n\to\infty} \frac{|\alpha - A|\left(\frac{x_n}{x_{n-1}}\right) + C}{\alpha\left(\frac{x_n}{x_{n-1}}\right) + \beta} = \frac{|\alpha - A| + C}{\alpha + \beta}.$$

Now,

$$\frac{|\alpha - A| + C}{\alpha + \beta} < 1 \Leftrightarrow |\alpha - A| < \alpha + \beta - C = A + B \Leftrightarrow -B < \alpha < 2A + B,$$

which is true by hypothesis. Thus, by Lemma 3, x_n converges to a finite limit and the proof is complete.

COROLLARY 3. (*a*) *In equation* (16) *assume that $A + B = \alpha + \beta$, $B > 0$ and $\alpha < 2A + B$. Then, every positive solution of* (16) *converges to a finite limit.*

(*b*) *In equation* (17) *assume that $B + C = \alpha + \beta$ and $\alpha < B$. Then, every positive solution of* (17) *converges to a finite limit.*

So far we have not considered whether the positive solutions of (1) under the conditions of Theorem 2 are monotone or not. If the mapping g is decreasing at the fixed point 1, i.e., if $g'(1) < 0$, then the attracting nature of 1 means that the ratios sequence r_n oscillates about 1. Thus, the sequence x_n cannot be eventually monotonic.

LEMMA 5. *The mapping g is decreasing on* $(0,\infty)$ *if* $\beta A \leq \alpha B$ *and* $C > 0$ *or if* $\beta A < \alpha B$ *and* $C \geq 0$.

Proof. The derivative of g is

$$g^{'}(r) = \frac{(2Ar + B)(\alpha r^2 + \beta r) - (2\alpha r + \beta)(Ar^2 + Br + C)}{(\alpha r^2 + \beta r)^2}.$$

So $g'(r) < 0$ when the numerator is negative. Multiplying terms in the numerator and rearranging them gives the requirement

$$(\beta A - \alpha B)r^2 - 2\alpha Cr - \beta C < 0. \tag{20}$$

This last inequality is clearly true for $r > 0$ under the stated hypotheses and the proof is complete.

Remark. (Non-monotonicity) In Theorem 2 or Corollary 3, solutions may or may not be monotonic. For instance, in Corollary 3(b), the conditions of Lemma 5 are satisfied so solutions are not eventually monotonic; they approach their limits in an oscillatory fashion (unless, of course, $x_0 = x_{-1}$). However, in Corollary 3(a) if $\beta A > \alpha B$ then it is clear that the left hand side of (20) is positive (given that $C = 0$ so that g is increasing for $r > 0$; thus the converging solutions are eventually monotonic in this case. For more details see Theorem 3 below; also see Corollary 4 which can be compared with Theorem 2.

LEMMA 6. *Assume that* $C > 0$. *Then:*

(a) *g has a unique pair of distinct, positive, period-*2 *points, namely,* $\rho_{\pm}$ *in* (12), *if and only if*

$$\alpha C > (\beta + A)B + 2A\sqrt{C(\beta + A)}. \tag{21}$$

(b) $\rho_- < 1 < \rho_+$ *if and only if*

$$\alpha C > (\beta + A)B + (A + C + \beta)A = AC + (A + \beta)(A + B). \tag{22}$$

(c) *The product of the period-*2 *points is given by*

$$\rho_- \rho_+ = \frac{C}{A + \beta}. \tag{23}$$

Proof. (a) The positive roots of the mapping ψ in (10) are the non-fixed point roots of $g^2(r)$; hence, these roots of ψ are the periodic points of g. As seen in the proof of Lemma 2, distinct real roots of ψ, i.e. the numbers $\rho_{\pm}$, exist and are positive if and only if (21) holds.

(b) From (12) we find that

$$\rho_- < 1 \Leftrightarrow \sqrt{(\alpha C - AB - \beta B)^2 - 4(A+\beta)A^2 C} > \alpha C - (2A+B)(A+\beta)$$

$$\rho_+ > 1 \Leftrightarrow \sqrt{(\alpha C - AB - \beta B)^2 - 4(A+\beta)A^2 C} > -[\alpha C - (2A+B)(A+\beta)].$$

Therefore, $\rho_- < 1 < \rho_+$ if and only if

$$\sqrt{\left(\alpha C - AB - \beta B\right)^2 - 4\left(A+\beta\right)A^2 C} > |\alpha C - (2A+B)(A+\beta)|$$

$$[\alpha C - (A+\beta)B]^2 - 4\left(A+\beta\right)A^2 C > \{[\alpha C - (A+\beta)B] - 2A(A+\beta)\}^2$$

$$-AC > -\alpha C + (A+\beta)B + (A+\beta)A.$$

The last expression is equivalent to (22). As might be expected from a comparison of (21) and (22), it is easy to verify that indeed

$$(A + C + \beta)A \geq 2A\sqrt{C(\beta + A)}$$

with equality holding if and only if $A + B = C$.

(c) The relation of (23) may be verified by direct multiplication or more quickly, by writing

$$\psi(r) = (A+\beta)A(r-\rho_-)(r-\rho_+)$$

to see, using of (10), that $(A+\beta)A\rho_-\rho_+ = AC$.

Lemma 7. *Assume that* $A,C > 0$, $\beta A \leq \alpha B$ *and* (21) *holds. Then, the two-cycle* $\{\rho_-,\rho_+\}$ *attracts all orbits of* g *in* $(0,\infty)$ *except for* $\bar{r}$.

Proof. Note that $g^2 = g \circ g^2$ is increasing on $(0,\infty)$ since by Lemma 5 g is decreasing there. Thus, every orbit of g^2 either converges to a fixed point or is unbounded [12]. Lemma 6 implies that $\{\rho_- \rho_+\}$ is the unique and positive two-cycle of g and Lemmas 1 and 2 imply that g has a unique positive fixed point $\bar{r}$ that is unstable. It follows that every orbit of g^2 converges to ρ_- or ρ_+ and the proof is completed.

Before presenting the next result, a few definitions and observations are needed. We say that a solution $\{x_n\}$ of (1) converges to 0 in an oscillatory fashion if $\{x_n\}$ converges to 0 but not eventually monotonically; i.e., $\lim_{n\to\infty} x_n = 0$ but $\{x_n\}$ is not an eventually decreasing sequence. We also say that $\{x_n\}$ converges to 0 in an oscillatory fashion if $\{1/x_n\}$ converges to 0 in an oscillatory fashion; thus $\lim_{n\to\infty} x_n = \infty$ but $\{x_n\}$ is not an eventually increasing sequence.

If $\{\rho_1,\rho_2,\ldots,\rho_k\}$ is one cycle of a positive periodic solution of (5) with period k, then defining $x_1 = \rho_1 x_0$ gives

$$x_2 = r_2 x_1 = \rho_2\rho_1 x_0, \cdots, x_k = \rho_k \cdots \rho_2\rho_1 x_0.$$

Thus, if the product $\rho_k \ldots \rho_2\rho_1 + 1$ then $x_k = x_0$, $x_{k+1} = x_1, \ldots, x_{2k} = x_k$; i.e., $\{x_1,\ldots,x_k\}$ is one cycle of a solution of (1) with period k. Conversely, if $\{x_1,\ldots,x_k\}$ is one cycle of a positive periodic solution of (1) with period k then with $\rho_i = x_i/x_{i-1}$ for $i = 1,\ldots,k$ we get

$$\rho_{k+1} = \frac{x_{k+1}}{x_k} = \frac{x_1}{x_0} = \rho_1, \quad \rho_k\rho_{k-1}\cdots\rho_1 = \frac{x_k}{x_{k-1}}\frac{x_{k-1}}{x_{k-2}}\cdots\frac{x_1}{x_0} = \frac{x_k}{x_0} = 1.$$

THEOREM 3. *Let $A,C > 0$, $\beta A \le \alpha B$ and assume that* (22) *holds.*

(a) *Let $A + B + C < \alpha + \beta$;*
  (i) *If $C = A + \beta$ then every solution of* (1) *not on the invariant ray $\bar{r}x$ converges to a periodic solution with period* 2.
  (ii) *If $C > A + \beta$ then every solution of* (1) *not on the invariant ray $\bar{r}x$ converges to ∞ in an oscillatory fashion.*
  (iii) *If $C + A + \beta$ and $(A + B + C)(A + \beta) < (\alpha + \beta)C$ then every solution of* (1) *not on the invariant ray $\bar{r}x$ converges to* 0 *in an oscillatory fashion.*
  (iv) *Solutions on the invariant ray $\bar{r}x$(i.e., $x_0 = \bar{r}x_{-1}$) converge to* 0 *monotonically.*

(*b*) *Let $A + B + C \ge \alpha + \beta$;*
Every solution of (1) *not on the invariant ray $\bar{r}x$converges to ∞ in an oscillatory fashion. If $A + B + C = \alpha + \beta$ then solutions on the invariant ray $\bar{r}x$ are stationary or constant solutions. If $A + B + C > \alpha + \beta$ then solutions on the invariant ray $\bar{r}x$ converge to ∞ eventually monotonically.*

Proof. (a), (i): By Lemmas 6 and 7, g has a globally attracting, positive two-cycle $\{\rho_-,\rho_+\}$ with $\rho_-\rho_+ = 1$. Thus, by (6) and the remarks preceding this theorem, solutions of (1) with $x_0/x_{-1} \ne \bar{r}$ converge to a period-two solution (the two limit points depend on the initial values).

(a), (ii): As in (a), (i) g has a globally attracting, positive two-cycle $\{\rho_-,\rho_+\}$ with $\rho_-\rho_+ > 1$. For sufficiently large $n \ge 1$, each of r_{2n} and r_{2n-1} is arbitrarily close to one of ρ_-,ρ_+. Without loss of generality assume that $r_{2n-1} \to \rho_-$ and $r_{2n} \to \rho_+$. Then, there is $k \ge 1$ and $1 < \gamma < \rho_-\rho_+$ such that

$r_{2n-1} < 1 < r_{2n}$ and $r_{2n-1}r_{2n} \ge \gamma > 1$ for all $n \ge k$.

Thus $x_{2n} = r_{2n}x_{2n-1} > x_{2n-1}$ and $x_{2n+1} = r_{2n+1}x_{2n} < x_{2n}$ for $n \ge k$ i.e., $\{x_n\}$ is an eventually oscillatory solution of (1). Further, for $n > k$

$$x_{2n} = x_{2k-1}\prod_{i=k}^{n}(r_{2i}r_{2i-1}) > x_{2k-1}\gamma^{n-k} \quad \text{and} \quad x_{2n+1} = x_{2k}\prod_{i=k}^{n}(r_{2i+1}r_{2i}) > x_{2k}\gamma^{n-k}.$$

Therefore, both $x_{2n},x_{2n+1} \to \infty$ as $n \to \infty$ as required.

(a), (iii): We first need to consider a consequence of inequality in (22); i.e.

$$A + B + C < \frac{\alpha C - AC}{A + \beta} + C = \frac{(\alpha + \beta)C}{A + \beta} = (\alpha + \beta)\rho_-\rho_+.$$

This inequality is stronger than the hypothesis $A + B + C < \alpha + \beta$ if $\rho_-\rho_+ < 1$. If this stronger inequality holds, then by modifying the argument used to prove (a), (ii) appropriately (e.g. reversing the obvious inequalities and using $\rho_-\rho_+ < \delta < 1$ instead of γ) shows that our claim is true.

(a), (iv): This is clear by our earlier observations about the invariant ray.

(b): As in the proof of (a), (iii) from (22) we obtain in this case

$$\alpha + \beta \leq A + B + C < \frac{(\alpha+\beta)C}{A+\beta} \Rightarrow C > A + \beta.$$

Thus $\rho_{-}\rho_{+} > 1$. The rest of the argument goes as in the proofs of (a), (ii) and (iv), thus completing the proof.

The next result shows, what happens when the inequality in Theorem 2 is reversed.

COROLLARY 4. *Assume that $A,C > 0$, $\beta A \leq \alpha B$ and $A + B + C = \alpha + \beta$. If $\alpha > 2A + B$ then every solution of* (1) *not on the diagonal* (*i.e.*, $x_0 \neq x_{-1}$) *converges to* ∞ *in an oscillatory fashion.*

Proof. From Lemma 4 it follows that $C > A + \beta$, since $\alpha = 2A + B$ implies $C = A + \beta$. Also multiplying the inequality by C gives

$$\alpha C > 2AC + BC = AC + (A+B)C > AC + (A+B)(A+\beta).$$

Thus (22), holds and we apply Theorem 3(b) to complete the proof.

We note that the hypotheses of Theorem 3 are not satisfied if $C = 0$. The next result shows that this is not a deficiency of Theorem 3.

COROLLARY 5. *Under the hypotheses of Corollary* 1, *equation* (16) *has no positive periodic solutions.*

Proof. Note that the function ψ in (10) takes the form

$$\psi(r) = \big(A+\beta\big)Ar^2 + \big(A+\beta\big)Br,$$

which has no positive roots; in fact, $\psi(r) > 0$ for $r > 0$. Hence, the only zeros of $g^2(r) - r$ are the fixed points of g and there are no points of period 2. Now, if (16) has a periodic solution then by the remarks preceding Theorem 3, g must also have periodic points. Therefore, since g is continuous, by the Sharkovski ordering (see, e.g., Ref. [13]) g has to have points of period 2, which is a contradiction.

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