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Dynamics of rational difference equations containing quadratic terms

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Second order rational difference equations with quadratic terms in their numerators and linear terms in their denominators exhibit a rich variety of dynamic behaviors. It is demonstrated that depending on the parameters and initial values, there can be globally attracting fixed points, coexisting periodic solutions or chaotic trajectories.

Keywords: Rational; Quadratic; Global attractivity; Periodicity; Chaos; Semiconjugate

AMS Subject Classification: 39A10; 39A11

1. Introduction

Consider the second order, rational difference equation

$$x_{n+1} = \frac{Ax_n^2 + Bx_nx_{n-1} + Cx_{n-1}^2 + Dx_n + Ex_{n-1} + F}{\alpha x_n + \beta x_{n-1} + \gamma}. \tag{1}$$

If $A = B = C = 0$, then both the numerator and the denominator are linear functions. The linear/linear case has been studied extensively [2,4,6,7,10–12]. The last three references list additional papers in this area. As a result of these studies much is understood about the behavior of this interesting class of nonlinear difference equations and their various applications. By contrast, no systematic study of rational equations containing quadratic terms has been conducted, although a few special instances of such equations have been considered previously. An equation of type (1) that is related to the Fibonacci numbers through the secant method for estimating the solutions of certain quadratic equations has been studied [10, p. 174] and the references cited therein. In Ref. [3] (or see [10, p. 155]) a rational equation which is not of type (1) but falls into the more general quadratic category that contains (1) has been studied.

Equations of type (1) also include rational equations that are the sum of a linear equation and a linear/linear rational equation, i.e.

$$x_{n+1} = ax_n + bx_{n-1} + \frac{qx_n + rx_{n-1} + s}{\alpha x_n + \beta x_{n-1} + \gamma} \tag{2}$$

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Equation (1) includes a large number of special cases that can be obtained by fixing some of the parameters or coefficients (in particular, by setting some equal to zero). Some of these cases are trivial and some others either reduce to the linear/linear case or to a linear case. In this paper we consider a few non-trivial cases involving at least one quadratic term in the numerator of (1). We study the global attractivity of a positive fixed point or equilibrium, the occurrence of periodic solutions and give conditions for the occurrence of chaotic behavior (in the sense of Li and Yorke [13]; also see [14]). Some of these types of behavior (e.g. multiple coexisting periods or chaos) are not possible for linear equations and are also not seen in linear/linear equations. A semiconjugate relation that links (1) to a first order rational equation facilitates some of our calculations.

2. Global attractivity

The following general result which involves coordinate-wise monotonicity is from Ref. [9] (but also see Refs. [8,10,12,15]). We use it here to give sufficient conditions for the global attractivity of the positive fixed point.

LEMMA 1. *Let I be an open interval of real numbers and suppose that $f \in C(I^m, \mathbb{R})$ is nondecreasing in each coordinate. Let $\bar{x} \in I$ be a fixed point of the difference equation*

$$x_{n+1} = f(x_n, x_{n-1}, \ldots, x_{n-m+1}) \tag{3}$$

and assume that the function $h(t) = f(t, \ldots, t)$ satisfies the conditions

$$h(t) > t \ \text{ if } \ t < \bar{x} \quad \text{and} \quad h(t) < t \ \text{ if } \ t > \bar{x}, \quad t \in I. \tag{4}$$

Then I is an invariant interval of (3) *and $\bar{x}$ attracts all solutions with initial values in I.*

THEOREM 1. *Assume that all parameters in* (1) *are non-negative and satisfy the following conditions:*

$$\alpha C \leq \beta B, \ \ \beta A \leq \alpha B, \ \ \alpha F \leq \gamma D, \ \ \beta F \leq \gamma E, \ \ |\beta D - \alpha E| \leq \gamma B \tag{5}$$

$$0 \leq A + B + C < \alpha + \beta, \quad \gamma \leq D + E \ \text{ with } \ F > 0 \ \text{ if } \ \gamma = D + E. \tag{6}$$

Then (1) *has a unique fixed point $\bar{x} > 0$ that attracts all positive solutions.*

Proof. We first show that if the inequalities (5) hold then the function

$$f(u,v) = \frac{Au^2 + Buv + Cv^2 + Du + Ev + F}{\alpha u + \beta v + \gamma} \tag{7}$$

is nondecreasing in each of its two coordinates u,v. This is demonstrated by computing the partial derivatives f_u and f_v and setting $f_u \geq 0$ and $f_v \geq 0$. By direct calculation $f_u \geq 0$ iff

$$\alpha Au^2 + 2\beta Auv + 2\gamma Au + (\beta B - \alpha C)v^2 + (\gamma B + \beta D - \alpha E)v + \gamma D - \alpha F \geq 0.$$

The above inequality holds for all $u, v > 0$ if

$$\alpha C \leq \beta B, \quad \gamma B + \beta D - \alpha E \geq 0, \quad \alpha F \leq \gamma D. \tag{8}$$

Similarly, $f_v \geq 0$ iff

$$\beta C v^2 + 2\alpha C u v + 2\gamma C v + (\alpha B - \beta A)u^2 + (\gamma B + \alpha E - \beta D)u + \gamma E - \beta F \geq 0$$

which is true for all $u,v > 0$ if

$$\beta A \leq \alpha B, \quad \gamma B + \alpha E - \beta D \geq 0, \quad \beta F \leq \gamma E \tag{9}$$

The middle inequalities in (8) and (9) combine into the single inequality $|\beta D - \alpha E| \leq \gamma B$. Therefore, (5) gives sufficient conditions for f to be nondecreasing in each of its coordinates.

Next, assume that (6) holds and define

$$a = \frac{A+B+C}{\alpha+\beta}, \quad b = \frac{D+E}{\alpha+\beta}, \quad c = \frac{F}{\alpha+\beta}, \quad d = \frac{\gamma}{\alpha+\beta}.$$

Then the function h in (4) takes the form

$$h(t) = \frac{at^2 + bt + c}{t+d}.$$

Now $\bar{x}$ is a fixed point of (1) if and only if $\bar{x}$ is a solution of the equation $h(t) = t$. This is equivalent to the quadratic

$$(1-a)t^2 - (b-d)t - c = 0. \tag{10}$$

Since by (6) $a < 1$ and $d \leq b$ with $c > 0$ if $d = b$, a unique positive fixed point is obtained as

$$\bar{x} = \frac{b - d + \sqrt{(b-d)^2 + 4(1-a)c}}{2(1-a)}; \quad \text{or multiplying by } \frac{\alpha+\beta}{\alpha+\beta}$$
$$\bar{x} = \frac{D + E - \gamma + \sqrt{(D+E-\gamma)^2 + 4(\alpha+\beta-A-B-C)F}}{2(\alpha+\beta-A-B-C)}. \tag{11}$$

Next, we verify that conditions (4) hold. Note that h may be written as

$$h(t) = \phi(t)t, \quad \text{where} \quad \phi(t) = \frac{at + b + c/t}{t+d}.$$

Since

$$\phi'(t) = \frac{ad - b - (c/t)(2 + d/t)}{(t+d)^2}$$

and by (6) $ad - b < d - b \leq 0$, it follows that ϕ is decreasing (strictly) for all t. Therefore, with $\phi(\bar{x}) = h(\bar{x})/\bar{x} = 1$ we find that

$$t < \bar{x} \text{ implies } h(t) > \phi(\bar{x})t = t,$$

$$t > \bar{x} \text{ implies } h(t) < \phi(\bar{x})t = t.$$

Now we may use Lemma 1 to complete the proof. □

Remark 1. The number of inequalities in Theorem 1 is partly necessitated by the 8 parameters in (1). Not all of the conditions in Theorem 1 are necessary; for example, Theorem 1 does not apply if $F > 0$ and $\gamma = 0$ in (1) since the condition $\alpha + \beta > 0$ together with the 3rd and the 4th inequalities in (5) imply $F = 0$ when $\gamma = 0$. However, for certain values of the other 6 parameters, numerical investigations indicate that a globally attracting positive equilibrium exists if $F > 0$ and $\gamma = 0$ in (1). The following result goes a step in that direction, again by using Lemma 1. In the next section this case is revisited using a different set of tools (see Theorem 4 and Remark 4).

THEOREM 2. *Assume that $D = E = \gamma = 0$ in* (1) *with other parameters non-negative and satisfying the following conditions:*

$$\alpha + \beta > A + B + C, \ \ 2A + B > \alpha, \ \ 2C + B > \beta, \ \ F > 0 \tag{12}$$

$$\beta A \leq \alpha B, \ \ \alpha C \leq \beta B. \tag{13}$$

(a) *If* (12) *holds then there is a unique positive equilibrium $\bar{x}$ and $0 < \delta < \bar{x}$ such that $\bar{x}$ attracts all solutions of* (1) *with initial values in the interval $(\bar{x} - \delta, \bar{x} + \delta)$.*
(b) *If both* (12) *and* (13) *hold then $\bar{x}$ attracts all solutions of* (1) *with initial values in* $(\bar{x} - \delta, \infty)$.

Proof.

(a) Let f and h be the two functions in the proof of Theorem 1. Then the fixed point of h is the unique positive solution $\bar{x}$ of

$$t = h(t) = \frac{at^2 + c}{t} \quad \text{with} \quad a = \frac{A + B + C}{\alpha + \beta} < 1, \quad c = \frac{F}{\alpha + \beta} > 0.$$

That is,

$$\bar{x} = \sqrt{\frac{c}{1 - a}} = \sqrt{\frac{F}{\alpha + \beta - A - B - C}}. \tag{14}$$

Next, note that $f_u(\bar{x}, \bar{x}) > 0$ iff

$$A\alpha\bar{x}^2 + 2A\beta\bar{x}^2 + (B\beta - C\alpha)\bar{x}^2 - F\alpha > 0. \tag{15}$$

With $\bar{x}$ as in (14), inequality (15) is equivalent to

$$\frac{(A\alpha + 2A\beta + B\beta - C\alpha)F}{\alpha + \beta - A - B - C} > F\alpha \quad \alpha(2A + B) + \beta(2A + B) > \alpha(\alpha + \beta)$$

or equivalently, $2A + B > \alpha$. Similarly, $f_v(\bar{x}, \bar{x}) > 0$ iff

$$(B\alpha - A\beta)\bar{x}^2 + 2C\alpha\bar{x}^2 + C\beta\bar{x}^2 - F\beta > 0 \tag{16}$$

and (16) is equivalent to $2C + B > \beta$. Since f_u and f_v are continuous functions on $(0, \infty)^2$ we conclude that there is $\delta > 0$ such that

$$f_u(u, v), f_v(u, v) > 0 \quad \text{for all} \quad u, v \in (\bar{x} - \delta, \bar{x} + \delta).$$

Next,

$$h(t) - t = -(1-a)t + \frac{c}{t} < 0 \text{ if } t > \sqrt{c/(1-a)} = \bar{x}$$

$$h(t) - t = -(1-a)t + \frac{c}{t} > 0 \text{ if } t < \sqrt{c/(1-a)} = \bar{x}$$

so that (4) in Lemma 1 is satisfied on (0,∞) and that lemma then completes the proof.

(b) Suppose that both (12) and (13) hold. Then for all $u, v > \bar{x}$,

$$\begin{aligned} f_u(u,v) &= A\alpha u^2 + 2A\beta uv + (B\beta - C\alpha)v^2 - F\alpha \\ &\geq A\alpha\bar{x}^2 + 2A\beta\bar{x}^2 + (B\beta - C\alpha)\bar{x}^2 - F\alpha > 0 \end{aligned}$$

and

$$\begin{aligned} f_v(u,v) &= (B\alpha - A\beta)u^2 + 2C\alpha uv + C\beta v^2 - F\beta \\ &\geq (B\alpha - A\beta)\bar{x}^2 + 2C\alpha\bar{x}^2 + C\beta\bar{x}^2 - F\beta > 0. \end{aligned}$$

Hence the arguments in the proof of Part (a) apply to the interval $(\bar{x} - \delta, \infty)$ and Lemma 1 once again completes the proof □.

When $F = 0$ and $\gamma > 0$ in (1) then the origin is a fixed point of (1) and the class of solutions can be expanded to include the non-negative solutions. The next theorem gives sufficient conditions for the global asymptotic stability of the origin in this case. The proof is based on the following result from Ref. [10, Theorem 4.3.1] that for convenience we quote as a lemma.

LEMMA 2. *Let $\bar{x}$ be a fixed point of the difference equation* (3) *in a closed, invariant set $T \subset \mathbb{R}^m$ and define*

$$M = \{(u_1, \ldots, u_m) : |f(u_1, \ldots, u_m) - \bar{x}| < \max\{|u_1 - \bar{x}|, \ldots, |u_m - \bar{x}|\} \cup \{(\bar{x}, \ldots, \bar{x})\}.$$

Then $(\bar{x}, \ldots, \bar{x})$ is asymptotically stable relative to the largest invariant subset S of $M \cap T$ such that S is closed in T.

THEOREM 3 (GLOBAL ASYMPTOTIC STABILITY OF THE ORIGIN). *Assume that $F = 0$ in* (1) *with all other parameters non-negative. If the following conditions are satisfied for some $\delta \in [0, 1]$:*

$$A + \delta B \leq \alpha, \quad C + (1-\delta)B \leq \beta, \quad D + E < \gamma \tag{17}$$

then the origin is a fixed point of (1) *that is asymptotically stable relative to* $[0, \infty)^2$.

Proof. The origin is a fixed point of (1) because $F = 0$ and also by (17) $\gamma > 0$. If $f(u,v)$ is given by (7) then for all $u, v \geq 0$

$$\begin{aligned} f(u,v) &\leq \frac{Au + B\min\{u,v\} + Cv + D + E}{\alpha u + \beta v + \gamma} \ \max\{u,v\} \\ &\leq \frac{(A + \delta B)u + [C + (1-\delta)B]v + D + E}{\alpha u + \beta v + \gamma} \ \max\{u,v\} \\ &< \max\{u,v\}, \quad \text{if}(u,v) \neq (0,0) \end{aligned}$$

where the strict inequality holds because of conditions (17). Now we apply Lemma 2 with $M = T = [0,\infty)^2$ to conclude the proof. □

3. Limit cycles and chaos

In the proof of Theorem 1 it is seen that the restriction of the function $f(u,v)$ in (7) to the diagonal $u = v$, namely the function h is again a function of the *same type*; i.e. both f and h have a quadratic numerator and a linear denominator. The monotonicity conditions of Lemma 1 ensured that global attractivity for h could be extended to f. Since the dynamics of the one dimensional map are easier to analyze, it is natural to wonder whether there are conditions that allow us to extend other properties of the one dimensional map (e.g. the existence of limit cycles or the occurrence of chaos) to the higher order case. In this section we discuss one such set of conditions using semiconjugacy [15].

First we consider a basic yet important consequence of having the 2nd degree terms in (1). The presence of these terms allows $(0,\infty)$ to be invariant even when some coefficients or parameters in (1) are negative. It is seen later in this paper that negative coefficients are often associated with complex trajectories, so the added flexibility in the next result is significant.

Lemma 3. *Let $A, C, F, \alpha, \beta, \gamma \geq 0$ in (1) with $A + C, \alpha + \beta > 0$.*

(a) *If $x_0, x_{-1} > 0$ and any one of the following conditions holds, then $x_n > 0$ for all $n \geq 1$:*

$$D, E \geq 0, \quad B > -2\sqrt{AC}, \tag{18}$$

$$B, E \geq 0, \quad D > -2\sqrt{AF}, \tag{19}$$

$$B, D \geq 0, \quad E > -2\sqrt{CF}. \tag{20}$$

When two out of three coefficients can be negative then the positivity conditions are:

$$B > -2\sqrt{AC}, \quad D > -2\sqrt{AF} \ \text{ and } \ E \geq 2\sqrt{CF}, \tag{21}$$

$$B > -2\sqrt{AC}, \quad E > -2\sqrt{CF} \ \text{ and } \ D \geq 2\sqrt{AF}, \tag{22}$$

$$D > -2\sqrt{AF}, \quad E > -2\sqrt{CF} \ \text{ and } \ B \geq 2\sqrt{AC}. \tag{23}$$

(b) *Let $\gamma > 0$. Then (a) holds with all strict $>$ changed to $\geq$.*

Proof.

(a) Suppose that (18) holds. Then setting $n = 0$ in (1) gives

$$x_1 \geq \frac{Ax_0^2 + Bx_0x_{-1} + Cx_{-1}^2}{\alpha x_0 + \beta x_{-1} + \gamma} > \frac{\left(\sqrt{A}x_0 - \sqrt{C}x_{-1}\right)^2}{\alpha x_0 + \beta x_{-1} + \gamma} \geq 0.$$

Hence $x_1 > 0$. By induction, it follows that $x_n > 0$ for all n. Next, if (19) holds then

$$x_1 \geq \frac{Ax_0^2 + Dx_0 + F}{\alpha x_0 + \beta x_{-1} + \gamma} > \frac{\left(\sqrt{A}x_0 - \sqrt{F}\right)^2}{\alpha x_0 + \beta x_{-1} + \gamma} \geq 0.$$

Again by induction $x_n > 0$ for all n. The proof is similar if (20) holds.
Now let (21) hold. Then

$$x_1 > \frac{Ax_0^2 - 2\sqrt{AC}x_0x_{-1} + Cx_{-1}^2 - 2\sqrt{AF}x_0 + 2\sqrt{CF}x_{-1} + F}{\alpha x_0 + \beta x_{-1} + \gamma}$$

$$= \frac{\left(-\sqrt{A}x_0 + \sqrt{C}x_{-1} + \sqrt{F}\right)^2}{\alpha x_0 + \beta x_{-1} + \gamma} \geq 0.$$

Hence, by induction $x_n > 0$ for all n. For (22) and (23) a similar argument applies with only minor modifications.

(b) This is clear since with $\gamma > 0$ equation (1) is well-defined for non-negative solutions. □

Now we define the one-dimensional, quadratic/linear rational map

$$g(t) = \frac{pt^2 + qt + s}{t + w}$$

and consider the difference equation

$$x_{n+1} = cx_n + f(x_n - cx_{n-1}), \quad |c| < 1. \tag{24}$$

Using $f = g$ in (24) and rearranging terms we obtain a difference equation of type (1) with $\alpha > 0$ and

$$\begin{aligned} &A = (p+c)\alpha,\ \ B = -c(2p+c)\alpha,\ \ C = pc^2\alpha,\ \ D = (q+wc)\alpha, \\ &E = -qc\alpha,\ \ F = s\alpha,\ \ \beta = -c\alpha,\ \ \gamma = w\alpha \end{aligned} \tag{25}$$

To reduce the amount of calculations without losing the variety of dynamic behaviors that we explore, we set $w = 0$ and study the following difference equation:

$$x_{n+1} = \frac{(p+c)x_n^2 - c(2p+c)x_nx_{n-1} + pc^2x_{n-1}^2 + qx_n - qcx_{n-1} + s}{x_n - cx_{n-1}}, \quad |c| < 1 \tag{26}$$

with initial values x_0, x_{-1} satisfying

$$x_0 > cx_{-1}. \tag{27}$$

Note that if $c \leq 0$ then in particular every pair of positive initial values satisfies (27); further, if $p + c > 0$ and $q \geq 0$ as well, then all of the coefficients are non-negative and therefore all solutions of (26) are positive. However, as indicated by Lemma 3 positive solutions may also occur with negative coefficients; see Remark 3 below for a sufficient condition that guarantees positive solutions when $q < 0$.

Equation (26) is equivalent to the system of first order equations

$$t_{n+1} = f(t_n) \tag{28}$$

$$x_{n+1} = cx_n + t_{n+1}. \tag{29}$$

This is an example of a triangular system. A general result on the structure of periodic solutions of such systems in terms of the periodic orbits of its two first order equations appears in Ref. [1]. Here, since the triangular system is specific in its second equation, we derive the needed relationships directly and also establish attractivity when $|c| < 1$.

Equations (28) and (29) also constitute a semiconjugate factorization of (1) under conditions (25). More precisely, the two dimensional mapping that is equivalent to (1) is said to be semiconjugate to the mapping g or to equation (28) with $f = g$ and a semiconjugate link provided by equation (29); see Ref. [15] for more background on this concept.

Remark 2. If $t_n > 0$ for all $n \geq k$ where k is a positive integer, then $x_{n+1} > cx_n$ for all $n \geq k$. This is evident from (29):

$$x_{k+1} = cx_k + t_{k+1} > cx_k.$$

This conclusion extends by induction to all points (x_n, x_{n+1}) in the plane with $n \geq k$. In particular if $f(t) > 0$ for all $t > 0$ and we start with $t_1 = x_1 - cx_0 > 0$ then the entire orbit (x_n, x_{n+1}) of (24) stays in the region $\{(u, v) : v > cu\}$.

If $\{t_n\}$ is a solution of the first order equation (28) then (29) has the following solution

$$x_n = c^n x_0 + \sum_{j=1}^{n} c^{n-j} t_j, \quad n \geq 1. \tag{30}$$

Using (30) we obtain results about the solutions of (1) based on the corresponding properties of solutions of the first order equation (28). The link between the two equations is furnished by the following result from Ref. [16] which we quote for convenience as a lemma.

Lemma 4. *Assume that* $|c| < 1$.

(a) *If for a given sequence* $\{t_n\}$ *of real numbers equation* (29) *has a solution* $\{x_n\}$ *of period pthen* $\{t_n\}$ *is periodic with period p. Conversely, if* $\{t_n\}$ *is a periodic sequence of real numbers with period p and*

$$\xi_i = \frac{1}{1 - c^p} \sum_{j=0}^{p-1} c^{p-j-1} \tau_{(i+j) \bmod p} \quad i = 0, 1, \ldots, p-1 \tag{31}$$

where $\{\tau_0, \ldots, \tau_{p-1}\}$ *is a cycle of* $\{t_n\}$ *then the solution* $\{x_n\}$ *of equation* (29) *with* $x_0 = \xi_0$ *and* $t_1 = \tau_0$ *has period p and* $\{\xi_0, \ldots, \xi_{p-1}\}$ *is a cycle of* $\{x_n\}$.

(b) *If* $\{t_n\}$ *is a periodic solution of the first order equation* (28) *with prime period p and a cycle* $\{\tau_0, \ldots, \tau_{p-1}\}$ *then* (24) *has a solution* $\{x_n\}$ *of prime period p with a cycle* $\{\xi_0, \ldots, \xi_{p-1}\}$ *given by* (31).

(c) *If f is continuous and $\{t_n\}$ is an attracting periodic solution of* (28) *then $\{x_n\}$ is an attracting periodic solution of* (24).

(d) *If the first order equation* (28) *has an invariant interval $[\mu, \nu]$ then every solution of the second order equation* (24) *is eventually contained in the planar compact, convex set:*

$$S_{\mu,\nu} = \{(x, y) : cx + \mu \le y \le cx + \nu\} \cap \left[-|c| - \frac{\max\{|\mu|, |\nu|\}}{1 - |c|}, |c| + \frac{\max\{|\mu|, |\nu|\}}{1 - |c|}\right]^2.$$

If (28) *is chaotic in $[\mu, \nu]$ (e.g. if f has a period-3 point) then* (24) *is chaotic in $S_{\mu,\nu}$. Here chaos is defined in the sense of Refs.* [13] *and* [14].

It is clear from Lemma 4 that as long as $|c| < 1$ the various properties of the first order equation (28) with $f = g$ directly lead to corresponding properties of (1) under conditions (25). The conclusions of Parts (a) and (b) in Lemma 4 hold as long as $|c| \neq 1$; however, neither Part (c) nor (d) hold if $|c| \ge 1$.

We point out that if $\bar{t}$ is a fixed point of (28) then by (31) the second order equation (24) has a corresponding fixed point

$$\bar{x} = \frac{\bar{t}}{1 - c}. \tag{32}$$

Lemma 5. *Assume that $s > 0$, $p \ge 0$ and consider*

$$g(t) = pt + q + \frac{s}{t}, \quad t > 0. \tag{33}$$

(a) *If $p > 0$ then the function g attains its global minimum value $2\sqrt{sp} + q$ on $(0, \infty)$ at $t = \sqrt{s/p}$. Hence for $p \ge 0$, $g(0, \infty) \subset (0, \infty)$ if and only if*

$$q > -2\sqrt{sp}.$$

(b) *The positive fixed points of g, when they exist, are solutions of the equation $\phi(t) = 0$ where*

$$\phi(t) = (p - 1)t^2 + qt + s.$$

There are three possible cases:

(i) *If $p = 1$ and $q < 0$ then g has a unique fixed point*

$$\bar{t} = \frac{s}{|q|} > 0.$$

(ii) *If $p < 1$ then g has a unique positive fixed point*

$$\bar{t} = \frac{q + \sqrt{q^2 + 4s(1 - p)}}{2(1 - p)}.$$

(iii) *If $p > 1$ and*

$$q \le -2\sqrt{s(p - 1)}$$

then g has up to two positive fixed points

$$\bar{t}_1 = \frac{|q| - \sqrt{q^2 - 4s(p-1)}}{2(p-1)} \leq \bar{t}_2 = \frac{|q| + \sqrt{q^2 - 4s(p-1)}}{2(p-1)}.$$

(c) *Let* $q > -2\sqrt{sp}$ *and* $g^2(t) = g(g(t))$. *Then*

$$g^2(t) = t \ \textit{iff} \ \ \phi(t)\psi(t) = 0$$

where

$$\psi(t) = [p(p+1)t^2 + q(p+1)t + sp]. \tag{34}$$

Proof.

(a) $\lim_{t\to 0^+} g(t) = \lim_{t\to\infty} g(t) = \infty$ and

$$g'(t) = p - \frac{s}{t^2}, \quad g''(t) = \frac{2s}{t^3}.$$

Hence, a global minimum $2\sqrt{sp}$ for g occurs on $(0,\infty)$ at $t = \sqrt{s/p}$.

(b) The fixed points of g are solutions of the equation $g(t) = t$, or equivalently of

$$0 = g(t) - t = tg(t) - t^2 = \phi(t).$$

Solving this quadratic equation readily gives the roots structure as claimed in (i)–(iii).

(c) The equation $g^2(t) = t$ written explicitly is

$$p^2 t + q(p+1) + \frac{sp}{t} + \frac{st}{pt^2 + qt + s} = t. \tag{35}$$

Since $pt^2 + qt + s$ has no real roots when $q > -2\sqrt{sp}$, the real solutions of (35) in $(0,\infty)$ are precisely those of

$$t(pt^2 + qt + s)g^2(t) = t^2(pt^2 + qt + s)$$

After multiplying out the terms and rearranging them, the following is obtained:

$$p(p^2 - 1)t^4 + q(p+1)(2p-1)t^3 + [q^2(p+1) + 2sp^2]t^2 - sp(2p+1)t + s^2 p = 0. \tag{36}$$

The roots of the quartic polynomial on the left hand side of (36) are precisely the solutions of the equation $g^2(t) = t$. The solutions of the latter equation obviously include the fixed points, or the roots of $\phi(t)$ so the quartic is divisible by $\phi(t)$. Dividing in the straightforward fashion yields the polynomial $\psi(t)$ as the quotient.

We now consider applications of Lemmas 3 and 4 to equation (26) with initial values (27). As an initial value problem, (26) and (27) are semiconjugate to the first order discrete initial value problem

$$t_{n+1} = pt_n + q + \frac{s}{t_n}, \quad t_0 = x_0 - cx_{-1} > 0. \tag{37}$$

The semiconjugate link is provided by the nonautonomous, linear first order equation (29). □

Remark 3.

(a) Let us check whether the results in Lemma 5 are consistent with Lemma 3 when $c < 0$ (so that $\beta > 0$). In this case, if $q < 0$ then $D = q < 0$ and $E = -cq < 0$ while all other parametes in (1) are non-negative. We now check to see if (23) in Lemma 3 holds. By Lemma 5(a) and conditions (25) when $\alpha = 1$

$$E = -cq = |c|q > -2|c|\sqrt{sp} = -2\sqrt{spc^2} = -2\sqrt{CF}.$$

Similarly, by conditions (25)

$$B \geq 2\sqrt{AC} \Leftrightarrow (2p + c)^2 \geq 4(p + c)p \Leftrightarrow c^2 \geq 0.$$

Finally,

$$-2\sqrt{AF} = -2\sqrt{(p + c)s} > -2\sqrt{sp}.$$

As it is possible that $-2\sqrt{sp} < q \leq -2\sqrt{(p + c)s}$ we conclude that one of three inequalities in (23) is not satisfied. We recall from Remark 2 that when $c < 0$ it may happen that $x_n < 0$ for some $n \geq 1$ (though $x_n > cx_{n-1}$). Of course, if $q > -2\sqrt{(p + c)s}$ and (23) is satisfied then $x_n > 0$ for all n.

(b) When $c = 0$ then (26) reduces to the first order equation (37). On the other hand, as $|c|$ approaches 1 the behavior of solutions of (26) becomes completely different from the first order case and generally, more complex (figure 1 below). Thus c, or $-\beta/\alpha$ by (25) is a focusing parameter that relates the first order equation to the second order one, with the two equations coinciding when $c = 0$; however, c does not affect the first order equation in any way.

The next result on the global attractivity of the positive fixed point supplements Theorems 1 and 2. In particular, it shows that the coefficients of the linear terms in the numerator of equation (1) can play a significant role in the behavior of solutions of (1).

THEOREM 4. *Assume that* $s > 0$, $0 \leq p \leq 1$ *and* $0 < |c| < 1$.

(a) *Let* $p < 1$ *and assume that*

$$q \geq -2p\sqrt{\frac{s}{p + 1}}. \tag{38}$$

Then the positive fixed point

$$\bar{x} = \frac{q + \sqrt{q^2 + 4s(1 - p)}}{2(1 - p)(1 - c)}$$

attracts all solutions of (26) *satisfying* (27).

(b) *Let* $p = 1$. *If*

$$-\sqrt{2s} \leq q < 0 \tag{39}$$

then all solutions of (26) *satisfying* (27) *converge to the positive fixed point*

$$\bar{x} = \frac{s}{q(c-1)}.$$

If

$$q \geq 0 \tag{40}$$

then every solution of (26) *satisfying* (27) *is unbounded.*

Proof.

(a) Because of Lemma 4 it is sufficient to prove that the fixed point $\bar{t} = \bar{x}(1-c)$ is a global attractor for (37). To show that $\bar{t}$ is attracting on $(0,\infty)$ we prove that

$$g^2(t) > t \ \text{ if } \ t < \bar{t} \ \text{ and } \ g^2(t) < t \ \text{ if } \ t > \bar{t}, \quad t > 0 \tag{41}$$

where $g^2(t) = g(g(t))$; see Ref. [15, Theorem 2.1.2]. By Lemma 5(c) the positive solutions of $g^2(t) = t$ are the positive zeros of $\phi(t)\psi(t)$. The only positive zero of $\phi(t)$ is the fixed point $\bar{t}$. Also ψ has no positive real roots; for $p > 0$ its discriminant is negative by (38):

$$q^2(p+1)^2 - 4sp^2(p+1) < 0.$$

Therefore, the only zero of $g^2(t) = t$ in $(0,\infty)$ is $\bar{t}$. From its explicit form in (35) we see that $g^2(t) \to \infty$ as $t \to 0^+$ so it must be that $g^2(t) > t$ if $t < \bar{t}$. Also, for t sufficiently large the last two terms of $g^2(t)$ are negligible and bounded above by $\varepsilon > 0$ such that $\varepsilon < (1-p^2)t - q(p+1)$ and so

$$g^2(t) \leq p^2 t + q(p+1) + \varepsilon = t - (1-p^2)t + q(p+1) + \varepsilon < t.$$

Thus (41) holds and it follows that $\bar{t}$ is a global attractor for (37). Therefore, $\bar{x}$ attracts all solutions of (26) satisfying (27).

(b) If $p = 1$ then the first inequality in (39) is (38) and the second is needed to obtain the unique fixed point $\bar{t} = -s/q$ for (37). Repeating an argument similar to the proof of Part (a) shows that (41) holds and thus $\bar{t}$ is a global attractor in this case too.

Finally, if (40) holds then $g(t) = t + q + s/t > t$ for all $t > 0$. Hence, for every $t_0 > 0$, $t_n = g^n(t_0) \to \infty$ monotonically as $n \to \infty$. Hence, (30) implies that $x_n \to \infty$. □

Remark 4. The hypotheses in Part (a) of the preceding theorem overlap those in Theorems 1 and 2. For example, using (25):

$$\alpha + \beta - A - B - C = 1 - 2c + c^2 - p(1 - 2c + c^2) = (1-p)(1-c)^2.$$

Therefore, if $c \neq 1$ then the inequality $p \leq 1$ is equivalent to $A + B + C \leq \alpha + \beta$. In particular, if $p = 1$ so that $A + B + C = \alpha + \beta$ then Theorem 4 shows that the existence of a globally attracting positive fixed point is possible only if the coefficient $D = q$ is negative. Unlike Theorems 1 and 2, negative coefficients can be allowed in Theorem 4 since the solutions of equation (26) can have negative values (also refer to Remark 3).

Theorem 4 also helps in checking the necessity of hypotheses. In relation to Theorem 2, when $q = w = 0$ (i.e. $D = E = \gamma = 0$) suppose that $c < 0$ and $-c < p$ so that the remaining coefficients in (26) are non-negative. Then (38) holds and when $p < 1$, by Theorem 4 the positive fixed point is globally attracting. Further, under conditions (25), the hypotheses in Theorem 4(a) imply only the first of inequalities (13); the second, as well as the middle two inequalities in (12) are implied if $p > (1 - c)/2$, or equivalently if $A = p + c > (1 + c)/2$. Therefore, evidently they are not necessary. On the other hand, since Theorem 4 applies to a special case of (1) where parameters are restricted, it should be viewed as supplementing, not replacing the eariler results.

THEOREM 5. *Assume that*

$$s, p > 0, \quad 0 < |c| < 1, \quad q < -2p\sqrt{\frac{s}{p+1}}. \tag{42}$$

Then equation (26) *has a positive periodic solution with prime period* 2 (*or a* 2-*cycle*)

$$\xi_0 = \frac{c\tau_0 + \tau_1}{1 - c^2}, \quad \xi_1 = \frac{c\tau_1 + \tau_0}{1 - c^2}$$

where

$$\tau_0 = \frac{|q| - \sqrt{q^2 - 4sp^2/(p+1)}}{2p}, \quad \tau_1 = \frac{|q| + \sqrt{q^2 - 4sp^2/(p+1)}}{2p}. \tag{43}$$

This 2-*cycle is asymptotically stable if in addition to* (42) *the following holds:*

$$q > -\frac{\sqrt{2sp(2p^2 + 2p + 1)}}{p+1} = -2p\sqrt{\frac{s}{p+1}}\sqrt{1 + \frac{1}{2p(p+1)}}. \tag{44}$$

Proof. Conditions (42) imply that the polynomial $\psi(t)$ in (34) has two positive roots which are computed easily as the numbers τ_0, τ_1 in (43). Since these are the non-fixed point solutions of the equation $g^2(t) = t$ it follows that $\{\tau_0, \tau_1\}$ is a 2-cycle of (37). Lemma 4(a) then gives the 2-cycle $\{\xi_0, \xi_1\}$ for (26).

By Lemma 4(c) the 2-cycle $\{\xi_0, \xi_1\}$ is attracting if $\{\tau_0, \tau_1\}$ is an attracting 2-cycle of (37); i.e. if

$$|g'(\tau_0)g'(\tau_1)| < 1.$$

See, Ref. [5]. With g given as in (33) the above inequality takes the form

$$\left|\left(p - \frac{s}{\tau_0^2}\right)\left(p - \frac{s}{\tau_1^2}\right)\right| < 1 \quad \left|p^2 + \frac{s^2}{\tau_0^2\tau_1^2} - \frac{sp(\tau_0^2 + \tau_1^2)}{\tau_0^2\tau_1^2}\right| < 1 \tag{45}$$

Note that

$$\tau_0 + \tau_1 = \frac{q}{p} \text{ and } \tau_0\tau_1 = \frac{s}{p+1} \Rightarrow \tau_0^2 + \tau_1^2 = (\tau_0 + \tau_1)^2 - 2\tau_0\tau_1 = \frac{q^2(p+1) - 2sp^2}{p^2(p+1)}.$$

Inserting these values in (45) and doing some straightforward calculations yield

$$\frac{4sp^2}{p+1} < q^2 < \frac{2sp(2p^2+2p+1)}{(p+1)^2}.$$

This completes the proof. □

Remark 5. If $s = p = 1$ then Theorems 4 and 5 indicate that a (globally) attracting fixed point $\bar{x} = 1/(c-1)q$ exists when

$$-\sqrt{2} \leq q < 0.$$

The fixed point $\bar{x}$ becomes unstable and the stable 2-cycle $\{\xi_0, \xi_1\}$ in Theorem 5 emerges over the range

$$-\sqrt{\frac{5}{2}} < q < -\sqrt{2}.$$

Thus as the bifurcation parameter q decreases and crosses $-\sqrt{2}$ a period-doubling bifurcation occurs for the second order equation (26). Then as q crosses $-\sqrt{5/2}$ a second period-doubling bifurcation destabilizes the 2-cycle and creates a stable 4-cycle. The mapping

$$g(t) = t + q + \frac{1}{t} \tag{46}$$

exhibits the usual bifurcations to higher periods that follow the Sharkovski ordering (see, e.g. [15, p. 34]) as the parameter q continues to decrease further. The requirement that g in (46) be positive puts a lower bound on q; in fact, by Lemma 5(a) it is necessary that $q > -2\sqrt{sp} = -2$.

The next result marks the emergence of a period 3 solution (through a tangent bifurcation rather than a period-doubling one).

LEMMA 6. *If $q = -\sqrt{3}$ then the function g in* (46) *has a unique set of positive period* 3 *points given by*

$$\tau_0 = \frac{2}{\sqrt{3}}\left(1 + \cos\frac{\pi}{9}\right), \quad \tau_1 = g(\tau_0), \quad \tau_2 = g(\tau_1). \tag{47}$$

Proof. A period 3 point is a solution of the equation $g^3(t) = t$, which is equivalent to

$$\frac{1}{t} + \frac{t}{t^2 - \sqrt{3}t + 1} + \frac{t(t^2 - \sqrt{3}t + 1)}{t(t - 2\sqrt{3})(t^2 - \sqrt{3}t + 1) + 2t^2 - \sqrt{3}t + 1} = 3\sqrt{3}.$$

Multiplying out and rearranging various terms, the above equation may be written as the polynomial equation

$$P(t) = 3\sqrt{3}t^7 - 39t^6 + 66\sqrt{3}t^5 - 168t^4 + 77\sqrt{3}t^3 - 57t^2 + 7\sqrt{3}t - 1 = 0.$$

By Lemma 5(b)(i) g has a unique positive fixed point at $1/\sqrt{3}$ and no negative fixed points so this is the only fixed point root of $P(t)$. Dividing we obtain

$$P(t) = 3\sqrt{3}\left(t - 1/\sqrt{3}\right)Q(t)$$

where

$$Q(t) = t^6 - 4\sqrt{3}t^5 + 18t^4 - \frac{38\sqrt{3}}{3}t^3 + 13t^2 - 2\sqrt{3}t + \frac{1}{3}.$$

Since all the fixed points are accounted for, the 6 roots of Q give two sets of period 3 points (if they are all real). These two sets are identical if Q is a perfect square, i.e.

$$Q(t) = (t^3 + \lambda t^2 + \omega t + \sigma)^2. \tag{48}$$

Indeed, by matching coefficients on both sides of (48) we find a set of numbers

$$\lambda = -2\sqrt{3}, \quad \omega = 3, \quad \sigma = -\frac{1}{\sqrt{3}}$$

for which (48) holds for all $t > 0$. Therefore,

$$P(t) = 3(\sqrt{3}t - 1)\left(t^3 - 2\sqrt{3}t^2 + 3t - \frac{1}{\sqrt{3}}\right)^2.$$

The roots of the cubic above can be found using the standard formula with radicals (see, e.g. Ref. [17]); one root using this formula is found to be

$$\tau_0 = \frac{1}{\sqrt{3}}\left(2 + \sqrt[3]{\frac{z}{2}} + \sqrt[3]{\frac{\bar{z}}{2}}\right) \quad \text{where} \quad z = 1 + i\sqrt{3} = 2\mathrm{e}^{i\pi/3}.$$

Therefore,

$$\tau_0 = \frac{1}{\sqrt{3}}\left(2 + e^{i\pi/9} + e^{-i\pi/9}\right) = \frac{1}{\sqrt{3}}\left(2 + 2\cos\frac{\pi}{9}\right)$$

as in (47).

The following theorem is an immediate consequence of Lemmas 4–6, Remark 3(a), the Sharkovski ordering and the “Chaos Theorem” of Li and Yorke; see Refs. [13–15]. □

THEOREM 6. *Let $p = s = 1$ in* (26).

(a) *If $q = -\sqrt{3}$ then* (26) *has a unique* 3-*cycle*

$$\xi_0 = \frac{c^2\tau_0 + c\tau_1 + \tau_2}{1 - c^3}, \quad \xi_1 = \frac{c^2\tau_1 + c\tau_2 + \tau_0}{1 - c^3}, \quad \xi_2 = \frac{c^2\tau_2 + c\tau_0 + \tau_1}{1 - c^3}$$

where τ_0, τ_1, τ_2 are given by (47).

(b) *If $-2 < q \le -\sqrt{3}$ then equation* (26) *has periodic solutions of all possible periods.*

(c) *For $-2 < q < -\sqrt{3}$ solutions of* (26) *exhibit chaotic behavior in the sense of Li and Yorke.*

(d) *If $q > -2\sqrt{1+c}$ then all solutions of* (26) *with positive initial values are positive; i.e.* $(0,\infty)$ *is invariant. In particular, this is the case for all $-2 < q < 0$ if $c > 0$.*

Figures 1 and 2 illustrate Theorem 6. The straight lines in figure 1 have equations

$$y = cx + \mu, \quad y = cx + \nu$$

where $\mu = g(1) = 0.2$ is the minimum value of g in (46) when $q = -1.8$ and $\nu = g(0.2) = 3.4$; refer to Lemmas 4(d) and 5(a). Figure 2 clearly shows the emergence of the 3-cycle at $q = -\sqrt{3}$ as well as the initial period-doubling bifurcations mentioned in Remark 5.

4. Future directions

The preceding discussion shows that the quadratic/linear rational difference equation (1) has a rich dynamical sturcture. However, the coverage here is far from exhaustive and many questions have been left unanswered. For equation (26) we omitted various cases here, e.g. when $p > 1$ (i.e. $A + B + C > \alpha + \beta$) or when $w > 0$ (i.e $\gamma > 0$). As Lemma 5 shows, two positive fixed points exist when $p > 1$ and this situation adds another layer of complexity to the second order equation; also when $w > 0$ the infinite discontinuity in g is shifted from the origin to w so some new situations can occur in addition to having to modify various calculations. These and similar cases can be studied in future papers.

The question is left open here as to what range of possible behaviors can occur in cases where a semiconjugate relation is either not known for (1) or is quite different from the one that resulted in (26), e.g. as in the case where D, E, F and γ are all zeros in (1). In particular, whether chaotic behavior can occur in the positive quadrant of the plane when all the coefficients in (1) are non-negative is an open question. A traditional approach could be fruitful in cases where semiconjugate relations are not known by seeking conditions under which the fixed point may be a snap-back repeller; see Ref. [14].

Going in a different direction, the quadratic/linear rational equation (1) is itself a special case of the quadratic/quadratic (or just quadratic) rational equation where both the numerator and the denominator may contain second-degree terms. Extending (2), quadratic rational difference equations include sums of two linear/linear rational difference equations.

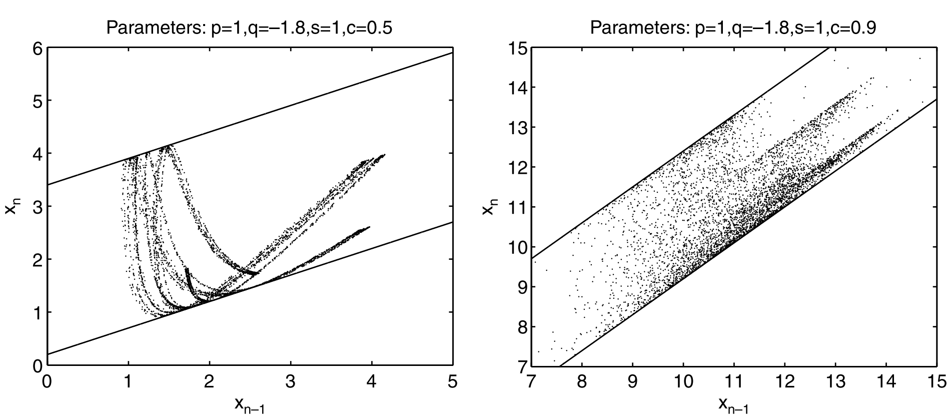

Figure 1. Orbits of equation (26).

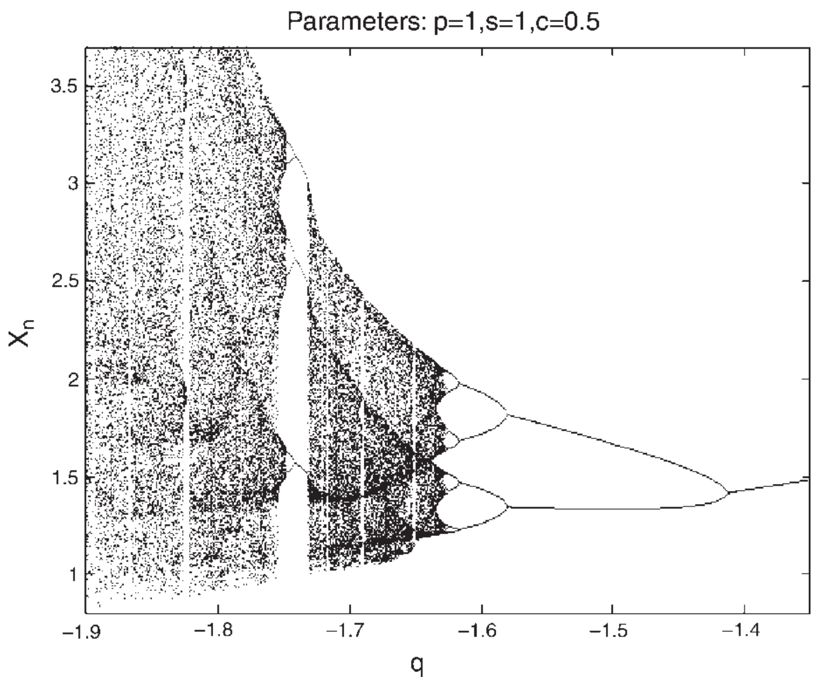

Figure 2. Bifurcation diagram for equation (26).

If we abbreviate the general case as QQR (or just QR) and refer to equation (1) as a QLR equation then naturally a LQR equation presents itself as a case worthy of consideration. LQR difference equations of certain types have been studied in the literature; see, e.g. Refs. [4,7,8]. However, like other QR equations there has been no systematic study of LQR equations.

All of these equations can be seen as extensions of the more familiar LLR (or LR) equations. From the existing literature we know that the LR type equations exhibit a more restricted variety of dynamic behavior than the QLR type and it should be interesting to learn if LQR equations possess as rich a dynamic profile as the QLR type. A QR type equation which is neither QLR nor LQR is studied in Refs. [3,10]. A long-term project could involve a systematic study of QR type equations that is similar to the investigation of the LR case in Ref. [12] where special cases are studied by setting some of the parameters equal to zero. Such a study may also include non-autonomous QR equations with variable coefficients.

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