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Global Stability and Boundedness in $x_{n+1} = cx_n + f(x_n - x_{n-1})$

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Dedicated to Saber Elaydi on the Occasion of His 60th Birthday.

For the equation in the title, we present conditions on the function *f* that are sufficient for the boundedness of all solutions, conditions that imply oscillation of all solutions and also conditions that imply the global asymptotic stability of the unique fixed point. In the latter case, we also specify conditions under which convergence to the equilibrium is monotonic.

Keywords: Second-order difference equation; Non-linear; Global asymptotic stability; Semicycle; Variation of constants; Boundedness

AMS Subject Classification Numbers: 39A10; 39A11

The non-linear, second order difference equation

$$x_{n+1} = cx_n + f(x_n - x_{n-1}), \quad x_0, x_{-1} \in \mathbb{R} \tag{1}$$

has its roots in the early macroeconomic models of the business cycle. The linear model in Ref. [5] as well as non-linear models such as those in Refs. [2] and [4] are all modeled by special cases of Eq. (1). For more details, some historical remarks and additional references see Ref. [9]. Throughout this paper we assume that $0 \leq c < 1$ and f is continuous on $\mathbb{R}$.

Depending on the conditions placed on the function f, Eq. (1) displays a considerable variety of different types of asymptotic behavior that range from stable (and unstable) global convergence to the equilibrium, to persistent oscillations (periodic, aperiodic and chaotic) which in some cases occur off-equilibrium; see Refs. [6,9,10], as well as the Remark after Lemma 3 and the Example below.

In this paper, we present a set of conditions on f that imply boundedness, as well as conditions that imply oscillations of all solutions. In addition, we discuss conditions that imply the global asymptotic stability of the equilibrium. Our results also partially establish Conjectures 1 and 2 in Ref. [8] under weaker hypotheses. All of the necessary background for this paper (fixed points, linearization, stability, semicycles, etc.) is found in standard texts such as Refs. [1] and [3].

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GLOBAL STABILITY

We begin with a discussion of the global asymptotic stability of the equilibrium. Note that Eq. (1) has a unique fixed point $\bar{x} = f(0)/(1-c)$, so if $f(0) = 0$ then the only fixed point of Eq. (1) is at the origin. We need a result from Ref. [7] which we quote here as a lemma.

LEMMA 1 *Let $g: \mathbb{R}^m \to \mathbb{R}$ be continuous and let $\bar{x}$ be an isolated fixed point of*

$$x_{n+1} = g(x_n, x_{n-1}, \ldots, x_{n-m}).$$

Also, assume that for some $\alpha \in (0,1)$ the set

$$A_\alpha = \{(u_1, \ldots, u_m) : |g(u_1, \ldots, u_m) - \bar{x}| \leq \alpha \max\{|u_1 - \bar{x}|, \ldots, |u_m - \bar{x}|\}\}$$

has a non-empty interior (i.e. g is not very steep near $\bar{x}$) and let r be the largest positive number such that $[\bar{x} - r, \bar{x} + r]^m \subset A_\alpha$. Then, $\bar{x}$ is exponentially stable relative to the interval $[\bar{x} - r, \bar{x} + r]$.

The function g in Lemma 1 is said to be a *weak contraction* on the set A_α. See Ref. [9] for a systematic development of the concepts of weak contractions (respectively, expansions) and their relationship to asymptotic stability (respectively, instability).

As a corollary to Lemma 1 we have the following general fact regarding the global attractivity of the equilibrium in Eq. (1).

COROLLARY 1 *If $|f(t)| \leq a|t|$ for all t and $0 < a < (1-c)/2$ then the origin is globally asymptotically stable in Eq. (1).*

Proof The inequality involving f in particular implies that $f(0) = 0$, so that the origin is the unique fixed point of Eq. (1). Define $g(x, y) = cx + f(x - y)$ and notice that

$$|g(x, y)| \leq c|x| + a|x - y| \leq (c + a)|x| + a|y| \leq (c + 2a) \max\{|x|, |y|\}.$$

Since $c + 2a < 1$ by assumption, it follows that g is a weak contraction on the entire plane and therefore, Lemma 1 implies that the origin is globally asymptotically (in fact, exponentially) stable in Eq. (1). □

Remarks (1) Noteworthy in the preceding result is the fact that restrictions on f are minimal, except of course for the upper bound on a. A consideration of linear $f(t) = at$ shows that some restriction on either f or on a is necessary to assure convergence. For instance, if $f(t) = t$ then solutions of Eq. (1) generally do not converge to zero for any value of c as specified above. Hence, if the conclusion of Corollary 1 is to hold, then it is necessary that $a \in (0,1)$. But alone this is not sufficient; for example, if $f(t) = -at$ where

$$\frac{1+c}{2} < a < 1$$

then the negative eigenvalue of the linearization of Eq. (1) will have a magnitude less than -1 and so the solutions of Eq. (1) will be typically unbounded.

(2) If $f(0) = 0$ and f is continuously differentiable with $|f'(t)| \leq a$ for all t or more generally, if f satisfies the Lipschitz inequality

$$|f(t) - f(s)| \leq a|t - s|, \quad t, s \in \mathbb{R}$$

then in particular (with $s = 0$) it is also true that $|f(t)| \le a|t|$ for all $t \in \mathbb{R}$. However, a function f satisfying the conditions of Corollary 1 need not be Lipschitz.

In the remainder of this section, we consider alternative conditions that improve the upper bound on a in Corollary 1. We begin with the next lemma which furnishes some useful inequalities.

LEMMA 2 *If there is $a > 0$ such that $|f(t)| \le a|t|$ for all t, then for all n,*

$$x_{n+1} - x_n \le (a + c - 1)(x_n - x_{n-1}), \quad \text{if } x_n \ge x_{n-1} \ge 0,$$
$$x_{n+1} - x_n \ge (a + c - 1)(x_n - x_{n-1}), \quad \text{if } x_n \le x_{n-1} \le 0.$$

Proof Suppose that $x_n \ge x_{n-1} \ge 0$. Then,

$$x_{n+1} - x_n = -(1 - c)x_n + f(x_n - x_{n-1}) \le (c - 1)x_n + a(x_n - x_{n-1})$$
$$\le (c - 1)(x_n - x_{n-1}) + a(x_n - x_{n-1}) = (a + c - 1)(x_n - x_{n-1}).$$

If $x_n \le x_{n-1} \le 0$, then the above argument holds when the inequalities are reversed in it. □

Recall that with the origin as an equilibrium, a positive (respectively, negative) semicycle of a solution $\{x_n\}$ of Eq. (1) is a collection of consecutive terms $\{x_{k+1}, x_{k+2}, \ldots, x_{k+l}\}$ where $l \ge 1$ and $x_{k+j} \ge 0$ (respectively, $x_{k+j} < 0$) for $1 \le j \le l$ and $x_k, x_{k+l+1} < 0$ (respectively, ≥ 0). The number l is the length of the semicycle. See Ref. [3] for more details and applications of semicycle analysis. The next result is an immediate corollary of Lemma 2.

COROLLARY 2 *Under the hypotheses of Lemma 2, if $0 < a \le 1 - c$ then the maximum (respectively, minimum) of a positive (respectively, negative) semicycle is achieved at the first or the second term of that semicycle.*

The condition on f in the next lemma prevents certain types of unusual oscillations from occurring (see the Remark following the lemma).

LEMMA 3 *If $tf(t) \ge 0$ for all t then every eventually non-negative and every eventually non-positive solution of Eq. (1) is eventually monotonic.*

Proof Suppose that $\{x_n\}$ is a solution of Eq. (1) that is eventually non-negative, i.e. there is $k > 0$ such that $x_n \ge 0$ for all $n \ge k$. Either $x_n \ge x_{n-1}$ for all $n > k$ in which case $\{x_n\}$ is eventually monotonic, or there is $n > k$ such that $x_n \le x_{n-1}$. In the latter case,

$$x_{n+1} = cx_n + f(x_n - x_{n-1}) \le cx_n \le x_n$$

so that by induction, $\{x_n\}$ is eventually non-increasing, hence monotonic. The argument for an eventually non-positive solution is similar and omitted. □

Remark (Off-equilibrium Oscillations) If the inequality mentioned in Lemma 3 does not hold, then it is possible for Eq. (1) to have oscillatory solutions that are eventually non-negative (or non-positive). For example, if

$$f(t) = \min\{1, |t|\}, \quad c = 0$$

then Eq. (1) has a stable period-3 solution $\{x_n\} = \{0, 1, 1\}$ which is clearly both non-negative and non-monotonic. Note also that here

$$\liminf_{n\to\infty} x_n = 0 < 1 = \limsup_{n\to\infty} x_n$$

so $\{x_n\}$ oscillates off the equilibrium value, which is the origin in this case. Off-equilibrium oscillations, which are obviously non-linear in nature, can occur in both convergent and non-convergent solutions of Eq. (1) and they may also be associated with chaos [9,10].

LEMMA 4 *Assume that $tf(t) \geq 0$ for all t. Let $\{x_n\}$ be a solution of Eq. (1) with at least two adjacent semicycles, one positive and one negative. Denote the first term at which the maximum of a positive semicycle occurs by x_M and in the succeeding (or preceding) negative semicycle, denote the first term at which the minimum occurs by x_m. Then $\{x_n\}$ is strictly decreasing (respectively, increasing) from x_{M+1} (respectively, x_{m+1}) to x_m (respectively, x_M).*

Proof Note that because of maximality, $x_{M+1} \leq x_M$ so we have

$$x_{M+2} = cx_{M+1} + f(x_{M+1} - x_M) \leq cx_{M+1} < x_{M+1}.$$

The same observation plus induction if necessary, shows that $x_{M+i+1} < x_{M+i}$ for as long as x_{M+i} is within the positive semicycle. Suppose that $x_{M+k+1} = x_{m-l} < 0$ is the first element of the negative semicycle. Then,

$$x_{m-l} = x_{M+k+1} = cx_{M+k} + f(x_{M+k} - x_{M+k-1}) < x_{M+k}$$

and also

$$x_{m-l+1} = cx_{m-l} + f(x_{m-l} - x_{M+k}) \leq cx_{m-l} < 0.$$

If $x_{m-l} \leq x_{m-l+1}$ then

$$x_{m-l+2} = cx_{m-l+1} + f(x_{m-l+1} - x_{m-l}) \geq cx_{m-l+1} > x_{m-l+1}$$

so it may be concluded that $x_{m-l} = x_m$ and the lemma is proved. If not, then

$$x_{m-l} > x_{m-l+1}$$

and a repeat of the above argument will establish that either $x_{m-l+1} = x_m$ or that

$$x_{m-l+1} > x_{m-l+2}$$

in which case we continue inductively until x_m is reached. On the upward path from x_m we have $x_{m+1} \geq x_m$ due to minimality and thus if $x_{m+1} < 0$, then

$$x_{m+2} = cx_{m+1} + f(x_{m+1} - x_m) \geq cx_{m+1} > x_{m+1}.$$

As in the downward case, one may use induction to show that x_n must increase strictly either indefinitely, or if there is an adjacent positive semicycle, then until x_n is non-negative. After that, until the maximum of the following positive semicycle is reached one shows a strictly increasing pattern in a manner similar to that for the negative semicycle, with inequalities reversed. □

THEOREM 1 *Assume that for all t, $tf(t) \geq 0$ and there is $a > 0$ such that $|f(t)| \leq a|t|$. If*

$$a < \frac{2-c}{3-c} \doteq d \quad \text{or} \quad a \leq 1 - c \ \text{ and } \ c \neq 0 \tag{2}$$

then the origin is globally asymptotically stable in Eq. (1).

Proof In the light of Lemma 3 we may assume that solutions of Eq. (1) oscillate about the origin. Let $\{x_n\}$ be such a solution and with Lemma 4 in mind, let

$$x_M \geq x_{M+1} > \cdots > x_{M+k} \geq 0 > x_{M+k+1}$$

be the segment stretching from the peak of a positive semicycle down to zero. Then we claim that

$$x_{M+i} - x_{M+i+1},\ \ x_{M+k},\ \ |x_{M+k+1}| \leq \mu x_M, \quad i = 0, 1, \ldots, k. \tag{3}$$

where $\mu = \max\{a, 1-c\}$. To establish this claim, note that

$$|x_{M+1} - x_M| = (1-c)x_M - f(x_M - x_{M-1}) \leq (1-c)x_M \leq \mu x_M$$

$$|x_{M+2} - x_{M+1}| \leq (1-c)x_{M+1} - a(x_{M+1} - x_M) \leq \mu x_{M+1} + \mu(x_M - x_{M+1}) = \mu x_M$$

$$|x_{M+3} - x_{M+2}| \leq (1-c)x_{M+2} - a(x_{M+2} - x_{M+1}) \leq \mu x_{M+1} \leq \mu x_M.$$

The above process inductively gives

$$|x_{M+k+1} - x_{M+k}| \leq \mu x_M.$$

In particular, this observation establishes the above claim. Next, by Lemma 4, let

$$0 > x_{M+k+1} = x_{m-l} > x_{m-l+1} > \cdots > x_{m-1} > x_m$$

represent the part of the succeeding negative semicycle that stretches from zero down to the minimum of the negative semicycle. We consider two cases:

Case 1 $a + c \leq 1,\ c \neq 0$; Corollary 2 implies that $x_m = x_{M+k+1}$ or $x_{m-1} = x_{M+k+1}$. In the first case,

$$|x_m| \leq \mu x_M = (1-c)x_M$$

and in the second case, Lemma 2 implies that

$$0 \geq x_m - x_{M+k+1} \geq (a+c-1)(x_{M+k+1} - x_{M+k}) \geq 0$$

so that $|x_m| = |x_{M+k+1}| \leq \mu x_M$.

Case 2 $a + c > 1$; for $j = 1, \ldots, l-1$, by Lemma 2

$$|x_{m-l+j+1} - x_{m-l+j}| \leq (a+c-1)|x_{m-l+j} - x_{m-l+j-1}|.$$

Further, from Eq. (3)

$$x_{M+k},\ |x_{M+k+1}| = |x_{m-l}| \leq |x_{M+k+1} - x_{M+k}| \leq \mu x_M = a x_M,$$

and also $|x_{m-l+1} - x_{m-l}| \leq a x_{m-l-1} = a x_{M+k} \leq a^2 x_M$. Writing

$$|x_m| = |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \cdots |x_{m-l+1} - x_{m-l}| + |x_{m-l}|,$$

and substituting for various terms, we obtain

$$|x_m| \leq a x_M \sum_{j=0}^{l-1} a(a+c-1)^j + a x_M \leq a x_M \left[1 + a \sum_{j=0}^{\infty} (a+c-1)^j\right]$$

$$= a\left[1 + \frac{a}{2-a-c}\right] x_M$$

If β is the coefficient of x_M, then $\beta < 1$ if and only if $a < (2-c)/(3-c)$.

The preceding arguments for Cases 1 and 2 show that under conditions (2),

$$|x_m| \leq \gamma x_M, \quad \textit{where} \;\; \gamma = \max\{\mu, \beta\} < 1. \tag{4}$$

Arguments similar to the above show that if

$$x_m \leq x_{m+1} < \cdots < x_{m+l} < 0 \leq x_{m+l+1}$$

is the segment stretching from the peak of the negative semicycle up to zero, then

$$x_{m+i+1} - x_{m+i}, \;\; |x_{m+l}|, \;\; x_{m+l+1} \leq \mu x_m, \quad i = 0, 1, \ldots, l.$$

Applying Corollary 2 and Lemma 2 in a manner similar to the above, we can show once again that if x_{M^*} is the peak of the positive semicycle following the current negative one, then under conditions (2),

$$x_{M^*} \leq \gamma |x_m|, \quad \text{where} \;\; \gamma = \max\{\mu, \beta\} < 1. \tag{5}$$

Now an obvious induction based on inequalities (4) and (5) establish the global attractivity of the origin. To complete the proof of the theorem, we show that the origin is stable. In the light of the preceding discussion which shows that the consecutive peaks decrease in magnitude, we need only show that the first peak (positive or negative) that may occur in each solution, approaches zero as the initial conditions x_0, x_{-1} approach zero. Let x_p be this initial peak for some $p \geq 0$ and note that if $c + a \leq 1$, then we have $|x_p| \leq |x_0|$ by Corollary 2, and stability follows. So we assume in the rest of the proof that $c + a > 1$ and consider a few cases.

Case 1 $\;x_0 \geq x_{-1} > 0$ or $x_0 \leq x_{-1} < 0$; for $0 \leq n \leq p$, $|x_n| \geq |x_0| > 0$ and

$$\frac{x_{n+1}}{x_n} = c + \frac{f(x_n - x_{n-1})}{x_n} \leq c + \frac{a(x_n - x_{n-1})}{x_n} = c + a - \frac{a}{\frac{x_n}{x_{n-1}}}.$$

Define $r_n = x_n/x_{n-1}$ and $z_0 = (x_0/x_{-1}) = r_0$ and note that $r_n, z_0 > 0$. Further, if for $1 \leq n \leq p$,

$$z_n = c + a - \frac{a}{z_{n-1}} \tag{6}$$

then

$$r_1 \leq a + c - \frac{\mathrm{a}}{z_0} = z_1$$

$$r_2 \leq a + c - \frac{a}{r_1} \leq a + c - \frac{a}{z_1} = z_2$$

and so on. Inductively, $r_n \leq z_n$ for $1 \leq n \leq p$. Further, since $|x_n| \geq |x_{n-1}|$ it follows that

$$1 \leq r_n \leq z_n < c + a, \quad 1 \leq n \leq p.$$

Therefore, $|x_n| \leq z_n |x_{n-1}|$ and we may conclude that

$$|x_p| \leq |x_0| \prod_{n=1}^{p} z_n \leq |x_0|(c + a)^p.$$

Because x_p is a semicycle peak, we note that $|x_{p+1}| \leq |x_p|$ so $r_{p+1} \leq 1$. Note that the function $\phi(z) = c + a - a/z$ is increasing and $\phi(z) < z$ if $z \geq 1$, so the terms z_n are decreasing in (6) as long as they are greater than 1. Let μ be the smallest index such that

$$\phi^{\mu}(c + a) < 1.$$

Then, μ is independent of x_{-1}, x_0 and $\mu \geq p$ since $z_1 = \phi(z_0) < c + a$. Further, if $K = (c + a)^{\mu}$, then

$$|x_p| \leq |x_0|(c + a)^p \leq K|x_0|.$$

It follows that if $|x_0| \to 0$ (so $|x_{-1}| \to 0$ also) then $|x_p| \to 0$.

Case 2 $x_0 > 0 \geq x_{-1}$; to avoid a trivial case, assume that $x_1 \geq x_0 > 0$ so that Case 1 obtains with shifted indices. Thus, defining $z_1 = x_1/x_0$ and arguing as in Case 1, we find that

$$x_p \leq x_1 \prod_{n=2}^{p+1} z_n \leq x_1(c + a)^p \leq K[cx_0 + f(x_0 - x_{-1})]$$

so that $x_p \to 0$ in this case as $x_0, x_{-1} \to 0$.

Case 3 $x_0 < 0 \leq x_{-1}$; as in Case 2, assume that $x_1 \leq x_0 < 0$ and conclude that

$$|x_p| \leq |x_1|(c + a)^p \leq K|cx_0 + f(x_0 - x_{-1})|.$$

Case 4 $0 \leq x_0 < x_{-1}$; in this case, assuming that $\{x_n\}$ is not monotonically decreasing to zero, there is $k \geq 1$ such that $x_k < 0 \leq x_{k-1}$ as in Case 3. Hence, with $p \geq k$, and $z_1 = x_{k+1}/x_k$ we have

$$|x_p| \leq |x_k|(c + a)^p \leq K|cx_{k-1} + f(x_{k-1} - x_{k-2})| \tag{7}$$

Further, for $1 \leq j \leq k - 1$ because $tf(t) \geq 0$, we have

$$x_j = cx_{j-1} + f(x_{j-1} - x_{j-2}) \leq cx_{j-1} \leq c^2 x_{j-2} \leq \cdots \leq c^j x_0$$

Therefore, the right hand side of Eq. (7) approaches zero as $x_0, x_{-1} \to 0$, as desired.

Case 5 $0 \geq x_0 > x_{-1}$; this case is handled similarly to Case 4, using Case 2.

Stability of the origin now follows from Cases 1 to 5 and the proof of the theorem is complete. □

COROLLARY 3 *If $tf(t) \geq 0$ and $|f(t)| \leq a|t|$ for all t where*

$$0 < a < \max\{1 - c, d\}$$

(e.g. if $a \leq 1/2$) then the origin is globally asymptotically stable.

The statement of Corollary 3 may be compared with that of Corollary 1. Next, we explore the effects of the relationship between $f(t)$ and the linear mapping *at* on the behavior of the solutions of Eq. (1).

THEOREM 2 *Let $b = (1 - \sqrt{1 - c})^2$. Then, the following are true:*

(a) If $\beta \geq \alpha > b$ and $\alpha|t| \leq |f(t)| \leq \beta|t|$ for all t, then every non-zero solution of Eq. (1) oscillates about the origin.

(b) If $tf(t) \geq 0$ and $|f(t)| \leq a|t|$ for some positive $a \leq b$ and all t, then every solution of Eq. (1) is eventually monotonic.

Proof (a) Assume on the contrary that Eq. (1) has a non-zero solution $\{x_n\}$ that is eventually monotonic. Then, there is $k \geq 1$ such that either (i) $x_k \geq x_{k+1} \geq \cdots$ or (ii) $x_k \leq x_{k+1} \leq \cdots$. We consider (i), since the arguments for (ii) will be similar. First, suppose that $x_n \geq 0$ for all $n \geq k$. We may assume that $x_n > 0$ for all $n \geq k$. For otherwise, $x_n = 0$ for $n \geq l$ and some least $l \geq k$. In particular, this implies that $f(-x_{l-1}) = 0$ with $x_{l-1} \neq 0$ which is not possible if $|f(t)| \geq \alpha|t|$. So we may define

$$r_j = \frac{x_{j+k}}{x_{j+k-1}}, \quad j = 1, 2, \ldots$$

Note that since $|f(t)| \geq \alpha|t|$ we have $tf(t) \geq 0$ and thus

$$r_{j+1} = c + \frac{f(x_{j+k} - x_{j+k-1})}{x_{j+k}} \leq c + \frac{\alpha(x_{j+k} - x_{j+k-1})}{x_{j+k}} = c + \alpha - \frac{\alpha}{r_j}. \tag{8}$$

If $\phi_\alpha(t) = c + \alpha - \alpha/t$ then ϕ_α is an increasing function of t for $t > 0$, so that by Eq. (8)

$$r_{j+1} \leq \phi_\alpha(r_j) \leq \phi_\alpha(\phi_\alpha(r_{j-1})) = \phi_\alpha^2(r_{j-1}) \leq \cdots \leq \phi_\alpha^j(r_1)$$

But since $\alpha > b$, it is easy to see that ϕ_α has no fixed points, which implies that $\phi_\alpha^l(r_1) \leq 0$ for some $l \geq 1$. But then we obtain the contradiction $r_{l+1} \leq 0$.

Therefore, in Case (i) above it must be that there is $n \geq k$ such that $x_n < 0$. Without loss of generality, we may assume that $x_k < 0$. Again defining r_j as above, we see that $r_j > 0$ for all j and also that

$$r_{j+1} \leq c + \beta - \frac{\beta}{r_j} = \phi_\beta(r_j).$$

The above inequality follows because $x_{j+k} < 0$ for all j and $|f(t)| \leq \beta|t|$. Now an argument similar to the above for α implies a contradiction in this case also. We conclude that no non-zero solution can be eventually monotonic.

(b) The inequality $a \leq b$ is equivalent to $(a+c)^2 \geq 4a$ so that

$$p = \frac{a + c - \sqrt{(a+c)^2 - 4a}}{2}$$

is a (positive) real number. In fact, it is easy to verify that p is a fixed point of the mapping $\phi_a(t) = c + a - a/t$ and $p < 1$. First, suppose that the initial values x_{-1}, x_0 are such that

$$0 \leq px_{-1} \leq x_0 \leq x_{-1} \tag{9}$$

and let $\{x_n\}$ be the solution of Eq. (1) that is generated by these initial values. If $x_n > 0$ for all n, then by Lemma 3 $\{x_n\}$ is monotonic (in fact, decreasing). Otherwise, there is an integer $m \geq 1$ such that $x_m \leq 0$ but $x_{m-1} > 0$. It follows that $x_{-1} \geq x_0 > \cdots > x_m$ and

$$0 \geq x_m = cx_{m-1} + f(x_{m-1} - x_{m-2}) \geq cx_{m-1} + a(x_{m-1} - x_{m-2})$$

so that

$$\frac{x_{m-1}}{x_{m-2}} \leq \frac{a}{a+c}. \tag{10}$$

Further,

$$\frac{a}{a+c} \geq \frac{x_{m-1}}{x_{m-2}} = c + \frac{f(x_{m-2} - x_{m-3})}{x_{m-2}} \geq c + a - \frac{ax_{m-3}}{x_{m-2}} = \phi_a\left(\frac{x_{m-2}}{x_{m-3}}\right)$$

Since ϕ_a and its inverse ϕ_a^{-1} are both increasing functions, the above inequality yields

$$\frac{x_{m-2}}{x_{m-3}} \leq \phi_a^{-1}\left(\frac{a}{a+c}\right). \tag{11}$$

Note that

$$\phi_a^{-1}(s) = \frac{a}{a+c-s}, \quad 0 \leq s < a+c$$

so the right hand side of Eq. (10) equals $\phi_a^{-1}(0)$. Hence, we may express the right hand side of Eq. (11) as $\phi_a^{-2}(0)$. This pattern continues inductively, since if for any $n = 2,\ldots,m$ we have

$$\frac{x_{m-n+1}}{x_{m-n}} \leq \phi_a^{-(n-1)}(0)$$

then

$$\phi_a^{-(n-1)}(0)x_{n-m} \geq cx_{m-n} + f(x_{m-n} - x_{m-n-1}) \geq cx_{m-n} + a(x_{m-n} - x_{m-n-1})$$

which yields

$$\frac{x_{m-n}}{x_{m-n-1}} \leq \frac{a}{a+c-\phi_a^{-(n-1)}(0)} = \phi_a^{-n}(0).$$

In particular, for $n = m$, we obtain

$$x_0 \leq \phi_a^{-m}(0)x_{-1}.$$

But, it is easy to see that the real sequence $\{\phi_a^{-n}(0)\}$ is strictly increasing towards p, which is also a fixed point of ϕ_a^{-1}. Therefore, $\phi_a^{-m}(0) < p$ and we obtain $x_0 \leq \phi_a^{-m}(0)x_{-1} < px_{-1}$ which contradicts Eq. (9).

A similar argument shows that if $x_{-1} \leq x_0 \leq px_{-1} < 0$, then $x_n < 0$ for all $n \geq 1$ and thus again the solution is monotonic (strictly increasing, in fact).

In the general case, the sequence starting from an arbitrary pair of initial values x_{-1}, x_0 if not monotonic, will have a term x_k which is either a positive maximum, i.e. $x_k \geq 0, x_{k-1}, x_{k+1}$ or a negative minimum, i.e. $x_k \leq 0, x_{k-1}, x_{k+1}$. Consider the positive maximum case. Then,

$$x_k \geq x_{k+1} = cx_k + f(x_k - x_{k-1}) \geq cx_k. \tag{12}$$

Now, if $a \leq b$ then $c \geq 2\sqrt{a} - a > a$ since $0 < a < 1$. Thus,

$$c - p = \frac{c - a + \sqrt{(a+c)^2 - 4a}}{2} > 0$$

i.e. $c > p$. From Eq. (12) it follows that $x_k \geq x_{k+1} > px_k > 0$ at which point the same argument as that given above for Eq. (9) applies and establishes that the solution $\{x_n\}$ is monotonically decreasing for $n > k$. A similar argument for the negative minimum case finally completes the proof. □

According to Theorem 2, the solutions of Eq. (1) tend to be oscillatory for c close to zero, but when c is close to 1 the solutions tend to be monotonic. The next corollary improves the conclusion of Corollary 3 for values of c near 1.

COROLLARY 4 *If $tf(t) \geq 0$ and $|f(t)| \leq a|t|$ for all t where*

$$0 < a < \max\{b,\ 1-c,\ d\}$$

then the origin is globally asymptotically stable.

The results of this section address Conjecture 2 in Ref. [8]. Though not fully settling that conjecture, the inequalities on a in Corollary 4 cover much of the unit square in the (c, a) parameter space and under a weaker restriction on f than in Ref. [8], i.e. $tf(t) \geq 0$ instead of f being non-decreasing. Based on numerical simulations, we conjecture that the rest of the unit square will also work out; i.e. the following stronger form of Conjecture 2 in Ref. [8] seems to be true:

CONJECTURE *If $tf(t) \geq 0$ and $|f(t)| \leq a|t|$ for all t where $0 < a < 1$, then the origin is globally asymptotically stable in Eq. (1).*

BOUNDEDNESS OF SOLUTIONS

The conditions in Theorem 2(a) imply oscillatory behavior for all solutions of Eq. (1). These oscillations may be bounded or unbounded. We now turn to the question of the boundedness of solutions. We say that Eq. (1) has an absorbing interval $[a, b]$ if for every set x_0, x_{-1} of initial values, the corresponding solution $\{x_n\}$ is eventually in $[a, b]$; that is, there is a positive integer $N = N(x_0, x_{-1})$ such that $x_n \in [a, b]$ for all $n \geq N$. Note that an absorbing interval is compact by definition. Clearly, if Eq. (1) has an absorbing interval, then every solution of Eq. (1) is bounded; however, the converse is not true (consider a linear map that has an eigenvalue of unit magnitude [9]).

The next theorem uses the method of variation of constants (or parameters [1]) to obtain a sufficient condition for the existence of an absorbing interval.

THEOREM 3 *Assume that there are constants $B > 0$ and $a \in [0, 1)$ such that*

$$|f(t) - at| \leq B \quad \text{for all } t. \tag{13}$$

Then Eq. (1) has an absorbing interval.

Proof Assume that Eq. (13) holds and let $\{y_n\}$ be any solution of Eq. (1). We show that there is a constant $M > 0$ independent of y_0, y_{-1} such that $|y_n| \leq M$ for all sufficiently large $n \geq 1$; in particular, $[-M, M]$ is an absorbing interval. To this end, define

$$r_n = f(y_n - y_{n-1}) - a(y_n - y_{n-1})$$

and note that $|r_n| \leq B$ for all n. Next, consider the non-homogeneous linear equation

$$x_{n+1} - (a+c)x_n + ax_{n-1} = r_n \tag{14}$$

and note that $\{y_n\}$ is a solution of Eq. (14). Hence,

$$y_n = x_n^{(h)} + x_n^{(p)}$$

where $\{x_n^{(h)}\}$ is the solution of the homogeneous equation corresponding to Eq. (14) and $\{x_n^{(p)}\}$ is a particular solution of Eq. (14). There are two possible cases:

Case 1 $(a+c)^2 - 4a \neq 0$; in this case

$$x_n^{(h)} = C_1\lambda_1^n + C_2\lambda_2^n$$

where the constants C_1, C_2 are determined by the initial conditions y_0, y_{-1} and $\lambda_1 \neq \lambda_2$ are the eigenvalues (real or complex)

$$\lambda_{1,2} = \frac{1}{2}\left[a + c \pm \sqrt{(a+c)^2 - 4a}\,\right].$$

Further, using the method of variation of constants, the particular solution $\{x_n^{(p)}\}$ is found to be

$$x_n^{(p)} = \frac{1}{\lambda_2 - \lambda_1}\sum_{j=0}^{n-1} r_{n-(j+1)}(\lambda_2^j - \lambda_1^j).$$

It follows that for all n

$$|y_n| \leq |C_1\lambda_1|^n + |C_2\lambda_2|^n + \frac{B}{|\lambda_2 - \lambda_1|}\left[\sum_{j=0}^{n-1}|\lambda_2|^j + \sum_{j=0}^{n-1}|\lambda_1|^j\right].$$

Hence, for all sufficiently large n

$$|y_n| \leq 1 + \frac{B}{|\lambda_2 - \lambda_1|}\left[\sum_{j=0}^{\infty}|\lambda_2|^j + \sum_{j=0}^{\infty}|\lambda_1|^j\right] = 1 + \frac{B}{|\lambda_2 - \lambda_1|(1-|\lambda_2|)(1-|\lambda_1|)} = M.$$

Case 2 $(a+c)^2 - 4a = 0$; in this case

$$x_n^{(h)} = C_1\lambda^n + C_2 n\lambda^n, \quad \lambda = \frac{a+c}{2} \in (0,\,1)$$

and

$$x_n^{(p)} = \sum_{k=0}^{n-1} r_k(n-k-1)\lambda^{n-k-2} = \sum_{j=1}^{n-1} r_{n-j-1}\, j\lambda^{j-1}.$$

Hence,

$$|y_n| \leq |C_1|\lambda^n + |C_2|n\lambda^n + B\sum_{j=1}^{n-1} j\lambda^j$$

Thus, for large enough n we have

$$|y_n| \leq 1 + B\sum_{k=1}^{\infty} j\lambda^j = 1 + B\frac{d}{d|\lambda|}\sum_{k=0}^{\infty}\lambda^j = 1 + \frac{B}{(1-|\lambda|)^2} = M.$$

This completes the proof. □

Theorem 3 complements a boundedness result in Ref. [6] and addresses Conjecture 1 in Ref. [8], proving it for a large class of functions f. Though not all functions allowed by that conjecture are permitted in Theorem 3, the latter covers functions that are not considered in Ref. [8] (indeed, in Theorem 3 it is not even necessary that f be continuous).

We close with an example that illustrates various uses of the preceding results and also suggests new directions for substantial work of a different nature on Eq. (1).

Example In Eq. (1), set

$$f(t) = \alpha t + \frac{\alpha t}{1 + |t|^p}, \quad \alpha > 0, \ p > 1.$$

The function $f(t)$ converges to the linear map αt for large $|t|$; in particular, f neither has a lower bound nor an upper bound. However, it is not hard to see that

$$|f(t) - \alpha t| = \frac{\alpha |t|}{1 + |t|^p} < \alpha.$$

Therefore, by Theorem 3 every solution of Eq. (1) is bounded when $\alpha < 1$. Further, since for all t

$$\alpha |t| < |f(t)| \leq 2\alpha |t|$$

by Theorem 2(a) every solution of Eq. (1) is oscillatory if $\alpha \geq b$. Also, by Corollary 4, the condition

$$\alpha < \frac{1}{2} \max\{b, 1 - c, d\}$$

ensures that the origin is globally asymptotically stable. Note that $f'(0) = 2\alpha$, so if $\alpha > 1/2$ then the origin is unstable (this may be inferred from the magnitudes of the eigenvalues of the linearization of Eq. (1) [9]). We can say more: By Corollary 3 and Theorem 2(a), the conditions

$$b \leq \alpha < \frac{1}{2} \max\{1 - c, d\}$$

imply that every solution of Eq. (1) converges to zero in an oscillatory fashion (these conditions generally require that c not be close to 1). Finally, Theorem 2(b) ensures the monotonic convergence of all solutions of Eq. (1) to zero if $\alpha \leq b/2$.

We also note that the function f of this example is non-decreasing if $p \leq 3 + 2\sqrt{2}$. In this case, numerical simulations and computer-generated, approximate bifurcation diagrams with α as the changing parameter indicate that all solutions of Eq. (1) are either periodic or almost periodic for various fixed values of c, p. When $p > 3 + 2\sqrt{2}$ (e.g. $p \geq 6$) and f is no longer monotonic, then substantial changes occur in the asymptotic behavior of the solutions of Eq. (1). More complicated forms of behavior seem to occur that for $\alpha > 1/2$ range from periodic to chaotic.

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