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Global attractivity in a rational delay difference equation with quadratic terms

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For the following rational difference equation with arbitrary delay and quadratic terms:

$$x_{n+1} = \frac{Ax_n^2 + Bx_n x_{n-k} + Cx_{n-k}^2 + Dx_n + Ex_{n-k} + F}{\alpha x_n + \beta x_{n-k} + \gamma},$$

we determine sufficient conditions on the parameter values which guarantee that the unique non-negative fixed point attracts all positive solutions. When the fixed point is the origin ($F = 0$), we show that it attracts all non-negative solutions of the more general equation

$$x_{n+1} = \frac{Ax_n^2 + Bx_n x_{n-k} + Cx_{n-k}^2 + D_1 x_n + D_2 x_{n-1} + \cdots + D_m x_{n-m+1}}{\alpha x_n + \beta x_{n-k} + \gamma},$$

where $m \in \{1, 2, \ldots\}$. We also show that altering some of the above conditions on parameters causes the origin to not be globally attracting.

Keywords: rational; delay; quadratic terms; global stability

2000 Mathematics Subject Classification: 39A10; 39A11

1. Introduction

Consider the rational difference equation

$$x_{n+1} = \frac{Ax_n^2 + Bx_n x_{n-k} + Cx_{n-k}^2 + Dx_n + Ex_{n-k} + F}{\alpha x_n + \beta x_{n-k} + \gamma}, \tag{1}$$

where $k \in \{1, 2, \ldots\}$, all parameters (coefficients and initial values) are non-negative and

$$A + B + C > 0, \quad \alpha + \beta > 0. \tag{2}$$

Note that if all initial values $x_0, x_{-1}, \ldots, x_{-k}$ are positive, then all solutions of (1) are positive.

For over a decade rational equations with linear expressions in both the numerator and the denominator have been studied methodically and extensively; see, for example Refs. [2,5,9]. However, rational equations with quadratic terms in the numerator or the denominator have not been studied systematically. These equations exhibit a rich variety of dynamic behaviours and offer substantial insights into rational difference equations.

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For $k = 1$, the dynamics of equation (1) were studied in Refs. [3,4]. In this paper, we first determine sufficient conditions on the parameter values that for arbitrary k imply the existence of a unique, positive fixed point for (1) that attracts all of its positive solutions.

When the fixed point is the origin, i.e. $F = 0$, the results in Ref. [7] extend the conditions in Ref. [3] for the global attractivity of the origin to arbitrary k. In this paper, we extend the conditions in Ref. [3] in a different direction to the more general equation

$$x_{n+1} = \frac{Ax_n^2 + Bx_n x_{n-k} + Cx_{n-k}^2 + D_1 x_n + D_2 x_{n-1} + \cdots + D_m x_{n-m+1}}{\alpha x_n + \beta x_{n-k} + \gamma}, \tag{3}$$

where $m \in \{1, 2, \ldots\}$, all the parameters are non-negative and (2) holds. We obtain sufficient conditions on the parameters that imply the origin is the unique non-negative fixed point of (3) which is globally asymptotically stable relative to $[0, \infty)^m$. Here, we may assume without loss of generality that $m > k$ in (3) since we may set $D_j = 0$ for various values of j and any value of m.

For general background and definitions of basic concepts, we refer to texts such as Refs. [5,8,9,10].

2. Global attractivity of the positive fixed point

First, we need the following coordinate-wise monotonicity result from Ref. [6].

Lemma 1. *Let I be an open interval of real numbers and suppose that $f \in C(I^m, \mathbb{R})$ is nondecreasing in each coordinate. Let $\bar{x} \in I$ be a fixed point of the difference equation*

$$x_{n+1} = f(x_n, x_{n-1}, \ldots, x_{n-m+1}), \tag{4}$$

and assume that the function $h(t) = f(t, \ldots, t)$ satisfies the conditions

$$h(t) > t \quad \text{if } t < \bar{x} \quad \text{and} \quad h(t) < t \quad \text{if } \; t > \bar{x}, \; t \in I. \tag{5}$$

Then I is an invariant interval of (4) *and $\bar{x}$ attracts all solutions with initial values in I.*

With $m = k + 1$ and $I = (0, \infty)$, we can apply Lemma 1 to equation (1) as in the next result. We note that the nontrivial case without quadratic terms, i.e.

$$A = B = C = 0, \quad \alpha + \beta > 0$$

is included in the hypotheses of this result.

Theorem 1. *Assume that all parameters in* (1) *are non-negative and satisfy the following conditions:*

$$\alpha C \leq \beta B, \quad \beta A \leq \alpha B, \quad \alpha F \leq \gamma D, \quad \beta F \leq \gamma E, \quad |\beta D - \alpha E| \leq \gamma B, \tag{6}$$

$$0 \leq A + B + C < \alpha + \beta, \quad \gamma \leq D + E \quad \text{with } \; F > 0 \quad \text{if } \gamma = D + E. \tag{7}$$

Then (1) *has a unique fixed point $\bar{x} > 0$ that attracts all positive solutions.*

Proof. We first show that if the inequalities (6) hold then the function

$$f(u_1, \ldots, u_{k+1}) = \frac{Au_1^2 + Bu_1u_{k+1} + Cu_{k+1}^2 + Du_1 + Eu_{k+1} + F}{\alpha u_1 + \beta u_{k+1} + \gamma} \tag{8}$$

is nondecreasing in each of its coordinates $u_1, \ldots, u_{k+1}$ (i.e. f is coordinate-wise monotonic). Since f in (8) is independent of $u_2, \ldots, u_k$ hence nondecreasing in those coordinates, we need only prove monotonicty in coordinates $i = 1, k+1$. For this purpose, we compute the partial derivatives $f_i = \partial f/\partial u_i$ and determine when each is non-negative. For $i = 1$, direct calculation shows that $f_1 = \partial f/\partial u_1 \geq 0$ iff

$$\alpha Au_1^2 + 2\beta Au_1u_{k+1} + (\beta B - \alpha C)u_{k+1}^2 + 2\gamma Au_1 + (\gamma B + \beta D - \alpha E)u_{k+1} + \gamma D - \alpha F \geq 0.$$

The above inequality holds for $u_1, u_{k+1} > 0$ if

$$\alpha C \leq \beta B, \quad \gamma B + \beta D - \alpha E \geq 0, \quad \alpha F \leq \gamma D. \tag{9}$$

Similarly, $f_{k+1} = \partial f/\partial u_{k+1} \geq 0$ if and only if

$$\beta Cu_{k+1}^2 + 2\alpha Cu_1u_{k+1} + (\alpha B - \beta A)u_1^2 + 2\gamma Cu_{k+1} + (\gamma B + \alpha E - \beta D)u_1 + \gamma E - \beta F \geq 0$$

which is true for all $u_1, u_{k+1} > 0$ if

$$\beta A \leq \alpha B, \quad \gamma B + \alpha E - \beta D \geq 0, \quad \beta F \leq \gamma E. \tag{10}$$

The middle inequalities in (9) and (10) combine into the single inequality $|\beta D - \alpha E| \leq \gamma B$. Therefore, we have shown that conditions (6) are sufficient for f to be nondecreasing in each of its coordinates.

Next, assume that (7) holds and define

$$a = \frac{A+B+C}{\alpha+\beta}, \quad b = \frac{D+E}{\alpha+\beta}, \quad c = \frac{F}{\alpha+\beta}, \quad d = \frac{\gamma}{\alpha+\beta}.$$

Then the function h in (5) takes the form

$$h(t) = \frac{at^2 + bt + c}{t+d}.$$

Now $\bar{x}$ is a fixed point of (1) if and only if $\bar{x}$ is a solution of the equation $h(t) = t$, i.e.

$$(1-a)\bar{x}^2 - (b-d)\bar{x} - c = 0. \tag{11}$$

Since by (7) $a < 1$ and $d \leq b$ with $c > 0$ if $d = b$, a unique positive fixed point is obtained as

$$\bar{x} = \frac{b-d+\sqrt{(b-d)^2+4(1-a)c}}{2(1-a)}; \quad \text{or multiplying by} \quad \frac{\alpha+\beta}{\alpha+\beta},$$
$$\bar{x} = \frac{D+E-\gamma+\sqrt{(D+E-\gamma)^2+4(\alpha+\beta-A-B-C)F}}{2(\alpha+\beta-A-B-C)}. \tag{12}$$

Next, we verify that conditions (5) hold. Note that h may be written as

$$h(t) = \phi(t)t, \quad \text{where} \quad \phi(t) = \frac{at + b + c/t}{t + d}.$$

Since

$$\phi'(t) = \frac{ad - b - (c/t)(2 + d/t)}{(t + d)^2}$$

and by (7) $ad - b < d - b \leq 0$, it follows that ϕ is decreasing (strictly) for all $t \in I$. Therefore, with $\phi(\bar{x}) = h(\bar{x})/\bar{x} = 1$ we find that

$$t < \bar{x} \text{ implies } h(t) > \phi(\bar{x})t = t,$$
$$t > \bar{x} \text{ implies } h(t) < \phi(\bar{x})t = t.$$

Now using Lemma 1 completes the proof. □

Remarks.

1. *Periodic solutions*. If some of the inequalities in Theorem 1 do not hold then its conclusion is easily seen to be false. In particular, if

$$A = D = F = 0 \quad \text{and} \quad B = \alpha, \quad C = \beta, \quad E = \gamma, \tag{13}$$

then the inequalities in (6) hold but those in (7) do not. If (13) holds then equation (1) reduces to $x_{n+1} = x_{n-k}$ in which case every positive solution (non-negative solution if $\gamma > 0$) is periodic with period $k + 1$.
2. Inequalities (6) are not necessary for global attractivity. In Ref. [1], a special case of (1) is discussed where

$$A = C = D = E = \gamma = 0, \quad \alpha = \beta = B > 0, \quad F \geq 0. \tag{14}$$

In this case, it is shown that every positive solution converges to the unique fixed point $\sqrt{F}$. Note that conditions (14) contradict some of the inequalities in (6) if $F > 0$.

3. Global asymptotic stability of the origin

When $F = 0$ and $\gamma > 0$ in (1) then the origin is a fixed point of (1) and the class of solutions can be expanded to include the non-negative solutions. The question arises as to whether the origin can be globally attracting for the non-negative solutions. In this section, we study the global attractivity of the origin for the more general equation (3). We need the following result from Ref. [10] (Theorem 4.3.1) that for convenience we quote as a lemma. Recall that a fixed point $\bar{x}$ is asymptotically stable relative to a set S if $\bar{x}$ is locally stable and attracts all orbits with their initial points in S.

LEMMA 2. *Let $\bar{x}$ be a fixed point of the difference equation* (4) *in a closed, invariant set $T \subset \mathbb{R}^m$ and define the set M as*

$$\{(u_1, \ldots, u_m) : |f(u_1, \ldots, u_m) - \bar{x}| < \max\{|u_1 - \bar{x}|, \ldots, |u_m - \bar{x}|\}\} \cup \{(\bar{x}, \ldots, \bar{x})\}.$$

Then $(\bar{x}, \ldots, \bar{x})$ is asymptotically stable relative to the largest invariant subset S of $M \cap T$ such that S is closed in T.

THEOREM 2. *Assume that the following conditions are satisfied for equation* (3) *for some $\delta \in [0, 1]$:*

$$A + \delta B \le \alpha, \quad C + (1 - \delta)B \le \beta, \quad D_1 + \cdots + D_m < \gamma. \tag{15}$$

Then the origin is the unique non-negative fixed point of (3) *and it is asymptotically stable relative to* $[0, \infty)^m$.

Proof. Let $D = D_1 + \cdots + D_m$. The origin is clearly a fixed point of (3) since $\gamma > D \ge 0$ by (15). Also by (15), $A + B + C \le \alpha + \beta$. If this inequality strict then there exists a negative fixed point

$$\bar{x} = -\frac{\gamma - D}{2(\alpha + \beta - A - B - C)},$$

while if $A + B + C = \alpha + \beta$ then there are no nonzero fixed points. It follows that the origin is the only non-negative fixed point of (3). Next, if

$$g(u_1, \ldots, u_m) = \frac{Au_1^2 + Bu_1u_{k+1} + Cu_{k+1}^2 + D_1u_1 + \cdots + D_mu_m}{\alpha u_1 + \beta u_{k+1} + \gamma}$$

with $m > k$ then defining $\mu = \max\{u_1, \ldots, u_m\}$ for all $u_1, \ldots, u_m \ge 0$

$$\begin{aligned} g(u_1, \ldots, u_m) &\le \frac{B \min\{u_1, u_{k+1}\} \max\{u_1, u_{k+1}\} + \mu(Au_1 + Cu_{k+1} + D)}{\alpha u_1 + \beta u_{k+1} + \gamma} \\ &\le \mu \frac{[B\delta + B(1 - \delta)]\min\{u_1, u_{k+1}\} + Au_1 + Cu_{k+1} + D}{\alpha u_1 + \beta u_{k+1} + \gamma} \\ &\le \mu \frac{(A + \delta B)u_1 + [C + (1 - \delta)B]u_{k+1} + D}{\alpha u_1 + \beta u_{k+1} + \gamma} \\ &< \mu, \quad \text{if } (u_1, \ldots, u_m) \ne (0, \ldots, 0), \end{aligned}$$

where the strict inequality holds because of conditions (15). Now we apply Lemma 2 with $S = M = T = [0, \infty)^m$ to conclude the proof. □

Next, we show that even when the origin is the only non-negative fixed point in equation (3) or its special case (1), it may not be globally attracting under certain conditions. Note that one of the inequalities in (15) is contradicted in Theorem 3 below, which we present after a few preliminaries. For further details in this direction, see Ref. [7]. Recall from (8) that

$$f(u_1, \ldots, u_{k+1}) = \frac{Au_1^2 + Bu_1u_{k+1} + Cu_{k+1}^2 + Du_1 + Eu_{k+1} + F}{\alpha u_1 + \beta u_{k+1} + \gamma}.$$

Also the partial derivatives f_{u_1} and $f_{u_{k+1}}$ of f used in the next result are

$$f_{u_1}(u_1, u_2, \ldots, u_{k+1}) = \frac{A\alpha u_1^2 + 2A\beta u_1u_{k+1} + 2A\gamma u_1 + (B\beta - C\alpha)u_{k+1}^2 + (B\gamma + D\beta - E\alpha)u_{k+1} + D\gamma}{(\alpha u_1 + \beta u_{k+1} + \gamma)^2} \tag{16}$$

and

$$f_{u_{k+1}}(u_1, u_2, \ldots, u_{k+1}) = \frac{C\beta u_{k+1}^2 + 2C\alpha u_1u_{k+1} + 2C\gamma u_{k+1} + (B\alpha - A\beta)u_1^2 + (B\gamma + E\alpha - D\beta)u_1 + E\gamma}{(\alpha u_1 + \beta u_{k+1} + \gamma)^2}. \tag{17}$$

DEFINITION 1.

1. *Let $\mathcal{N} \in (0,\infty)^{k+1}$ denotes the region in which the partial derivatives of f satisfy $f_{u_1} \geq 0$ and $f_{u_{k+1}} \geq 0$.*
2. *Let $\mathcal{V} \in (0,\infty)^{k+1}$ denotes the region in which $f_{u_1} \leq 0$.*
3. *Let $\mathcal{Y}$* denotes the set $\{(u_1, u_2, \ldots, u_k, u_{k+1}) \in (0,\infty)^{k+1} : u_1 = 0\}$.

LEMMA 3. *Assume that the following inequalities hold for the parameters of equation* (1)*:*

$$F = 0, \quad D + E < \gamma, \quad A + B + C \leq \alpha + \beta.$$

Then, in Y, we have $f(0, u_2, \ldots, u_k, a) \leq f(0, u_2, \ldots, u_k, b)$, for any $0 \leq a < b$.

Proof. Let $(u_1, u_2, \ldots, u_{k+1}) \in \mathcal{Y}$. Then $u_1 = 0$. If we let $u_{k+1} = a$ and we assume that $0 < a < b$, then $f(0, u_2, \ldots, u_k, a) \leq f(0, u_2, \ldots, u_k, b)$, where

$$f(0, u_2, \ldots, u_k, a) \leq f(0, u_2, \ldots, u_k, b) \Leftrightarrow \frac{Ca^2 + Ea}{\beta a + \gamma} \leq \frac{Cb^2 + Eb}{\beta b + \gamma}$$

$$\Leftrightarrow C\beta a^2b + C\gamma a^2 + E\beta ab + E\gamma a \leq C\beta ab^2 + C\gamma b^2 + E\beta ab + E\gamma b$$

$$\Leftrightarrow C\beta ab(b - a) + C\gamma(b + a)(b - a) + E\gamma(b - a) \geq 0,$$

which is true by assumption (with equality holding if $C = E = 0$). □

THEOREM 3. *Assume that the following inequalities hold for the parameters of equation* (1):

$$F = 0, \quad D + E < \gamma, \quad A + B + C \leq \alpha + \beta, \quad \beta < C, \quad \alpha C < \beta(A + B), \quad B\beta < C\alpha$$

$$\text{or} \quad B\gamma + D\beta - E\alpha < 0, \quad B\alpha \geq A\beta \quad \text{and} \quad B\gamma + E\alpha - D\beta \geq 0.$$

Then

$$\lim_{n\to\infty} x_n = \infty$$

for every solution $\{x_n\}$ *of* (1) *with initial values* $x_0, x_{-1}, \ldots, x_{-k} \in (\mu, \infty)$ *where*

$$\mu = \max\left\{\frac{\gamma - E}{C - \beta}, p, q\right\},$$

and p and q are defined as follows:

$$p = \begin{cases} \frac{-(2A\gamma+B\gamma+D\beta-E\alpha)+\sqrt{\mathcal{D}_p}}{2(A\alpha+2A\beta+B\beta-C\alpha)}, & \mathcal{D}_p \geq 0, \\ 0, & \mathcal{D}_p < 0, \end{cases}$$

and

$$q = \begin{cases} \frac{-(A\gamma+B\gamma+D\beta-E\alpha)+\sqrt{\mathcal{D}_q}}{2(A\beta+B\beta-C\alpha)}, & \mathcal{D}_q \geq 0, \\ 0, & \mathcal{D}_q < 0, \end{cases}$$

with

$$\mathcal{D}_p = (2A\gamma + B\gamma + D\beta - E\alpha)^2 - 4(A\alpha + 2A\beta + B\beta - C\alpha)D\gamma,$$

$$\mathcal{D}_q = (A\gamma + B\gamma + D\beta - E\alpha)^2 - 4(A\beta + B\beta - C\alpha)D\gamma.$$

Proof. Under our assumptions $f_{u_{k+1}} \geq 0$ on $(0, \infty)^{k+1}$. We observe that the condition $C\alpha < \beta(A + B)$ and equation (16) imply the following:

(O1): $f_{u_1}(u_1, u_2, \ldots, u_k, u_{k+1}) > 0$ for all $u_1 \geq u_{k+1} > p$, and thus $\mathcal{N} \supset \{(u_1, u_2, \ldots, u_k, u_{k+1}) \in (0, \infty)^{k+1} : u_1 \geq u_{k+1} > p\}$, where $\mathcal{N}$ is as defined in Definition 1.

(O2): Let $u_1 = u_{k+1} = v > q$ and let $u_2, \ldots, u_k > 0$ be arbitrary. Then $f(v, u_2, \ldots, u_k, u_k, v) > f(0, u_2, \ldots, u_k, v)$, where

$$f(v, u_2, \ldots, u_k, v) > f(0, u_2, \ldots, u_k, v) \Leftrightarrow \frac{(A + B + C)v^2 + (D + E)v}{(\alpha + \beta)v + \gamma}$$

$$> \frac{Cv^2 + Ev}{\beta v + \gamma} \Leftrightarrow (A + B + C)v(\beta v + \gamma) + (D + E)(\beta v + \gamma)$$

$$> (Cv + E)[(\alpha + \beta)v + \gamma]$$

$$\Leftrightarrow (A\beta + B\beta - C\alpha)v^2 + (A\gamma + B\gamma + D\beta - E\alpha)v + D\gamma > 0,$$

which is true since the condition that $C\alpha < \beta(A+B)$ holds (so that $A\beta + B\beta - C\alpha > 0$) and by the definition of q

We make further observations.

(O3): Let $\mathcal{V}, \mathcal{N}$ be as defined in Definition 1. Note that $\mathcal{V} \cup N = (0, \infty)^{k+1}$

1. Suppose that $(u_1, u_2, \ldots, u_k, u_{k+1}) \in \mathcal{V}$. Then $f_{u1} \leq 0$, by definition, but also $f_{u_{k+1}} \geq 0$, where the conditions that $\beta < C$; $B\beta < C\alpha$ or $B\gamma + D\beta - E\alpha < 0$; and $B\alpha \geq A\beta$ and $B\gamma + E\alpha - D\beta \geq 0$ all hold. Hence, if $u_1 < u_{k+1}$, then

$$f(u_1, \ldots, u_k, u_{k+1}) > f(u_1, u_2, \ldots, u_k, u_1) \Rightarrow$$

$$f(u_1, \ldots, u_k, u_{k+1}) > \min\{f(u_1, \ldots, u_k, u_1), f(u_{k+1}, u_2, \ldots, u_{k+1})\}.$$

2. Suppose that $(u_1, u_2, \ldots, u_k, u_{k+1}) \in \mathcal{N}$ (so that $f_{u_1} \geq 0, f_{u_{k+1}} \geq 0$). The following is true:
   (a) If $u_1 < u_{k+1}$, then

$$\begin{aligned} f(u_1, u_2, \ldots, u_k, u_{k+1}) &> f(u_1, u_2, \ldots, u_k, u_1) \\ &\geq \min\{f(u_1, \ldots, u_k, u_1), f(u_{k+1}, u_2, \ldots, u_{k+1})\}. \end{aligned}$$

   (b) If $u_1 \geq u_{k+1}$, then

$$\begin{aligned} f(u_1, u_2, \ldots, u_k, u_{k+1}) &\geq f(u_{k+1}, u_2, \ldots, u_k, u_{k+1}) \\ &\geq \min\{f(u_1, \ldots, u_k, u_1), f(u_{k+1}, u_2, \ldots, u_{k+1})\}. \end{aligned}$$

3. Thus, in general,

$$f(u_1, u_2, \ldots, u_k, u_{k+1}) \geq \min\{f(u_1, u_2, \ldots, u_k, u_1), f(u_{k+1}, u_2, \ldots, u_k, u_{k+1})\} \quad \text{for}$$

$$(u_1, u_2, \ldots, u_k, u_{k+1}) \in (0, \infty)^{k+1}$$

(O4): Let $u_1 = 0$ and $u_{k+1} = v > (\gamma - E)/(C - \beta)$, and let $u_2, \ldots, u_k \in (0, \infty)$ be arbitrary. Then $f(0, u_2, \ldots, u_k, v) > v$, since given the condition that $\beta < C$, we have

$$f(0, u_2, \ldots, u_k, v) > v \Leftrightarrow \frac{Cv^2 + Ev}{\beta v + \gamma} > v \Leftrightarrow \frac{\gamma - E}{C - \beta} < v.$$

(O5): $\mu > 0$ since $(\gamma - E/C - \beta) > 0$ by the conditions that $D + E < \gamma$ and $\beta < C$.

Combining Observations (O1)–(O4), we have that, for $u_1, u_{k+1} > \mu$ and $u_2, \ldots, u_k \in (0, \infty)$,

$$\begin{aligned} f(u_1, u_2, \ldots, u_k, u_{k+1}) &\geq \min\{f(u_1, u_2, \ldots, u_k, u_1), f(u_{k+1}, u_2, \ldots, u_k, u_{k+1})\} \\ &> \min\{f(0, u_2, \ldots, u_k, u_1), f(0, u_2, \ldots, u_k, u_{k+1})\} > \min\{u_1, u_{k+1}\} \\ &\geq \min\{u_1, u_2, \ldots, u_k, u_{k+1}\} \end{aligned}$$

Therefore, in summary, we can say that

$$\begin{aligned} &f(u_1, u_2, \ldots, u_k, u_{k+1}) > \min\{u_1, u_2, \ldots, u_k, u_{k+1}\}, \quad \text{for} \\ &(u_1, u_2, \ldots, u_{k+1}) \in (\mu, \infty)^{k+1}. \end{aligned} \tag{18}$$

Now let $\{x_n\}_{n=-k}^{\infty}$ be a positive solution of equation (1), with $k \in \{1, 2, \ldots\}$, and suppose that $x_{-k}, \ldots, x_{-1}, x_0 \in (\mu, \infty)$. Next define

$$x_n^* \stackrel{\text{def}}{=} \min\{x_n, x_{n-1}, \ldots, x_{n-k}\}, \quad \text{for all} \quad n \geq 0.$$

Clearly, $\min\{x_n, x_{n-k}\} \geq \min\{x_n, x_{n-1}, \ldots, x_{n-k}\}$. Thus, given equation (18) and the assumption that $(x_0, x_{-1}, \ldots, x_{-k}) \in (\mu, \infty)^{k+1}$, we have

$$x_1 = f(x_0, x_{-1}, \ldots, x_{-k}) > \min\{x_0, x_{-1}, \ldots, x_{-k}\} = x_0^* > \mu.$$

By induction,

$$x_{n+1} = f(x_n, x_{n-1}, \ldots, x_{n-k}) > \min\{x_n, x_{n-1}, \ldots, x_{n-k}\} = x_n^* > \mu, \quad \text{for all} \quad n \geq 0, \tag{19}$$

and so equation (18) holds for $(u_1, u_2, \ldots, u_{k+1}) = (x_n, x_{n-1}, \ldots, x_{n-k})$ for all $n \geq 0$. Also, from equation (19), we have $x_{n+1}^* = \min\{x_{n+1}, x_n, \ldots, x_{n-k+1}\} \geq x_n^*$, for all $n \geq 0$. This means that $\{x_n^*\}_{n=0}^{\infty}$ is a nondecreasing sequence, and so there exists $\mu < L \leq \infty$ such that $\lim_{n\to\infty} x_n^* = L$ and $x_n^* \leq L (x_n^* < L \text{ if } L = \infty)$ for all $n \geq 0$. We show that $L = \infty$ by contradiction. Assume that $L < \infty$. Let us define the function $\phi : [0, \infty) \to \mathbf{R}$ as follows:

$$\phi(x) \stackrel{\text{def}}{=} [\beta(L + x) - C(L - x)](L - x) + \gamma(L + x) - E(L - x).$$

Then, from the conditions in the hypotheses and the fact that L is positive (by Observation (O5)), we have that $\phi(0) = [\beta - C]L^2 + [\gamma - E]L < 0$. By the continuity of ϕ on $\mathbb{R}$, there exists $\varepsilon > 0$, $\varepsilon < L$ such that $\phi(\varepsilon) < 0$. Thus, we have the following:

$$\frac{C(L - \varepsilon)^2 + E(L - \varepsilon)}{\beta(L - \varepsilon) + \gamma} > L + \varepsilon. \tag{20}$$

Now, there exists $N \geq 0$ such that for all $n \geq N$,

$$x_{n-k}, \ldots, x_{n-1}, x_n \geq x_n^* > L - \varepsilon. \tag{21}$$

It follows from equations (20) and (21) and Lemma 3 that

$$\frac{Cx_{n-k}^2 + Ex_{n-k}}{\beta x_{n-k} + \gamma} \geq \frac{C(L-\varepsilon)^2 + E(L-\varepsilon)}{\beta(L-\varepsilon)+\gamma} > L + \varepsilon.$$

Hence, from observation (O4) by induction, $x_{n-k}, \ldots, x_{n-1}, x_n > L + \varepsilon$ for all $n \geq N$; and so

$$L = \lim_{n\to\infty} x_n^* = \lim_{n\to\infty} \min\{x_n, x_{n-1}, \ldots, x_{n-k}\} \geq \lim_{n\to\infty} \min\{L+\varepsilon, L+\varepsilon, \ldots, L+\varepsilon\}$$

$$= L + \varepsilon,$$

which is impossible and the proof is complete. □

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