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Global attractivity in a quadratic-linear rational difference equation with delay

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We investigate the global behaviour of non-negative solutions of the following rational difference equation with arbitrary delay and quadratic terms in its numerator:

$$x_{n+1} = \frac{Ax_n^2 + Bx_nx_{n-k} + Cx_{n-k}^2 + Dx_n + Ex_{n-k}}{\alpha x_n + \beta x_{n-k} + \gamma},$$

where all coefficients are non-negative and $A + B + C + D + E > 0$ and $\gamma > 0$. In this case, the origin is the only non-negative fixed point and we establish the asymptotic stability of the origin relative to an invariant set and behaviour of positive solutions outside that invariant set. We also state conditions implying that the invariant set is $[0, \infty)^{k+1}$, i.e. the origin is a global attractor of all non-negative solutions.

Keywords: rational difference equation; arbitrary delay; quadratic terms; global attractor

1. Introduction

We study the asymptotic behaviour of positive solutions of the quadratic-linear rational difference equation with arbitrary delay,

$$\begin{cases} x_{n+1} = \dfrac{Ax_n^2 + Bx_nx_{n-k} + Cx_{n-k}^2 + Dx_n + Ex_{n-k}}{\alpha x_n + \beta x_{n-k} + \gamma}, & n = 0, 1, \ldots, \\ k \in \{1, 2, \ldots\}, \end{cases} \tag{1}$$

where all parameters are non-negative, with $A + B + C + D + E > 0$ and $\gamma > 0$. We determine sufficient conditions on the parameter values that guarantee the following:

(i) The asymptotic stability of a unique non-negative fixed point, the origin, of equation (1) relative to an invariant set.
(ii) The behaviour of solutions outside of that invariant set.

Our results in this paper considerably strengthen and extend Theorems 1 and 3 in Ref. [1] that were proved for the second-order equation

$$x_{n+1} = \frac{Ax_n^2 + Bx_nx_{n-1} + Cx_{n-1}^2 + Dx_n + Ex_{n-1} + F}{\alpha x_n + \beta x_{n-1} + \gamma},$$

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where F is non-negative. For the past decade or so there has been a burgeoning library of systematically and thoroughly studied rational equations with linear expressions in both the numerator and the denominator; see, for example, [3–5]. However, rational equations with quadratic terms in the numerator or the denominator have not been systematically studied; see Refs. [1] and [2] for some comments and references. These equations exhibit a rich variety of dynamic behaviours and offer substantial insights – and significant challenges – into the dynamics of rational difference equations.

2. Preliminaries

In this section, we consider equation (1) and define $f : [0,\infty)^{k+1} \longrightarrow [0,\infty)$ as

$$f(u_1,u_2,\ldots,u_{k+1}) \stackrel{\text{def}}{=} \frac{Au_1^2 + Bu_1u_{k+1} + Cu_{k+1}^2 + Du_1 + Eu_{k+1}}{\alpha u_1 + \beta u_{k+1} + \gamma}.$$

We find sufficient conditions for the origin to be asymptotically stable relative to some invariant set. Below is a list of all such conditions which will be useful in determining the appropriate invariant set for each given theorem.

(C1) $A, B, C, D, E, \alpha, \beta \geq 0$ and $\gamma > 0$.
(C2) $A + B + C \leq \alpha + \beta$.
(C3) $D + E < \gamma$.
(C4) $A \leq \alpha$ and $C \leq \beta$.
(C5) $A > \alpha$.
(C6) $C > \beta$.
(C7) $B\beta \geq C\alpha$, $B\alpha \geq A\beta$, and $|D\beta - E\alpha| \leq B\gamma$.
(C8) (a) Either $B\beta < C\alpha$ or $B\gamma + D\beta - E\alpha < 0$.
(b) $B\alpha \geq A\beta$ and $B\gamma + E\alpha - D\beta \geq 0$.
(C9) (a) Either $B\alpha < A\beta$ or $B\gamma + E\alpha - D\beta < 0$.
(b) $B\beta \geq C\alpha$ and $B\gamma + D\beta - E\alpha \geq 0$.
(C10) (a) Either $B\beta < C\alpha$ or $B\gamma + D\beta - E\alpha < 0$.
(b) Either $B\alpha < A\beta$ or $B\gamma + E\alpha - D\beta < 0$.
(C11) $(A + B)\beta > C\alpha$.
(C12) $(B + C)\alpha > A\beta$.

In our determination of conditions that are sufficient to guarantee convergence of positive solutions equation (1) to the origin, we start by defining regions in $(0,\infty)^{k+1}$ based on whether or not f_{u_1} and $f_{u_{k+1}}$ are increasing or decreasing; but, first observe that $f_{u_2} = f_{u_3} = \cdots = f_{u_k} = 0$ in $(0,\infty)^{k+1}$. So, we define the following (not necessarily disjoint) sets and points:

Definition 1.

1. *Let $\mathcal{N} \subseteq (0,\infty)^{k+1}$ denote the region in which $f_{u_1} \geq 0$ and $f_{u_{k+1}} \geq 0$.*
2. *Let $\mathcal{V} \subseteq (0,\infty)^{k+1}$ denote the region in which $f_{u_1} \leq 0$.*
3. *Let $\mathcal{H} \subseteq (0,\infty)^{k+1}$ denote the region in which $f_{u_{k+1}} \leq 0$.*
4. *Let $\mathcal{D}$ denote the set $\{(u_1,u_2,\ldots,u_k,u_{k+1}) \in (0,\infty)^{k+1} : u_1 = u_{k+1}\}$.*
5. *Let $\mathcal{Y}$ denote the set $\{(u_1,u_2,\ldots,u_k,u_{k+1}) \in (0,\infty)^{k+1} : u_1 = 0\}$.*
6. *Let $\mathcal{X}$ denote the set $\{(u_1,u_2,\ldots,u_k,u_{k+1}) \in (0,\infty)^{k+1} : u_{k+1} = 0\}$.*

Important for the sequel is the following notation:

NOTATION 2. *Given* $(u_1, u_2, \ldots, u_{k+1}) \in (0, \infty)^{k+1}$ *define*

$$u^*(u_1, u_2, \ldots, u_{k+1}) = \max\{u_1, u_2, \ldots, u_{k+1}\},$$

and where when it is obvious from the context, we make the abuse of notation

$$u^* = u^*(u_1, u_2, \ldots, u_{k+1}) = \max\{u_1, u_2, \ldots, u_{k+1}\}.$$

We need the following lemmas for our main result.

LEMMA 3. *Suppose that Conditions* $(C1)$–$(C3)$ *hold in equation* (1). *Then, the following is true:*

(i) *In* $\mathcal{D}$, *we have* $f(m, u_2, \ldots, u_k, m) < f(M, u_2, \ldots, u_k, M)$, *for any* $0 \le m < M$.
(ii) *In* $\mathcal{Y}$, *we have* $f(0, u_2, \ldots, u_k, m) \le f(0, u_2, \ldots, u_k, M)$, *for any* $0 \le m < M$.
(iii) *In* $\mathcal{X}$, *we have* $f(m, u_2, \ldots, u_k, 0) \le f(M, u_2, \ldots, u_k, 0)$, *for any* $0 \le m < M$.
(iv) *Suppose* $(u_1, u_2, \ldots, u_{k+1}) \in \mathcal{N}$, *and let* $M \ge u_1, u_{k+1}$. *Then,* $f(u_1, u_2, \ldots, u_k, u_{k+1}) \le f(M, u_2, \ldots, u_k, M)$.
(v) *Suppose* $(u_1, u_2, \ldots, u_{k+1}) \in \mathcal{V}$, *and let* $M \ge u_{k+1}$. *Then,* $f(u_1, u_2, \ldots, u_k, u_{k+1}) \le f(0, u_2, \ldots, u_k, M)$.
(vi) *Suppose* $(u_1, u_2, \ldots, u_{k+1}) \in \mathcal{H}$, *and let* $M \ge u_1$. *Then,* $f(u_1, u_2, \ldots, u_k, u_{k+1}) \le f(M, u_2, \ldots, u_k, 0)$.

Proof. We first compute $f_{u_1}(u_1, u_2, \ldots, u_{k+1})$ and $f_{u_{k+1}}(u_1, u_2, \ldots, u_{k+1})$ and obtain the following:

$$f_{u_1}(u_1,u_2,\ldots,u_{k+1}) = \frac{A\alpha u_1^2 + 2A\beta u_1 u_{k+1} + 2A\gamma u_1 + (B\beta - C\alpha)u_{k+1}^2 + (B\gamma + D\beta - E\alpha)u_{k+1} + D\gamma}{(\alpha u_1 + \beta u_{k+1} + \gamma)^2}, \tag{2}$$

and

$$f_{u_{k+1}}(u_1,u_2,\ldots,u_{k+1}) = \frac{C\beta u_{k+1}^2 + 2C\alpha u_1 u_{k+1} + 2C\gamma u_{k+1} + (B\alpha - A\beta)u_1^2 + (B\gamma + E\alpha - D\beta)u_1 + E\gamma}{(\alpha u_1 + \beta u_{k+1} + \gamma)^2}. \tag{3}$$

We next consider the following six cases.

Case 1. $(u_1, u_2, \ldots, u_{k+1}) \in \mathcal{D}$. Then, $u_1 = u_{k+1}$. If we let $u_1 = u_{k+1} = m$ and we assume that $0 < m < M$, then $f(m, u_2, \ldots, u_k, m) < f(M, u_2, \ldots, u_k, M)$, where

$$f(m,u_2,\ldots,u_k,m)<f(M,u_2,\ldots,u_k,M) \Longleftrightarrow$$
$$\frac{(A+B+C)m^2+(D+E)m}{(\alpha+\beta)m+\gamma}<\frac{(A+B+C)M^2+(D+E)M}{(\alpha+\beta)M+\gamma} \Longleftrightarrow$$
$$(A+B+C)(\alpha+\beta)m^2M+(A+B+C)\gamma m^2+(D+E)(\alpha+\beta)mM+(D+E)\gamma m$$
$$<(A+B+C)(\alpha+\beta)mM^2+(A+B+C)\gamma M^2+(D+E)(\alpha+\beta)mM+(D+E)\gamma M\Longleftrightarrow$$
$$(A+B+C)(\alpha+\beta)mM(M-m)+(A+B+C)\gamma(M+m)(M-m)+(D+E)\gamma(M-m)>0,$$

which is true by assumption and (C1).

Case 2. $(u_1, u_2, \ldots, u_{k+1}) \in \mathcal{Y}$. Then, $u_1 = 0$. If we let $u_{k+1} = m$ and we assume that $0 < m < M$, then $f(0, u_2, \ldots, u_k, m) \leq f(0, u_2, \ldots, u_k, M)$, where

$$f(0, u_2, \ldots, u_k, m) \leq f(0, u_2, \ldots, u_k, M) \Longleftrightarrow \frac{Cm^2 + Em}{\beta m + \gamma} \leq \frac{CM^2 + EM}{\beta M + \gamma} \Longleftrightarrow$$
$$C\beta m^2 M + C\gamma m^2 + E\beta mM + E\gamma m \leq C\beta mM^2 + C\gamma M^2 + E\beta mM + E\gamma M \Longleftrightarrow$$
$$C\beta mM(M - m) + C\gamma(M + m)(M - m) + E\gamma(M - m) \geq 0,$$

which is true by assumption and (C1; with equality holding if $C = E = 0$).

Case 3. $(u_1, u_2, \ldots, u_{k+1}) \in \mathcal{X}$. Then, $u_{k+1} = 0$. If we let $u_1 = m$ and we assume that $0 < m < M$, then $f(0, u_2, \ldots, u_k, m) \leq f(0, u_2, \ldots, u_k, M)$, where

$$f(m, u_2, \ldots, u_k, 0) \leq f(M, u_2, \ldots, u_k, 0) \Longleftrightarrow \frac{Am^2 + Dm}{\alpha m + \gamma} \leq \frac{AM^2 + DM}{\alpha M + \gamma} \Longleftrightarrow$$
$$A\alpha m^2 M + A\gamma m^2 + D\alpha mM + E\gamma m \leq A\alpha mM^2 + C\gamma M^2 + D\alpha mM + E\gamma M \Longleftrightarrow$$
$$A\alpha mM(M - m) + A\gamma(M + m)(M - m) + D\gamma(M - m) \geq 0,$$

which is true by assumption and (C1; with equality holding if $A = D = 0$).

Case 4. $(u_1, u_2, \ldots, u_{k+1}) \in \mathcal{N}$. Then, we have $f(u_1, u_2, \ldots, u_{k+1}) \leq f(u^*, u_2, \ldots, u_k, u^*)$, as shown in the following subcases.

Subcase a. $u_1 = u_{k+1}$. We are done, by Case 1 above.

Subcase b. $u_1 < u_{k+1}$. We have that $u_{k+1} = u^*$. Observe that, from equations (2) and (3), $f_{u_1}(u, u_2, \ldots, u_k, u_{k+1})$ is non-decreasing in its first argument on the interval $(u_1, u_{k+1}) = (u_1, u^*)$, and, by Definition 1, $f_{u_1}(u_1, u_2, \ldots, u_k, u_{k+1}) \geq 0$. Thus, $f_{u_1}(u, u_2, \ldots, u_k, u_{k+1}) \geq 0$ for all $u \in (u_1, u_{k+1}) = (u_1, u^*)$, and so, with $u_1 < u_{k+1} = x^*$, we have $f(u_1, u_2, \ldots, u_k, u_{k+1}) \leq f(u^*, u_2, \ldots, u_k, u^*)$.

Subcase c. $u_1 > u_{k+1}$. We have that $u_1 = x^*$. Observe that, from equations (2) and (3), $f_{u_{k+1}}(u_1, u_2 \ldots, u_k, v)$ is non-decreasing in its $(k+1)$st argument on the interval $(u_{k+1}, u_1) = (u_{k+1}, u^*)$ and, by Definition 1, $f_{u_{k+1}}(u_1, u_2, \ldots, u_k, u_{k+1}) \geq 0$. Thus,

$f_{u_1}(u_1, u_2, \ldots, u_k, v) \geq 0$ for all $u \in (u_{k+1}, u_1)$, and so, with $x^* = u_1 > u_{k+1}$, we have $f(u_1, u_2, \ldots, u_k, u_{k+1}) \leq f(u^*, u_2, \ldots, u_k, u^*)$.

It then follows from Subcases a–c and Case 1 that $f(u_1, u_2, \ldots, u_k, u_{k+1}) \leq f(M, u_2, \ldots, u_k, M)$, for any $M \geq u_1, u_{k+1}$.

Case 5. $(u_1, u_2, \ldots, u_{k+1}) \in \mathcal{V}$. Observe that, from equations (2) and (3), $f_{u_1}(u, u_2, \ldots, u_k, u_{k+1})$ is non-decreasing in its first argument on the interval $(0, u_1)$ and, by Definition 1, $f_{u_1}(u_1, u_2, \ldots, u_k, u_{k+1}) \leq 0$. Thus, $f_{u_1}(u, u_2, \ldots, u_k, u_{k+1}) \leq 0$ for all $u \in (0, u_1)$, and so we have $f(u_1, u_2, \ldots, u_k, u_{k+1}) \leq f(0, u_2, \ldots, u_k, u_{k+1})$. Furthermore, by Case 2, we have for any $M \geq u_{k+1}$, $f(0, u_2, \ldots, u_k, u_{k+1}) \leq f(0, u_2, \ldots, u_k, M)$, and thus $f(u_1, u_2, \ldots, u_k, u_{k+1}) \leq f(0, u_2, \ldots, u_k, M)$.

Case 6. $(u_1, u_2, \ldots, u_{k+1}) \in \mathcal{H}$. Observe that, from equations (2) and (3), $f_{u_1}(u_1, u_2, \ldots, u_k, v)$ is non-decreasing in its $(k+1)$st argument on the interval $(0, u_{k+1})$ and, by Definition 1, $f_{u_1}(u_1, u_2, \ldots, u_k, u_{k+1}) \leq 0$. Thus, $f_{u_1}(u_1, u_2, \ldots, u_k, v) \leq 0$ for all $v \in (0, u_{k+1})$, and so we have $f(u_1, u_2, \ldots, u_k, u_{k+1}) \leq f(u_1, u_2, \ldots, u_k, 0)$. Furthermore, by Case 3, we have for any $M \geq u_1$, $f(u_1, u_2, \ldots, u_k, 0) \leq f(M, u_2, \ldots, u_k, 0)$, and thus $f(u_1, u_2, \ldots, u_k, u_{k+1}) \leq f(M, u_2, \ldots, u_k, 0)$. □

Remark 4.

(i) If Condition (C6) holds, then, by Condition (C2), Condition (C5) does not hold and we have that $A < \alpha$. If Condition (C5) holds, then, by Condition (C2), Condition (C6) does not hold and we have that $C < \beta$.

(ii) If Condition (C7) holds, then neither Condition (C5) nor (C6) can hold. Otherwise, either we would have $A > \alpha$, which would imply that $B > \beta$, thereby contradicting Condition (C2); or we would have $C > \beta$, which would imply that $B > \alpha$, thereby contradicting Condition (C2).

(iii) If Condition (C8) holds, then Condition (C5) cannot hold. Otherwise, we would have that $A > \alpha$ implies that $B > \beta$ in Part (b) of Condition (C8), which would contradict Condition (C2).

(iv) Similarly, if Condition (C9) holds, then Condition (C6) cannot hold. Otherwise, we would have that $C > \beta$ implies that $B > \alpha$ in Part (b) of Condition (C9), which would contradict Condition (C2).

(v) If Condition (C6) holds, then we must have $B\beta < C\alpha$, for otherwise $B\beta \geq C\alpha$ and $C > \beta$ implies $B > \alpha$, which contradicts Condition (C2). Therefore, if Condition (C6) holds, then Condition (C8) or Condition (C10) holds and $\mathcal{V} \neq \emptyset$ for u_1 sufficiently small and u_{k+1} sufficiently large.

(vi) If Condition (C5) holds, then we must have $B\alpha < A\beta$, for otherwise $B\alpha \geq A\beta$ and $A > \alpha$ implies $B > \beta$, which contradicts Condition (C2). Therefore, if Condition (C5) holds, then Condition (C9) or Condition (C10) holds and $\mathcal{H} \neq \emptyset$ for u_1 sufficiently large and u_{k+1} sufficiently small.

(vii) From Statements # (i)–(vi), we have that the only possible combinations of Conditions (C4)–(C10) are the following: (a) Conditions (C7) and (C4); (b) Conditions (C8) and (C4); (c) Conditions (C8) and (C6); (d) Conditions (C9) and (C4); (e) Conditions (C9) and (C5); (f) Conditions (C10) and (C4); (g) Conditions (C10) and (C5) and/or (C6).

Given Definition 1 and Remark 4, we have the following.

LEMMA 5.

(i) *If Condition* $(C7)$ *holds, then* $\mathcal{N} = (0,\infty)^{k+1}$ *and* $f(u_1, u_2, \ldots, u_{k+1}) < u^*$*, for* $(u_1, u_2, \ldots, u_{k+1}) \in (0,\infty)^{k+1}$.

(ii) *If Condition* $(C8)$ *holds, then* $\mathcal{N} \cup \mathcal{V} = (0,\infty)^{k+1}$ *(with possibly* $\mathcal{N} = \emptyset$ *or* $\mathcal{V} = \emptyset$*) and*

$$f(u_1, u_2, \ldots, u_k, u_{k+1}) < u^* \begin{cases} \textit{for } u_1, u_{k+1} \in (0,\infty), & \textit{if Condition (C4) holds}, \\ \textit{for } u_1, u_{k+1} \in \left(0, \frac{\gamma - E}{C - \beta}\right), & \textit{if Condition (C6) holds}. \end{cases}$$

(iii) *If Condition* $(C9)$ *holds, then* $\mathcal{N} \cup \mathcal{H} = (0,\infty)^{k+1}$ *(with possibly* $\mathcal{N} \neq \emptyset$ *or* $\mathcal{H} = \emptyset$*) and*

$$f(u_1, \ldots, u_{k+1}) < u^* \begin{cases} \textit{for } u_1, u_{k+1} \in (0,\infty), & \textit{if Condition (C4) holds}, \\ \textit{for } u_1, u_{k+1} \in \left(0, \frac{\gamma - D}{A - \alpha}\right), & \textit{if Condition (C5) holds}. \end{cases}$$

(iv) *If Condition* $(C10)$ *holds, then* $\mathcal{N} \cup \mathcal{V} \cup \mathcal{H} = (0,\infty)^{k+1}$ *(with possibly* $\mathcal{N} = \emptyset$*,* $\mathcal{V} = \emptyset$*, or* $\mathcal{H} = \emptyset$*) and*

$$f(u_1, \ldots, u_{k+1}) < u^* \begin{cases} \textit{for } u_1, u_{k+1} \in (0,\infty), & \textit{if Condition (C4) holds}, \\ \textit{for } u_1, u_{k+1} \in \left(0, \frac{\gamma - E}{C - \beta}\right), & \textit{if Condition (C6), but not (C5), holds}, \\ \textit{for } u_1, u_{k+1} \in \left(0, \frac{\gamma - D}{A - \alpha}\right), & \textit{if Condition (C5), but not (C6), holds}, \\ \textit{for } u_1, u_{k+1} \in & \\ \quad \left(0, \max\left\{\frac{\gamma - D}{A - \alpha}, \frac{\gamma - E}{C - \beta}\right\}\right), & \textit{if Conditions (C5) and (C6) hold}. \end{cases}$$

Proof. We look at each of Statements (i)–(iv), separately.

Statement (i): Suppose that Condition (C7) holds. Then, it follows from equations (2) and (3) and Condition (C1) that $f_{u_1}(u_1, \ldots, u_{k+1}) > 0$ and $f_{u_{k+1}}(u_1, \ldots, u_{k+1}) > 0$ for all $(u_1, \ldots, u_{k+1}) \in (0,\infty)^{k+1}$. Thus, by Definition 1, $\mathcal{N} = (0,\infty)^{k+1}$. Since $u_1, u_2, \ldots, u_{k+1} \leq u^*$, then by Lemma 3, for $(u_1, u_2, \ldots, u_{k+1}) \in (0,\infty)^{k+1}$, we have $f(u_1, u_2, \ldots, u_k, u_{k+1}) \leq f(u^*, u^2, \ldots, u_k, u^*)$. Also, it follows from conditions (C1)–(C3) that

$$\begin{aligned} f(u^*, u_2, \ldots, u_k, u^*) &= \frac{(A+B+C)(u^*)^2 + (D+E)u^*}{(\alpha+\beta)u^* + \gamma} \\ &= \frac{(A+B+C)(u^*)^2 + (D+E)u^*}{(\alpha+\beta)u^* + \gamma} \cdot u^* < u^*. \end{aligned}$$

Therefore, if Condition (C7) holds, we have $f(u_1, u_2, \ldots, u_k, u_{k+1}) \leq f(u^*, u_2, \ldots, u_k, u^*) < u^*$.

Statement (ii): Suppose that Condition (C8) holds. Then, it follows from equations (2) and (3) and Condition (C1) that, for any given $u_2, \ldots, u_k \in [0,\infty)$,

1. $f_{u_1}(0, u_2, \ldots, u_k, 0) > 0$,
2. $\lim_{v \to \infty} f_{u_1}(0, u_2, \ldots, u_k, v) = -\infty$,
3. $f_{u_{k+1}}(u, u_2, \ldots, u_k, v) > 0$ for all $u, v \in [0,\infty)$.

Thus, by Definition 1,

1. if $\mathcal{N} \neq \emptyset$, then $\mathcal{N}$ contains the origin and bisector set;
2. $\mathcal{N} \cup \mathcal{V} = (0, \infty)^{k+1}$ (with possibly $\mathcal{N} = \emptyset$ or $\mathcal{V} = \emptyset$).

If $(u_1, u_2, \ldots, u_k, u_{k+1}) \in \mathcal{N}$, then by Lemma 3 and Statement (i), we have $f(u_1, u_2, \ldots, u_k, u_{k+1}) \leq f(u^*, u_2, \ldots, u_k, u^*) < u^*$. If $(u_1, u_2, \ldots, u_{k+1}) \in \mathcal{V}$, then by Lemma 3, we have $f(u_1, u_2, \ldots, u_k, u_{k+1}) \leq f(0, u_2, \ldots, u_k, u_{k+1})$; and, by Condition (C3) and Remark 4, Statement (ii) is true.

Statement (*iii*): Suppose that Condition (C9) holds. Then, it follows from equations (2) and (3) and Condition (C1) that, for any given $u_2, \ldots, u_k \in [0, \infty)$,

1. $f_{u_1}(0, u_2, \ldots, u_k, 0) \geq 0$,
2. $\lim_{u \to \infty} f_{u_{k+1}}(u, u_2, \ldots, u_k, 0) = -\infty$,
3. $f_{u_1}(u, u_2, \ldots, u_k, v) \geq 0$ for all $(u, v) \in [0, \infty)$.

Thus, by Definition 1,

1. If $\mathcal{N} \neq \emptyset$, then $\mathcal{N}$ contains the origin and bisector
2. $\mathcal{N} \cup \mathcal{H} = (0, \infty)^{k+1}$ (with possibly $\mathcal{N} = \emptyset$ or $\mathcal{H} = \emptyset$).

If $(u_1, u_2, \ldots, u_{k+1}) \in \mathcal{N}$, then by Lemma 3 and Statement (i), we have $f(u_1, u_2, \ldots, u_k, u_{k+1}) \leq f(u^*, u^*, \ldots, u^*, u^*) < u^*$. If $(u_1, u_2, \ldots, u_{k+1}) \in \mathcal{H}$, then by Lemma 3, we have $f(u_1, u_2, \ldots, u_k, u_{k+1}) \leq f(u_1, u_2, \ldots, u_k, 0)$; and, by Condition (C3) and Remark 4, Statement (iii) is true.

Statement (*iv*): Suppose that Condition (C10) holds. Then, it follows from equations (2) and (3) and Condition (C1) that, for any given $u_2, \ldots, u_k \in [0, \infty)$,

1. $f_{u_1}(0, u_2, \ldots, u_k, 0) \leq 0$,
2. $\lim_{v \to \infty} f_{u_1}(0, u_2, \ldots, u_k, v) = -\infty$,
3. $f_{u_{k+1}}(0, u_2, \ldots, u_k, 0) \leq 0$,
4. $\lim_{u \to \infty} f_{u_{k+1}}(u, u_2, \ldots, u_k, 0) = -\infty$.

Thus, by Definition 1,

1. If $\mathcal{N} \neq \emptyset$, then $\mathcal{N}$ contains the origin and the bisector
2. $\mathcal{N} \cup \mathcal{V} \cup \mathcal{H} = (0, \infty)^{k+1}$ (with possibly $\mathcal{N} = \emptyset$, $\mathcal{V} = \emptyset$, or $\mathcal{H} = \emptyset$).

From a combination of the arguments used to establish Statements (ii) and (iii), we see that Statement (iv) is true. □

In order to prove the two main results below, we need the following two definitions, which are motivated by Lemmas 3 and 5, Remark 4, and Definition 1.

Definition 6.

1. *Let Condition* (C_∞) *represent all parameter values such that any one of these four* (*sets of*) *conditions holds: a. Condition* (*C*7)*; b. Conditions* (*C*8) *and* (*C*4)*; c. Conditions* (*C*9) *and* (*C*4)*; d. Conditions* (*C*10) *and* (*C*4).
2. *Let Condition* (C_y) *represent all parameter values such that any one of these two* (*sets of*) *conditions holds: a. Conditions* (*C*8) *and* (*C*6)*; b. Conditions* (*C*10) *and* (*C*6*; with or without Condition* (*C*5) *also holding*).

3. *Let Condition* (C_x) *represent all parameter values such that any one of these two (sets of) conditions holds: a. Conditions* $(C9)$ *and* $(C5)$*; b. Conditions* $(C10)$ *and* $(C5$*; with or without Condition* $(C6)$ *also holding).*

Note that Conditions (C_y) *and* (C_x) *are not mutually exclusive.*

A special case of a result in which parameters satisfy Condition (C_∞) is Theorem 3 in Ref. [1] where the conditions with $\delta \in [0,1]$, $A + \delta B \leq \alpha$, $C + (1-\delta)B \leq \beta$ and $(D + E < \gamma)$, imply that $A \leq \alpha$ and $C \leq \beta$ (i.e. Condition (C4)). as may occur along with Condition (C7), (C8), (C9) or (C10). Theorem 3 then shows that the trivial equilibrium solution of equation (1), with delay $k = 1$ (although, clearly, this result holds for $k > 1$ also) is globally asymptotically stable.

Based on the definitions given in Definition 6, we define the following.

Definition 7.

$$t \overset{\text{def}}{=} \begin{cases} \infty, & \textit{if Condition } (C_\infty) \textit{ holds}, \\ \frac{\gamma - E}{C - \beta}, & \textit{if Condition } (C_y) \textit{ only holds}, \\ \frac{\gamma - D}{A - \alpha}, & \textit{if Condition } (C_x) \textit{ only holds}. \\ \max\left\{\frac{\gamma - D}{A - \alpha}, \frac{\gamma - E}{C - \beta}\right\} & \textit{if Conditions } (C_y) \textit{ and } (C_x) \textit{ hold}. \end{cases} \tag{4}$$

3. Results

We now show that the origin is asymptotically stable with respect to the open rectangle, $(0, t)^{k+1}$, under one of three sets of conditions, C_∞, C_y, C_x, whose primary components are Conditions (C7) and (C10), (C8) and (C10), and (C9) and (C10), respectively.

Theorem 8 (Convergence to the Origin). *Suppose that Conditions* $(C1)$*–*$(C3)$ *hold in equation* (1)*. Then, the origin is a fixed point of* f *and the following is true:*

(i) *The origin is asymptotically stable with respect to* $(0, \infty)^{k+1}$ *if, in addition to having Conditions* $(C1)$*–*$(C3)$ *hold, we have Condition* (C_∞) *hold.*

(ii) *The origin is asymptotically stable with respect to*

$$\left(0, \frac{\gamma - E}{C - \beta}\right)^{k+1}$$

if, in addition to having Conditions $(C1)$*–*$(C3)$ *hold, we have Condition* $(C_y$*; and not Condition* $C_x)$ *hold.*

(iii) *The origin is asymptotically stable with respect to*

$$\left(0, \frac{\gamma - D}{A - \alpha}\right)^{k+1}$$

if, in addition to having Conditions $(C1)$*–*$(C3)$ *hold, we have Condition* (C_x) *(and not Condition* (C_y)*) hold.*

(iv) *The origin is asymptotically stable with respect to*

$$\left(0, \max\left\{\frac{\gamma - D}{A - \alpha}, \frac{\gamma - E}{C - \beta}\right\}\right)^{k+1}$$

if, in addition to having Conditions $(C1)$*–*$(C3)$ *hold, we have both Conditions* (C_y) *and* (C_x) *hold.*

Proof. First of all, we can summarize the results in Lemma 3 as follows: For $(u_1, u_2, \ldots, u_k, u_{k+1}) \in (0,t)^{k+1}$, with t as defined in Definition 7),

$$f(u_1, u_2, \ldots, u_k, u_{k+1}) < u^* \quad \text{for } (u_1, u_2, \ldots, u_{k+1}) \in (0,t)^{k+1}. \tag{5}$$

Next let $\{x_n\}_{n=-k}^{\infty}$ be a positive solution of equation (1), with $k \in \{1, 2, \ldots\}$ and suppose that $x_{-k+1}, \ldots, x_{-1}, x_0 \in (0,t)$, where t as defined in Definition 6. Next define

$$x_n^* \stackrel{\text{def}}{=} \max\{x_n, x_{n-1}, \ldots, x_{n-k}\}, \quad \text{for all } n \geq 0.$$

Clearly, $\max\{x_n, x_{n-k}\} \leq \max\{x_n, x_{n-1}, \ldots, x_{n-k}\}$. Thus, given equation (5) and the assumption that $(x_0, x_{-1}, \ldots, x_{-k}) \in (0,\infty)^{k+1}$, we have

$$x_1 = f(x_0, x_{-1}, \ldots, x_{-k}) \leq \max\{x_0, x_{-1}, \ldots, x_{-k}\} = x_0^* < t.$$

By induction,

$$x_{n+1} = f(x_n, x_{n-1}, \ldots, x_{n-k}) \leq \max\{x_n, x_{n-1}, \ldots, x_{n-k}\} = x_n^* < t, \quad \text{for all } n \geq 0, \tag{6}$$

and so equation (5) holds for $(u_1, u_2, \ldots, u_{k+1}) = (x_n, x_{n-1}, \ldots, x_{n-k})$ for all $n \geq 0$. Also, from equation (6), we have $x_{n+1}^* = \max\{x_{n+1}, x_n, \ldots, x_{n-k+1}\} \leq x_n^*$, for all $n \geq 0$. This means that $\{x_n^*\}_{n=0}^{\infty}$ is a nonincreasing sequence, and so there exists $L \geq 0$ such that $\lim_{n\to\infty} x_n^* = L$ and $x_n^* \geq L$ for all $n \geq 0$. We show that $L = 0$ by contradiction. Assume that $L > 0$. Let us define three functions, $\phi_i : [0,\infty) \to \mathbf{R}$, for $i = 1, 2, 3$, as follows:

$$\begin{cases} \phi_1(x) \stackrel{\text{def}}{=} [(\alpha+\beta)(L-x) - (A+B+C)(L+x)](L+x) + \gamma(L-x) - (D+E)(L+x), \\ \phi_2(x) \stackrel{\text{def}}{=} [\beta(L-x) - C(L+x)](L+x) + \gamma(L-x) - E(L+x), \\ \phi_3(x) \stackrel{\text{def}}{=} [\alpha(L-x) - A(L+x)](L+x) + \gamma(L-x) - D(L+x). \end{cases}$$

Then, from Conditions (C1)–(C3) and our assumption that $L > 0$, we have that

$$\begin{cases} \phi_1(0) = [(\alpha+\beta) - (A+B+C)]L^2 + [\gamma - (D+E)]L > 0, & \text{if Condition } (C_\infty) \text{ holds}, \\ \phi_2(0) = [\beta - C]\,L^2 + [\gamma - E]\,L > 0, & \text{if Condition } (C_y) \text{ holds}, \\ \phi_3(0) = [\alpha - A]\,L^2 + [\gamma - D]\,(L) > 0, & \text{if Condition } (C_x) \text{ holds}. \end{cases}$$

By the continuity of each of the functions, ϕ_1, ϕ_2 and ϕ_3, on $\mathbf{R}$, there exists $\epsilon > 0$ such that $\phi_i(\epsilon) > 0$ for $i = 1, 2, 3$. Thus, we have the following:

$$\frac{(A+B+C)(L+\epsilon)^2 + (D+E)(L+\epsilon)}{(\alpha+\beta)(L+\epsilon) + \gamma} < L - \epsilon. \tag{7}$$

$$\frac{C(L+\epsilon)^2 + E(L+\epsilon)}{\beta(L+\epsilon) + \gamma} < L - \epsilon. \tag{8}$$

$$\frac{A(L+\epsilon)^2 + D(L+\epsilon)}{\alpha(L+\epsilon) + \gamma} < L - \epsilon. \tag{9}$$

Now, there exists $N \geq 0$ such that for all $n \geq N$,

$$x_{n-k}, \ldots, x_{n-1}, x_n \leq x_n^* \leq L + \epsilon. \tag{10}$$

We next consider the following three cases:

Case 1. $(x_n, x_{n-1}, \ldots, x_{n-k}) \in \mathcal{N} \cap (0,t)^{k+1}$. It follows from equations (7) and (10) and Lemma 3 that

$$\frac{Ax_n^2 + Bx_nx_{n-k} + Cx_{n-k}^2 + Dx_n + Ex_{n-k}}{\alpha x_n + \beta x_{n-k} + \gamma} \leq \frac{(A+B+C)(L+\epsilon)^2 + (D+E)(L+\epsilon)}{(\alpha+\beta)(L+\epsilon)+\gamma}$$
$$< L - \epsilon.$$

Case 2. $(x_n, x_{n-1}, \ldots, x_{n-k}) \in \mathcal{V} \cap (0,t)^{k+1}$. It follows from equations (8) and (10) and Lemma 3 that

$$\frac{Cx_{n-k}^2 + Ex_{n-k}}{\beta x_{n-k} + \gamma} \leq \frac{C(L+\epsilon)^2 + E(L+\epsilon)}{\beta(L+\epsilon)+\gamma} < L - \epsilon.$$

Case 3. $(x_n, x_{n-1}, \ldots, x_{n-k}) \in \mathcal{H} \cap (0,t)^{k+1}$. It follows from equations (9) and (10) and Lemma 3 that

$$\frac{Ax_n^2 + Dx_n}{\alpha x_n + \gamma} \leq \frac{A(L+\epsilon)^2 + D(L+\epsilon)}{\alpha(L+\epsilon)+\gamma} < L - \epsilon.$$

Hence, by induction, $x_{n-k}, \ldots, x_{n-1}, x_n < L - \epsilon$ for all $n \geq N$; and so

$$L = \lim_{n\to\infty} x_n^* = \lim_{n\to\infty} \max\{x_n, x_{n-1}, \ldots, x_{n-k}\} \leq \lim_{n\to\infty} \max\{L-\epsilon, L-\epsilon, \ldots, L-\epsilon\} = L - \epsilon,$$

which is impossible. The proof is complete. □

We also have some results on the global behaviour of solutions outside of $(t,\infty)^{k+1}$ under certain conditions.

THEOREM 9 (INFINITE LIMITS). *Let* $\{x_n\}_{n=-k}^{\infty}$ *be a solution of equation* (1), *and assume that Conditions* (C1)–(C3) *hold in Equation* (1). *Let*

$$\mathcal{D}_{p_1} = (2A\gamma + B\gamma + D\beta - E\alpha)^2 - 4(A\alpha + 2A\beta + B\beta - C\alpha)D\gamma,$$

$$\mathcal{D}_{q_1} = (A\gamma + B\gamma + D\beta - E\alpha)^2 - 4(A\beta + B\beta - C\alpha)D\gamma;$$

$$\mathcal{D}_{p_2} = (2C\gamma + B\gamma + E\alpha - D\beta)^2 - 4(C\beta + 2C\alpha + B\alpha - A\beta)E\gamma,$$

$$\mathcal{D}_{q_2} = (B\gamma + C\gamma + E\alpha - D\beta)^2 - 4(B\alpha + C\alpha - A\beta)E\gamma.$$

Then, define the following:

$$p_1 \stackrel{\text{def}}{=} \begin{cases} \frac{-(2A\gamma+B\gamma+D\beta-E\alpha)+\sqrt{\mathcal{D}_{p_1}}}{2(A\alpha+2A\beta+B\beta-C\alpha)}, & \text{if } \mathcal{D}_{p_1} \geq 0, \\ 0, & \text{if } \mathcal{D}_{p_1} < 0, \end{cases}$$

$$q_1 \stackrel{\text{def}}{=} \begin{cases} \frac{-(A\gamma+B\gamma+D\beta-E\alpha)+\sqrt{\mathcal{D}_{q_1}}}{2(A\beta+B\beta-C\alpha)}, & \text{if } \mathcal{D}_{q_1} \geq 0, \\ 0, & \text{if } \mathcal{D}_{q_1} < 0, \end{cases}$$

$$m_1 \stackrel{\text{def}}{=} \max\left\{\frac{\gamma - E}{C - \beta}, p_1, q_1\right\};$$

$$p_2 \stackrel{\text{def}}{=} \begin{cases} \frac{-(2C\gamma+B\gamma+E\alpha-D\beta)+\sqrt{\mathcal{D}_{p_2}}}{2(C\beta+2C\alpha+B\beta-A\beta)}, & \text{if } \mathcal{D}_{p_2} \geq 0, \\ 0, & \text{if } \mathcal{D}_{p_2} < 0, \end{cases}$$

$$q_2 \stackrel{\text{def}}{=} \begin{cases} \frac{-(B\gamma+C\gamma+E\alpha-D\beta)+\sqrt{\mathcal{D}_{q_2}}}{2(B\alpha+C\alpha-A\beta)}, & \text{if } \mathcal{D}_{q_2} \geq 0, \\ 0, & \text{if } \mathcal{D}_{q_2} < 0, \end{cases}$$

$$m_2 \stackrel{\text{def}}{=} \max\left\{\frac{\gamma - E}{C - \beta}, p_2, q_2\right\}.$$

Then, we have the following:

(i) *If Conditions* $(C6)$, $(C8)$ *and* $(C11)$ *hold, and* $x_{-k}, \ldots, x_{-1}, x_0 \in (m_1, \infty)$, *then* $\lim_{n\to\infty} x_n = \infty$.

(ii) *If Conditions* $(C5)$, $(C9)$ *and* $(C12)$ *hold, and* $x_{-k}, \ldots, x_{-1}, x_0 \in (m_2, \infty)$, *then* $\lim_{n\to\infty} x_n = \infty$.

Proof. We consider Statements (i) and (ii), separately.

Statement (*i*): First of all, observe that Condition (C11) and Equation (2) imply the following.

(**O**1) $f_{u_1}(u_1, u_2, \ldots, u_k, u_{k+1}) > 0$ for all $u_1 \geq u_{k+1} > p_1$, and thus $\mathcal{N} \supset \{(u_1, u_2, \ldots, u_k, u_{k+1}) \in (0,\infty)^{k+1} : u_1 \geq u_{k+1} > p_1\}$, where $\mathcal{N}$ is as defined in Definition 1.

(**O**2) Let $u_1 = u_{k+1} = v > q_1$ and let $u_2, \ldots, u_k > 0$ be arbitrary. Then, $f(v, u_2, \ldots, u_k, u_k, v) > f(0, u_2, \ldots, u_k, v)$, where

$$f(v, u_2, \ldots, u_k, v) > f(0, u_2, \ldots, u_k, v) \Longleftrightarrow \frac{(A+B+C)v^2 + (D+E)v}{(\alpha+\beta)v+\gamma} > \frac{Cv^2 + Ev}{\beta v + \gamma} \Longleftrightarrow$$
$$(A+B+C)v(\beta v+\gamma) + (D+E)(\beta v+\gamma) > (Cv+E)[(\alpha+\beta)v+\gamma] \Longleftrightarrow$$
$$(A\beta + B\beta - C\alpha)v^2 + (A\gamma + B\gamma + D\beta - E\alpha)v + D\gamma > 0,$$

which is true since Condition (C11) holds (so that $A\beta + B\beta - C\alpha > 0$) and by the definition of q_1.

We make further observations.

(**O**3) Let $\mathcal{V}$ and $\mathcal{N}$ be as defined in Definition 1.

1. Suppose that $(u_1, u_2, \ldots, u_k, u_{k+1}) \in \mathcal{V}$. Then, $f_{u_1} \le 0$, by definition, but also $f_{u_{k+1}} \ge 0$ by Remark 4, where Conditions (C8) and (C6) hold. Hence, if $u_1 < u_{k+1}$, then

$$f(u_1, u_2, \ldots, u_k, u_{k+1}) > f(u_1, u_2, \ldots, u_k, u_1) \text{ and} f(u_{k+1}, u_2, \ldots, u_k, u_{k+1}) \Longrightarrow$$

$$f(u_1, u_2, \ldots, u_k, u_{k+1}) > \min\{f(u_1, u_2, \ldots, u_k, u_1), f(u_{k+1}, u_2, \ldots, u_k, u_{k+1})\}.$$

2. Suppose that $(u_1, u_2, \ldots, u_k, u_{k+1}) \in \mathcal{N}$ (so that $f_{u_1} \ge 0, f_{u_{k+1}} \ge 0$). The following is true:
   a. If $u_1 < u_{k+1}$, then

$$\begin{aligned} f(u_1, u_2, \ldots, u_k, u_{k+1}) &> f(u_1, u_2, \ldots, u_k, u_1) \\ &\ge \min\{f(u_1, u_2, \ldots, u_k, u_1), f(u_{k+1}, u_2, \ldots, u_k, u_{k+1})\}. \end{aligned}$$

   b. If $u_1 \ge u_{k+1}$, then

$$\begin{aligned} f(u_1, u_2, \ldots, u_k, u_{k+1}) &\ge f(u_{k+1}, u_2, \ldots, u_k, u_{k+1}) \\ &\ge \min\{f(u_1, u_2, \ldots, u_k, u_1), f(u_{k+1}, u_2, \ldots, u_k, u_{k+1})\}. \end{aligned}$$

3. Thus, in general,

$$f(u_1, u_2, \ldots, u_k, u_{k+1}) \ge \min\{f(u_1, u_2, \ldots, u_k, u_1), f(u_{k+1}, u_2, \ldots, u_k, u_{k+1})\}.$$

   for $(u_1, u_2, \ldots, u_k, u_{k+1}) \in (0, \infty)^{k+1}$.

(**04**) Let $u_1 = 0$ and $u_{k+1} = v > (\gamma - E)/(C - \beta)$, and let $u_2, \ldots, u_k \in (0, \infty)$ be arbitrary. Then, $f(0, u_2 \ldots u_k, v) > v$, where, given Condition (C6), we have

$$f(0, u_2, \ldots, u_k, v) > v \iff \frac{Cv^2 + Ev}{\beta v + \gamma} \iff \frac{\gamma - E}{C - \beta}.$$

(05) $m_1 > 0$ since $(\gamma - E)/(C - \beta) > 0$ by Conditions (C3) and (C6).

Combining observations (**O1**)–(**O4**), we have that, for $u_1, u_{k+1} > m_1$ and $u_2, \ldots, u_k \in (0, \infty)$,

$$\begin{aligned} f(u_1, u_2, \ldots, u_k, u_{k+1}) &\ge \min\{f(u_1, u_2, \ldots, u_k, u_1), f(u_{k+1}, u_2, \ldots, u_k, u_{k+1})\} \\ &> \min\{f(0, u_2, \ldots, u_k, u_1), f(0, u_2, \ldots, u_k, u_{k+1})\} \\ &> \min\{u_1, u_{k+1}\} \ge \min\{u_1, u_2, \ldots, u_k, u_{k+1}\} \end{aligned}$$

Therefore, in summary, we can say that

$$f(u_1, u_2, \ldots, u_k, u_{k+1}) > \min\{u_1, u_2, \ldots, u_k, u_{k+1}\}, \quad \text{for } (u_1, u_2, \ldots, u_{k+1}) \in (m_1, \infty)^{k+1}. \tag{11}$$

Now let $\{x_n\}_{n=-k}^{\infty}$ be a positive solution of equation (1), with $k \in \{1, 2, \ldots\}$ and suppose that $x_{-k+1}, \ldots, x_{-1}, x_0 \in (m_1, \infty)$, where m_1 is as defined above. Next define

$$x_n^* \stackrel{\text{def}}{=} \min\{x_n, x_{n-1}, \ldots, x_{n-k}\}, \quad \text{for all } n \ge 0.$$

Clearly, $\min\{x_n, x_{n-k}\} \ge \min\{x_n, x_{n-1}, \ldots, x_{n-k}\}$. Thus, given equation (11) and the assumption that $(x_0, x_{-1}, \ldots, x_{-k}) \in (m_1, \infty)^{k+1}$, we have

$$x_1 = f(x_0, x_{-1}, \ldots, x_{-k}) > \min\{x_0, x_{-1}, \ldots, x_{-k}\} = x_0^* > m_1.$$

By induction,

$$x_{n+1} = f(x_n, x_{n-1}, \ldots, x_{n-k}) > \min\{x_n, x_{n-1}, \ldots, x_{n-k}\} = x_n^* > m_1, \quad \text{for all } n \geq 0, \tag{12}$$

and so equation (11) holds for $(u_1, u_2, \ldots, u_{k+1}) = (x_n, x_{n-1}, \ldots, x_{n-k})$ for all $n \geq 0$. Also, from equation (12), we have $x_{n+1}^* = \min\{x_{n+1}, x_n, \ldots, x_{n-k+1}\} \leq x_n^*$, for all $n \geq 0$. This means that $\{x_n^*\}_{n=0}^{\infty}$ is a non-decreasing sequence and so there exists $m_1 < L \leq \infty$ such that $\lim_{n\to\infty} x_n^* = L$ and $x_n^* \leq L (x_n^* < L \text{ if } L = \infty)$ for all $n \geq 0$. We show that $L = \infty$ by contradiction. Assume that $L < \infty$. Let us define the function $\phi : [0, \infty) \to \mathbf{R}$ as follows:

$$\phi(x) \overset{\text{def}}{=} [\beta(L + x) - C(L - x)](L - x) + \gamma(L + x) - E(L - x).$$

Then, from Conditions (C1)–(C3), (C6) and (C8) and the fact that L is positive (by observation (**O5**)), we have that $\phi(0) = [\beta - C]L^2 + [\gamma - E]L < 0$. By the continuity of ϕ on $\mathbf{R}$, there exists $\epsilon > 0$ such that $\phi(\epsilon) < 0$. Thus, we have the following:

$$\frac{C(L+\epsilon)^2 + E(L-\epsilon)}{\beta(L-\epsilon) + \gamma} > L + \epsilon. \tag{13}$$

Now, there exists $N \geq 0$ such that for all $n \geq N$,

$$x_{n-k}, \ldots, x_{n-1}, x_n \geq x_n^* > L - \epsilon. \tag{14}$$

It follows from equations (13) and (14) and Lemma 3, Case 2, that

$$\frac{Cx_{n-k}^2 + Ex_{n-k}}{\beta x_{n-k} + \gamma} \geq \frac{C(L-\epsilon)^2 + E(L-\epsilon)}{\beta(L-\epsilon) + \gamma} > L + \epsilon.$$

Hence, by induction, $x_{n-k}, \ldots, x_{n-1}, x_n > L + \epsilon$ for all $n \geq N$, and so

$$L = \lim_{n\to\infty} x_n^* = \lim_{n\to\infty} \min\{x_n, x_{n-1}, \ldots, x_{n-k}\} \geq \lim_{n\to\infty} \min\{L+\epsilon, L+\epsilon, \ldots, L+\epsilon\} = L + \epsilon,$$

which is impossible. The proof for Statement (i) is complete.

Statement (*ii*): The proof here is similar to that of Statement (i), with the exception that

$$\phi(x) \overset{\text{def}}{=} [\alpha(L + x) - A(L - x)](L - x) + \gamma(L + x) - D(L - x).$$

Hence, we omit the proof for Statement (ii). □

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