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Difference equations with absolute values

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We study a class of multiparameter difference equations that contain the absolute value of a difference. Using two different semiconjugate factorizations, we obtain precise information about the solutions of these equations for various ranges of parameter values.

Keywords: Absolute value; Monotonic solutions; Non-autonomous equations; Semiconjugate relations

1. Introduction

Consider the second-order, 3-parameter difference equation

$$x_{n+1} = |ax_n - bx_{n-1}| + cx_n. \qquad (1)$$

The case $c = 0$ is studied extensively in [3] and [5] where several different types of behavior that bifurcate with changing parameters are shown to occur for equation (1). Periodic solutions, unbounded solutions and convergent (monotonic as well as non-monotonic) solutions all exist depending on the parameter values a, b; further, solutions with qualitatively different behavior may coexist with the same set of parameter values. Another special case of equation (1) where $a = b$ and $0 \leq c < 1$ is studied in [6] where various criteria for the global asymptotic stability of the origin are given, including conditions for monotonic and non-monotonic convergence.

Here we study equation (1) and its generalizations under different restrictions on parameter values than considered in previous studies. With the new parameter ranges, a semiconjugate relation facilitates the derivation of explicit solutions. Also with the aid of another semiconjugate relation we obtain results concerning the solutions of equation (1) when the first semiconjugate relation does not hold so as to highlight some differences that exist between the two cases. For background material and related issues, see [2,4] and [7].

2. The basic second order equation

In the case, $b = ac$, equation (1) can be written as

$$x_{n+1} = |a||x_n - cx_{n-1}| + cx_n. \qquad (2)$$

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Making the substitution

$$x_n - cx_{n-1} = t_n \tag{3}$$

reduces equation (2) to the first order equation

$$t_{n+1} = |a||t_n|. \tag{4}$$

Evidently solutions of equation (2) with real initial values x_0, x_{-1} coincide with the solutions of the uncoupled system of first order equations (3) and (4). We first solve equation (4) with $t_0 = x_0 - cx_{-1}$ to get

$$t_n = |t_0||a|^n, \quad n = 1, 2, \ldots \tag{5}$$

Using this solution, in the first order equation (3) inductively yields a solution

$$x_n = x_0 c^n + |t_0| \sum_{k=1}^{n} c^{n-k} |a|^k \tag{6}$$

for equation (2). Upon adding the geometric series and rearranging terms, we obtain the first part of the following comprehensive result on equation (2). The remaining parts are then easily established using the first part. In the following and elsewhere, "almost all solutions" means solutions generated by all initial values outside a set of Lebesgue measure zero in the phase space.

THEOREM 2.1 (*a*) *The general solution of equation* (2) *is given by*

$$x_n = (x_0 - s_0)c^n + s_0|a|^n, \quad |a| \neq c, \quad s_0 = \frac{|at_0|}{|a| - c}$$

and in the exceptional case where $|a| = c$ *we get*

$$x_n = (x_0 + |t_0|n)c^n.$$

(*b*) *If* $\max\{|a|, |c|\} < 1$ *then every solution of equation* (2) *converges to zero.*

(*c*) *If* $\max\{|a|, |c|\} = 1$ *and* $c < 1$, *then every solution of equation* (2) *is bounded. If* $c > -1$, *then almost all solutions of equation* (2) *converge to a nonzero constant. If* $c = -1$, *then non-trivial solutions of equation* (2) *converge to one of the following period*-2 *sequences*

$$\left[x_0 + \frac{|at_0|}{|a| + 1}\right](-1)^n, \quad |a| < 1$$

$$\left[x_0 + \frac{|t_0|}{2}\right](-1)^n + \frac{|t_0|}{2}, \quad |a| = 1.$$

(*d*) *If* $\max\{|a|, |c|\} > 1$ *or* $c = |a| = 1$, *then almost all solutions of equation* (2) *are unbounded. If* $|c| \le 1 < |a|$ *then all unbounded solutions approach infinity eventually monotonically.*

Equations (3) and (4) constitute a semiconjugate factorization of equation (2) in the sense of [7, Section 3.2]. Here, the factor map $f(t) = |a||t|$ is linked via the map $H(x, y) = x - cy$ to the standard vectorization or unfolding of equation (2). Although it is not easy to discover semiconjugacies in general, it is worth noting that when a particular equation can be split or factorized, a semiconjugate relation can often be identified. In fact, another semiconjugate factorization which applies to equation (1) generally (not just to the special case (2)) is obtained by dividing equation (1) on both sides by x_n to get

$$\frac{x_{n+1}}{x_n} = \left|a - \frac{bx_{n-1}}{x_n}\right| + c.$$

This latter equation can be written as

$$r_{n+1} = c + \left|a - \frac{b}{r_n}\right|, \quad n = 0, 1, 2, \ldots \tag{7}$$

if we define $r_n = x_n/x_{n-1}$ for every $n \geq 0$. We thus have a map $H(x, y) = x/y$ linking equation (1) to the factor map that defines the dynamics of the ratios r_n, namely,

$$\phi(r) = c + \left|a - \frac{b}{r}\right|, \quad r \neq 0.$$

The advantage of equation (7) over equation (4) is that equation (7) applies when b is not equal to ac. A disadvantage of equation (7) is that it does not account for those solutions of equation (1) that pass through zero. Rather than finding an explicit solution for equation (7) we use it in a qualitative fashion, as was done in [3,5,6].

The next theorem, which may be compared with Theorem 2.1, concerns the case $b \neq ac$. But first, we present a useful lemma on the behavior of the iterates of the mapping ϕ or of the solutions of equation (7) in this case.

LEMMA 2.2 (*a*) *If $a, c > 0$ and $0 < b < ac$ then the mapping ϕ has a unique fixed point $r^* > b/a$ that attracts all positive orbits of ϕ.*

(*b*) *If $0 < a < c$ and $b > ac$ then the mapping ϕ has a unique fixed point $\bar{r} \in (0, b/a)$ that attracts all positive orbits of ϕ.*

Proof (a) It is clear that ϕ attains its global minimum value c uniquely at $r = b/a$; further, if $b < ac$ then $(a + c)^2 - 4b > (a - c)^2$ so ϕ has a unique fixed point

$$r^* = \frac{a + c + \sqrt{(a + c)^2 - 4b}}{2} > c > \frac{b}{a}.$$

Since $b < ac$, for all $r > 0$ we have $\phi(r) \geq c > b/a$. Thus for any sequence of ratios $r_n = x_n/x_{n-1}$ we have $r_n > b/a$ for $n \geq 1$. Hence, we can eliminate the absolute value from equation (7) and rewrite it as a Riccati equation

$$r_n = \frac{(a + c)r_n - b}{r_n}.$$

It is not difficult to see that every solution of this equation converges to r^* (or see [1] or Theorem A.4 in [4]).

(b) If $0 < a < c$ and $b > ac$ then ϕ has a positive fixed point

$$\bar{r} = \frac{c - a + \sqrt{(c - a)^2 + 4b}}{2} < \frac{b}{a}.$$

We note that $\bar{r}$ is the unique fixed point of ϕ since ϕ is strictly decreasing if $r < b/a$ and for $r \geq b/a$ we have

$$r > a + c - \frac{b}{r} = \phi(r) \tag{8}$$

if $0 < a < c$ and $b > ac$. From equation (8), it also follows that if r_n is any solution of equation (7) then there is $N \geq 1$ such that $r_N < b/a$. Further, $r_{n+1} > b/a$ for $n \geq N$ if

$$r_n < r' = \frac{ab}{a^2 + b - ac}. \tag{9}$$

We note that $\phi(r') = b/a$. However, equation (9) cannot hold because $r' < c \leq r_n$ for all $n \geq 1$. Note that after multiplying and rearranging terms, the inequality $r' < c$ is seen to be equivalent to $b(c - a) > ac(c - a)$ which is true under the hypotheses in this case. We have established that $r_n \in (c, b/a)$ for $n \geq N$ so the absolute value may be eliminated from equation (7) to get the Riccati equation

$$r_n = \frac{(c - a)r_n + b}{r_n}.$$

Now from [1] or Theorem A.4 in [4] it follows that $r_n \to \bar{r}$ as $n \to \infty$.

Theorem 2.3 (*a*) *Let* $0 < a, c < 1$ *and* $0 < b < ac$. *Then every positive solution of equation* (1) *converges to zero eventually monotonically if*

$$b > a + c - 1. \tag{10}$$

But if $a + c > 1$ *and*

$$b < a + c - 1 \tag{11}$$

then every positive solution of equation (1) *approaches infinity eventually monotonically.*

In either case, if $x_0, x_{-1} > 0$ *then for all* $n \geq 1$ *there are suitable constants* k_1, k_2 *such that*

$$x_n = k_1 \left(\frac{a + c - \sqrt{(a + c)^2 - 4b}}{2} \right)^n + k_2 \left(\frac{a + c + \sqrt{(a + c)^2 - 4b}}{2} \right)^n. \tag{12}$$

(*b*) *Let* $0 < a < c$ *and* $b > ac$. *Then every positive solution of equation* (1) *converges to zero eventually monotonically if*

$$b < 1 - c + a \tag{13}$$

in which case $c < 1$ *also. But if*

$$b > 1 - c + a \tag{14}$$

then every positive solution of equation (1) *approaches infinity eventually monotonically.*

In either case, for all n sufficiently large, there are suitable constants k_1, k_2 *such that*

$$x_n = k_1 \left(\frac{c - a - \sqrt{(c - a)^2 + 4b}}{2} \right)^n + k_2 \left(\frac{c - a + \sqrt{(c - a)^2 + 4b}}{2} \right)^n. \tag{15}$$

Proof (a) First, note that if $0 < a, c < 1$ then $a(1 - c) < 1 - c$ so

$$a + c - 1 < ac.$$

If r^* is the fixed point in Lemma 2.2(a), then it follows that if equation (10) holds then $r^* < 1$ while if equation (11) holds with $a + c > 1$ then $r^* > 1$. Thus, if x_n is any positive solution of equation (1) then since $x_{n+1} = r_{n+1}x_n$ we see from Lemma 2.2 that for all large n, the sequence x_n is decreasing to zero or increasing to infinity depending on whether equation (10) or (11) holds, respectively. The explicit expression for x_n is obtained by dropping the absolute value from equation (1) and solving the resulting linear equation

$$x_{n+1} = (a + c)x_n - bx_{n-1}$$

with initial values x_0, x_1 to get the explicit form in equation (12). We note that, the two eigenvalues of the linear equation that are shown in equation (12), the one that is *not* equal to r^* is always between 0 and 1 under the hypotheses of this theorem.

(b) By Lemma 2.2(b), the fixed point $\bar{r}$ attracts all positive orbits of ϕ. Now if equation (13) holds, then $\bar{r} < 1$ while equation (14) implies that $\bar{r} > 1$. The stated conclusions are therefore true. Also equation (13) plus the condition $b > ac$ gives $c < 1$; for otherwise equation (13) would yield

$$b - a < 1 - c \leq 0$$

which would contradict $b > ac \geq a$. Therefore, convergence to zero under condition equation (13) requires that $c < 1$. On the other hand, with equation (14) it is possible that $c < 1$ so unbounded solutions may still occur when $a, c < 1$.

Since every orbit r_n of ϕ approaches $\bar{r}$, there is a positive integer N such that $r_n < b/a$ for all $n \geq N$. This means that $a < b/r_n$ for $n \geq N$ and the absolute value may thus be removed from equation (1) to solve the resulting linear equation

$$x_{n+1} = (c - a)x_n + bx_{n-1}$$

with initial values x_{N-1}, x_N to get the explicit form (15). □

Remark 2.4 When $b = a + c - 1$, i.e. the missing value in Theorem 2.3(a), then $r^* = 1$ and from equation (12) it follows that every solution of equation (1) approaches a constant, namely, k_2 in a monotonic fashion. Similarly, when $b = 1 - c + a$ then $\bar{r} = 1$ while the other eigenvalue shown in equation (15) equals $-b$. Since here $0 < b < 1$ we see that every solution of equation (1) approaches the constant k_2 in an oscillatory fashion.

Remark 2.5 Theorem 2.3 provides a set of restrictions on the three parameters of equation (1) that permit the removal (eventually, for large n) of the absolute value from equation (1), one way or the other. Thus we were able to obtain the explicit formulas (12) and (15) for the asymptotic behavior of the solutions of equation (1). Such flexibility does not exist for all parameter values; for example, if $c < a$ and $b > ac$ then the absolute value may continue to affect the asymptotic behavior in the long term because of the existence of multiple fixed points or of periodic orbits for the mapping ϕ. The analysis would then require using methods similiar to those discussed in [3] for the case $c = 0$.

3. Higher order and other generalizations

The next result concerns a delay version (and a generalization) of the second order equation (2). In this case, it is illuminating to write the solution in terms of its k constituent subsequences.

THEOREM 3.1 *The general solution of the equation*

$$x_{n+1} = a|x_n - cx_{n-k}| + cx_{n-k+1}, \quad k \geq 1, \quad a > 0, \quad c \neq 0 \tag{16}$$

with a fixed delay k is given in terms of the subsequences

$$x_{km+j} = (x_{j-k} - s_j)c^{m+1} + s_j a^{k(m+1)}, \quad m = 0, 1, 2, \ldots \tag{17}$$

when $a \neq c^{1/k}$ where for each $j = 1, 2, \ldots, k$,

$$s_j = \frac{|x_0 - cx_{-k}|a^j}{a^k - c}.$$

When $a = c^{1/k}$ we have for $j = 1, 2, \ldots, k$,

$$x_{km+j} = [x_{j-k} + |x_0 - cx_{-k}|c^{-1+j/k}(m+1)]c^{m+1}. \tag{18}$$

Proof As in the preceding section, we find the solutions of the linear non-homogeneous equation of order k given by the semiconjugate link

$$x_n - cx_{n-k} = |t_0|a^n, \quad t_0 = x_0 - cx_{-k}. \tag{19}$$

The eigenvalues of the homogeneous part are all the k-th roots of the real number c so it is possible to express the solutions of equation (16) in the customary way using trigonometric functions. However, a more informative form of the solution can be obtained by introducing the variables

$$y_m^{(j)} = x_{km+j}, \quad j = 1, 2, \ldots, k, \quad m = 0, 1, 2, \ldots$$

in equation (19) to obtain k independent subsequences each of which satisfies a first order equation

$$y_m^{(j)} - cy_{m-1}^{(j)} = |t_0|a^{km+j}, \quad y_0^{(j)} = x_j = cx_{j-k} + |t_0|a^j.$$

Solving this equation for each fixed j and some straightforward calculations give equation (17) or (18) as appropriate.

The next result extends Theorem 2.1 to a non-autonomous version of equation (2). Given the fundamentally non-homogeneous nature of equation (3), this is a natural extension of Theorem 2.1.

THEOREM 3.2 *Let a_n, b_n, d_n be given sequences of real numbers with $a_n \geq 0$ and $b_{n+1} + d_n \geq 0$ for all $n \geq 0$. The general solution of*

$$x_{n+1} = a_n|x_n - cx_{n-1} + b_n| + cx_n + d_n, \quad c \neq 0 \tag{20}$$

is given by

$$x_n = x_0 c^n + \sum_{k=1}^{n} c^{n-k}\left(d_{k-1} + |t_0|\prod_{j=0}^{k-1} a_j + \sum_{i=1}^{k-1}(b_i + d_{i-1})\prod_{j=i}^{k-1} a_j\right). \tag{21}$$

where $t_0 = x_0 - cx_{-1} + b_0$.

Proof Equation (20) has a semiconjugate factorization as

$$x_n - cx_{n-1} + b_n = t_n, \quad t_n = a_{n-1}|t_{n-1}| + b_n + d_{n-1}. \tag{22}$$

The second equation in (22) may be solved recursively to get

$$t_n = |t_0|\prod_{j=0}^{n-1} a_j + b_n + d_{n-1} + \sum_{i=1}^{n-1}(b_i + d_{i-1})\prod_{j=i}^{n-1} a_j. \tag{23}$$

Substituting equation (23) in the first equation in (22) and using another recursive argument yields equation (21).

In order to efficiently extract some new information from Theorem 3.2 we consider a special case in which $b_{n+1} + d_n = 0$ in equation (20).

COROLLARY 3.3 (*a*) *Let* a_n, b_n *be given sequences of real numbers with* $a_n \geq 0$ *for all* $n \geq 0$. *Then the general solution of*

$$x_{n+1} = a_n|x_n - cx_{n-1} - b_n| + cx_n + b_{n+1}, \quad c \neq 0 \tag{24}$$

is given by

$$x_n = x_0 c^n + \sum_{k=1}^{n} c^{n-k}\left(b_k + |t_0|\prod_{j=0}^{k-1} a_j\right) \tag{25}$$

where $t_0 = x_0 - cx_{-1} - b_0$.

(*b*) *If* $|c| < 1$ *and* $\prod_{k=0}^{n} a_k$ *and* b_n *are bounded, then all solutions of equation* (24) *are bounded.*

(*c*) *If* $|c| < 1$ *and* $\prod_{j=0}^{n} a_j \to a$ *and* $b_n \to b$ *as* $n \to \infty$ *then every solution of equation* (24) *converges to the real number*

$$\frac{b + a|t_0|}{1 - c}.$$

(*d*) *If* $a_N = 0$ *for some least* $N \geq 0$, *then for all* $n \geq N + 1$ *the solution* x_n *of equation* (24) *reduces to*

$$x_n = c^{n-N}x_N + \sum_{k=N+1}^{n} c^{n-k}b_k. \tag{26}$$

In particular, if $|c| < 1$ *and* $a_n = 0$ *for some* $n \geq 0$, *then each solution of equation* (24) *converges to a solution of*

$$y_{n+1} = cy_n + b_{n+1}.$$

Proof (a) This is an immediate consequence of Theorem 3.2.

(b) Taking the absolute value of equation (25) gives the following

$$|x_n| \leq |x_0||c|^n + \sum_{k=0}^{n-1} |c|^k(B + A|t_0|) \leq |x_0 c| + \frac{B + A|t_0|}{1 - |c|}$$

where $A = \sup_{n\geq 1}\prod_{k=0}^{n-1}|a_k|$ and $B = \sup_{k\geq 1}|b_k|$. Boundedness follows.

(c) Let p_n be any convergent sequence of real numbers with limit p. Then by a straightforward argument it may be established that

$$\lim_{n\to\infty}\sum_{k=1}^{n} c^{n-k}p_k = \frac{p}{1 - c}.$$

Thus the proof is completed by taking limits in equation (25) with p_n representing b_n or $\prod_{k=0}^{n-1}a_k$.

(d) If there is a least positive integer N such that $a_N = 0$, then for all $n \geq N + 1$, equation (23) reduces to $t_n = b_n + d_{n-1} = 0$. Using this in the first equation in (22) gives equation (26). The last assertion now follows from equation (26).

Remark 3.4 The situation in Corollary 3.3(d) is an interesting consequence of having a variable coefficient a_n; it says in effect that if the minimum of a_n is zero and $|c| < 1$, then the coefficients a_n (and thus the absolute value) have no effects on the solutions of equation (24) aysmptotically. We emphasize that the conclusion of Corollary 3.3(d) is *not* valid in the more general context of Theorem 3.2.

Another interesting consequence of variable a_n is the possibility that every solution of equation (24) or more generally of equation (20) may converge to a finite real number even if $a_n > 1$ for all n. For example, if $|c| < 1$ then each solution of the equation

$$x_{n+1} = cx_n + 2^{1/n!}\left|x_n - cx_{n-1} - \frac{n}{n+1}\right| + \frac{n+1}{n+2}$$

by Corollary 3.3(c) converges to the real number

$$\frac{1 + 2^e|x_0 - cx_{-1}|}{1 - c}$$

since for all $n \geq 1$,

$$\lim_{n\to\infty}\prod_{k=0}^{n-1} 2^{1/k!} = 2^{\sum_{k=0}^{\infty} 1/k!} = 2^e.$$

Remark 3.5 Some of the preceding methods also apply to certain equations that do not explicitly involve the absolute value. For example, arguing as we did in Theorem 3.2 and Corollary 3.3, it follows that the solutions of the quadratic equation

$$(z_{n+1} - cz_n + b_{n+1})^2 = a^2(z_n - cz_{n-1} - b_n)^2 \tag{27}$$

are given by

$$z_n = z_0c^n + \sum_{k=1}^{n} c^{n-k}(b_k + |t_0|a^k\sigma_k)$$

where σ_n is an arbitrary sequence in the set $\{-1,1\}$. This latter sequence originiates in the semiconjugate factor $t_{n+1}^2 = a^2t_n^2$ of equation (27) which is equivalent to

$$|t_{n+1}| = |a||t_n|. \tag{28}$$

It is evident that the general solution of equation (28) has the form $t_n = |t_0|a^n\sigma_n$.

4. Conclusion and future directions

We obtained several results about difference equations containing the absolute value of a difference. Many of these results can be further extended, and some of these generalizations are straightforward. But there are also less routine problems that exist in various cases. For example, the behavior of solutions of equation (1) in cases (i.e. for parameter values) not discussed here or in [3,5] or [6] remain to be determined. In fact, one may choose to study the more general equation

$$x_{n+1} = |ax_n - bx_{n-1}| + cx_n + \mathrm{d}x_{n-1}$$

and obtain results based on various subsets of the 4 dimensional parameter space, rather like we did here or in [3,6].

Going in a different direction, we may study non-autonomous equations such as

$$x_{n+1} = |a_n x_n - b_n x_{n-1}|$$

about which relatively little is known, in spite of the remarkable simplicity of the equation and its similarity to a linear equation. Both of the preceding equations are amenable to analysis using ratios, a procedure that was effective in Theorem 2.2 as well as in various previous work. Ratios however, are not as effective for generalizations of the above equations to order 3 and greater so new methodology may need to be developed for equations whose order is greater than 2.

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