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Journal of Difference Equations and Applications

ISSN: 1023-6198 (Print) 1563-5120 (Online) Journal homepage: http://www.tandfonline.com/loi/gdea20

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To cite this article: C.M. Kent & H. Sedaghat (2004) Convergence, Periodicity and Bifurcations for the Two-parameter Absolute-Difference Equation, Journal of Difference Equations and Applications, 10:9, 817-841, DOI: 10.1080/10236190410001685049

To link to this article: http://dx.doi.org/10.1080/10236190410001685049

Published online: 22 Aug 2006.

Journal of Difference Equations and Applications,
Vol. 10, No. 9, 10 August 2004, pp. 817–841

Convergence, Periodicity and Bifurcations for the Two-parameter Absolute-Difference Equation

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(Received 21 December 2003; In final form 11 February 2004)

The two-parameter absolute-difference equation $x_{n+1} = |ax_n - bx_{n-1}|$ is studied. Based on the parameter values a, b and a pair of initial values, we consider the existence and bifurcations of solutions having one or more of the following properties: (i) unbounded, (ii) convergent (to zero or to a positive constant), (iii) monotonic, (iv) periodic and (v) non-periodic oscillatory. The semiconjugate first order equation satisfied by the ratios $\{x_n/x_{n-1}\}$ is used to significant advantage for points (a, b) in certain regions of the parameter plane. Some open problems and conjectures are presented.

Keywords: Global asymptotic stability; Oscillatory; Bifurcations; Ratios; Semiconjugate; The golden mean

AMS Subject Classification Numbers: 39A11; 39A10

Consider the second-order difference equation

$$x_{n+1} = |ax_n - bx_{n-1}|, \quad a, b \geq 0, \quad n = 0, 1, 2, \ldots. \tag{1}$$

An equation such as this may appear implicitly in smooth difference equations (or difference relations) that are in the form of e.g. quadratic polynomials. In such cases, the solutions of the quadratic equation include the solutions of Eq. (1) so statements can be made with regard to the existence and stability of the solutions for the more general, smooth equations based on the solutions of Eq. (1). We may assume that the initial values x_{-1}, x_0 in Eq. (1) are non-negative and for non-triviality, at least one is positive. Dividing both sides of Eq. (1) by x_n we obtain a ratios equation

$$\frac{x_{n+1}}{x_n} = \left| a - \frac{bx_{n-1}}{x_n} \right|,$$

which can be written as

$$r_{n+1} = \left| a - \frac{b}{r_n} \right|, \quad n = 0, 1, 2, \ldots, \tag{2}$$

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Journal of Difference Equations and Applications
ISSN 1023-6198 print/ISSN 1563-5120 online © 2004 Taylor & Francis Ltd
http://www.tandf.co.uk/journals
DOI: 10.1080/10236190410001685049

where we define $r_n = x_n/x_{n-1}$ for every $n \geq 0$. We may think of Eq. (2) as the recursion $r_{n+1} = \phi(r_n)$ where ϕ is the piecewise smooth mapping

$$\phi(r) = \left|a - \frac{b}{r}\right|, \quad r > 0.$$

In this format, solutions $\{r_n\}$ of Eq. (2) can be written as $r_n = \phi^n(r_0)$ for $n \geq 1$. Note that Eq. (2) is a *first order* equation and a one-dimensional semicongjugate of Eq. (1) with the ratio *x/y* as a link map; see Ref. [6]. Also see Refs. [2,3] for an application of ratio links. Additional background material for this paper may be found in Refs. [1,6]. The special case where $a = b = 1$ in Eqs. (1) and (2) is studied in Ref. [9] where the following is proved:

THEOREM A *Let $a = b = 1$ in Eq.* (1) *and let $\mathbb{Q}^+$ denote the set of all non-negative rational numbers.*

(a) *If $x_0/x_{-1} \notin \mathbb{Q}^+$ then the corresponding solution $\{x_n\}$ of Eq.* (1) *converges to zero.*
(b) *If $x_0/x_{-1} \in \mathbb{Q}^+$ or $x_{-1} = 0$ then the corresponding solution $\{x_n\}$ of Eq.* (1) *has period* 3 *eventually and for all large n it has the form*

$$\{0, \alpha, \alpha, 0, \alpha, \alpha, 0, \ldots\},$$

where $\alpha > 0$.
(c) *Equation* (2) *has a p-periodic solution $\{r_1, \ldots, r_p\}$ for every $p \neq 3$ given by*

$$r_1 = \frac{1+\sqrt{5}}{2}, \quad r_2 = \frac{\sqrt{5}-1}{\sqrt{5}+1} \quad (p = 2),$$

$$r_1 = \frac{1}{2}\left[y_{p-4} + \sqrt{y_{p-4}^2 + 4y_{p-4}y_{p-1}}\right], \quad r_k = \frac{y_{k-4}r_1 - y_{k-2}}{y_{k-3} - y_{k-5}r_1}, \quad 2 \leq k \leq p, \quad (p \geq 4),$$

where y_n is the n-th Fibonacci number; i.e. $y_{n+1} = y_n + y_{n-1}$ for $n \geq -2$ where we define

$$y_{-3} = -1, \quad y_{-2} = 1.$$

(d) *The mapping ϕ has a scrambled set* S*; hence, if $\{x_n\}$ is a solution of Eq.* (1) *with initial value ratio $x_0/x_{-1} \in S$, then the sequence $\{x_n/x_{n-1}\}$ of consecutive ratios is chaotic.*

Although Eq. (1) is a rather simple equation, Theorem A suggests that for some values of the parameters a, b it exhibits interesting dynamics. In this paper, we study the asymptotic behavior of Eq. (1) for different parameter values $a, b \geq 0$. For comparison, it is interesting to note that Eq. (1) relates to certain other simple equations such as

$$u_{n+1} = au_n + bu_{n-1} \quad v_{n+1} = \max\{av_n, bv_{n-1}\} \quad w_{n+1} = \min\{aw_n, bw_{n-1}\} \tag{3}$$

through the relations

$$|\alpha - \beta| = 2\max\{\alpha, \beta\} - (\alpha + \beta) = \max\{\alpha, \beta\} - \min\{\alpha, \beta\}.$$

In contrast to Eq. (1), each of the equations in (3) produces solutions with simple behaviors for all parameter values a, b. Some indication of the major difference between these equations and Eq. (1) may be had by looking at their semiconjugate factors through ratio links:

$$\phi_u(r) = a + \frac{b}{r}, \quad \phi_v(r) = \max\left\{a, \frac{b}{r}\right\}, \quad \phi_w(r) = \min\left\{a, \frac{b}{r}\right\}.$$

It is evident from a quick examination of these mappings that only the factor ϕ which was defined for Eq. (1) is capable of generating complex behavior on the half-line. Indeed, an important feature of ϕ is its nonsmooth critical point at b/a whose pre-image is the origin, namely, the point of discontinuity of ϕ. This source of complexity is lacking in the factors ϕ_u, ϕ_v and ϕ_w all of which are non-increasing maps.

THE UNIT SQUARE $a, b \leq 1$

We first consider the parameter values (a, b) in the unit square. The function f (or equivalently, its vectorization V_f) in Lemma 1 below is said to be a *weak contraction* on the set A_α; see Ref. [6] for a general theory of weak contractions and weak expansions as well as a proof of the lemma.

LEMMA 1 Let $f : \mathbb{R}^m \to \mathbb{R}$ *be continuous and let* $\bar{x}$ *be an isolated fixed point of*

$$x_{n+1} = f(x_n, x_{n-1}, \ldots, x_{n-m}).$$

Let $V_f(u_1, \ldots, u_m) = (f(u_1, \ldots, u_m), u_1, \ldots, u_{m-1})$ *and for* $\alpha \in (0, 1)$ *define the set*

$$A_\alpha = \{(u_1, \ldots, u_m) : |f(u_1, \ldots, u_m) - \bar{x}|\} \leq \alpha \max\{|u_1 - \bar{x}|, \ldots, |u_m - \bar{x}|\}.$$

If S is a subset of A_α *such that* $V_f\,(S) \subset S$ *and* $(\bar{x}, \ldots, \bar{x}) \in S$, *then* $(\bar{x}, \ldots, \bar{x})$ *is asymptotically* (*in fact, exponentially*) *stable relative to S.*

COROLLARY 1 *If* $a, b < 1$ *then the origin is globally asymptotically stable for Eq.* (1).

Proof Define $f(x, y) = |ax - by|$ and note that for $x, y \in [0, \infty)^2$ we have

$$f(x, y) \leq \max\{ax, by\} \leq \max\{a, b\}\max\{x, y\}.$$

Hence f is a weak contraction on $[0, \infty)^2$ and Lemma 1 applies. The proof is completed upon recalling that for each solution $\{\mathrm{x_n}\}$ of Eq. (1) $x_n \geq 0$ for all $n \geq 1$. □

LEMMA 2 *Let* $0 < a, b \leq 1$ *and either* $a = 1$ *or* $b = 1$. *For each solution* $\{x_n\}$ *of Eq.* (1) *if* $x_k > x_{k-1} > 0$ *for some* $k \geq 0$ *then* $x_n < x_k$ *for all* $n > k$.

Proof First, let $b = 1$ and $0 < a \leq 1$. If $ax_k > x_{k-1}$ then

$$x_{k+1} = ax_k - x_{k-1} < ax_k \leq x_k. \tag{4}$$

From this we infer that $ax_{k+1} < ax_k \leq x_k$ So

$$x_{k+2} = x_k - ax_{k+1} < x_k. \tag{5}$$

Now from Eqs. (4) and (5) it follows that

$$x_{k+3} \leq \max\{ax_{k+2}, x_{k+1}\} < x_k.$$

The last step is easily extended by induction to all $n > k$. Next, assume that $ax_k \leq x_{k-1}$. Then

$$x_{k+1} = x_{k-1} - ax_k < x_{k-1} < x_k$$

and it follows that $ax_{k+1} < ax_k \leq x_k$. This again implies Eq. (5) and we argue as before.

Now, assume that $a = 1$ and $0 < b \leq 1$. Since $x_k > x_{k-1} \geq bx_{k-1} > 0$ by hypothesis, it follows that

$$0 < x_{k+1} = x_k - bx_{k-1} < x_k.$$

If $x_{k+1} \geq bx_k$, then $x_{k+2} = x_{k+1} - bx_k < x_{k+1}$ so that

$$x_{k+3} \leq \max\{x_{k+2}, bx_{k+1}\} < x_k.$$

Inductively, $x_n > x_k$ for $n > k$. If $x_{k+1} < bx_k$ then

$$x_{k+2} = bx_k - x_{k+1} < bx_k \leq x_k,$$

which implies $x_{k+3} < x_k$ and by induction the proof is completed. □

THEOREM 1 *For all* $(a, b) \in [0, 1]^2$ *except at the three boundary points* $(1, 1)$, $(1, 0)$ *and* $(0, 1)$, *the origin is globally asymptotically stable for Eq.* (1).

Proof In light of Corollary 1 it remains only to consider the boundaries where either $a = 1$ or $b = 1$. For such points (a, b), first assume that $0 < ab < 1$ and let $\{x_n\}$ be a solution of Eq. (1). Either $x_n < x_{n-1}$ for all n in which case $\mathrm{x_n}$ converges to zero monotonically, or there is $k_1 \geq 0$ such that $x_{k_1} > x_{k_1 - 1} > 0$. In the latter case, Lemma 2 implies that $x_n < x_{k_1}$ for all $n > k_1$. If the sequence $\{x_n\}$ is not eventually decreasing, then there is an increasing sequence k_i of positive integers such that

$$x_{k_1} > x_{k_2} > \cdots > x_{k_i} > \cdots$$

and for $i = 1, 2, 3, \ldots$

$$x_n < x_{k_i} \quad \textit{if } k_i < n \leq k_{i+1}.$$

These facts imply that $x_n \to 0$ as $n \to \infty$. Stability of the origin follows from Lemma 2 also since $x_n \leq \max\{x_0, x_{-1}\}$ for all $n \geq 1$.

It remains for us to examine the three boundary points mentioned in the statement of the theorem. At $(a, b) = (1, 1)$ Theorem A shows that the origin is not globally attracting. At $(a, b) = (1, 0)$, Eq. (1) reduces to the trivial equation $x_{n+1} = x_n$ whose solutions are the constants x_0. At $(a, b) = (0, 1)$ each solution of $x_{n+1} = x_{n-1}$ trivially has period 2: $\{x_{-1}, x_0\}$.

Since for $a = 0$(1) reduces to $x_{n+1} = bx_{n-1}$ and for $b = 0$ it reduces to $x_{n+1} = ax_n$, it is evident that along the rays from the origin through $(0, 1)$ and through $(1, 0)$ all solutions of Eq. (1) are unbounded when $b > 1$ or $a > 1$ respectively. Therefore, the points $(0, 1)$ and $(1, 0)$ are bifurcation points on their respective rays. The same is true for the ray through $(1, 1)$, though this is much harder to verify. In the remainder of this paper we consider the behavior of solutions of Eq. (1) when (a, b) is outside the unit square. □

THE REGION $b \leq a^2/4$

The semiconjugate factor map ϕ on the half-line can be used to further explore the behavior of solutions in both convergent and non-convergent cases. The mapping ϕ always has a fixed point $\bar{r}$ through its left half

$$\phi_{\mathrm{L}}(r) = \frac{b}{r} - a, \quad 0 < r \leq \frac{b}{a}.$$

Solving $\phi_{\mathrm{L}}(r) = r$ gives the value of $\bar{r}$ as

$$\bar{r} = \frac{1}{2}\left[\sqrt{a^2 + 4b} - a\right].$$

It may be noted that $\bar{r}$ is an unstable fixed point for all positive values of a, b, since

$$\left|\phi_{\mathrm{L}}'(\bar{r})\right| = \frac{b}{\bar{r}^2} = \frac{b}{b - a\bar{r}} > 1.$$

The right half of ϕ, namely,

$$\phi_{\mathrm{R}}(r) = a - \frac{b}{r}, \quad r \geq \frac{b}{a}$$

can have up to two fixed points if the equation $\phi_{\mathrm{R}}(r) = r$ possesses real solutions. Solving the latter equation yields two values

$$r_1^* = \frac{1}{2}\left[a - \sqrt{a^2 - 4b}\right], \quad r_2^* = \frac{1}{2}\left[a + \sqrt{a^2 - 4b}\right]$$

both of which are real if and only if $a^2 \geq 4b$. Stated differently, ϕ has up to two additional fixed points for all points (a, b) on or below the parabola $b = a^2/4$ in the parameter plane. On this parabola,

$$r_1^* = r_2^* = \frac{a}{2}$$

and the single fixed point of ϕ is semistable. In the sub-parabolic region $b < a^2/4$,

$$\frac{b}{a} < r_1^* < r_2^*$$

with r_1^* unstable and r_2^* asymptotically stable as may be readily verified. The tangent bifurcation of the factor map ϕ that produces r_1^*, r_2^* when a parameter point (a, b) crosses the parabola $b = a^2/4$ corresponds to significant behavioral changes in the solutions of Eq. (1) as well. In this section, we examine the region below the parabola, beginning with the following lemma whose straighforward proof is omitted.

LEMMA 3 *Let $b \leq a^2/4$.*

(a) *$r_2^* \leq a$ with equality holding if and only if $b = 0$.*
(b) *Let $a < 2$. Then $r_1^* < 1$; also $r_2^* < 1$ if and only if $b > a - 1$. Further, $r_2^* = 1$ if $b = a - 1$.*
(c) *Let $a \geq 2$. Then $r_2^* \geq 1$; also $r_1^* \leq 1$ if and only if $b \leq a - 1$. Further, $r_1^* = 1$ if $b = a - 1$.*

In Lemma 3 notice that the line $b = a - 1$ is tangent to the parabola $b = a^2/4$ at $a = 2$.

LEMMA 4 *Let $b \leq a^2/4$ and let $\{x_n\}$ be a solution of Eq. (1) such that $x_k \geq r_1^* x_{k-1}$ for some integer $k \geq 0$.*

(a) *If $b \neq a^2/4$ then there are real constants c_1, c_2 such that*

$$x_n = c_1(r_1^*)^n + c_2(r_2^*)^n, \quad n \geq k. \tag{6}$$

(b) *If* $b = a^2/4$ *then there are real constants* c_1, c_2 *such that*

$$x_n = (c_1 + c_2 n)(a/2)^n, \quad n \geq k \tag{7}$$

with $r_1^* = r_2^* = a/2$.

Proof By hypothesis, $r_k = x_k/x_{k-1} \geq r_1^*$. If $r_n = \phi(r_{n-1})$ for $n > k$, then since $r_n \geq r_1^*$ for $n \geq k$, we may write

$$\frac{x_{n+1}}{x_n} = r_{n+1} = \phi_R(r_n) = a - \frac{bx_{n-1}}{x_n}$$

i.e. $x_{n+1} = ax_n - bx_{n-1}$ for $n \geq k$. The eigenvalues of the preceding linear equation are none other than r_1^* and r_2^* so Eq. (6) or Eq. (7) holds as appropriate for $n \geq k$ and suitable constants c_1, c_2. □

THEOREM 2 *Let* $a < 2$.

(a) *If* $a - 1 < b \leq a^2/4$ *then every nontrivial solution of Eq.* (1) *converges to zero eventually monotonically.*
(b) *If* $b = a - 1$ *then every solution of Eq.* (1) *converges to a non-negative constant eventually monotonically.*
(c) *If* $b < a - 1$ *with* $a > 1$ *then any solution* $\{x_n\}$ *of Eq.* (1) *for which* $x_k > r_1^* x_{k-1}$ *for some integer* $k \geq 0$ *is unbounded and stricly increasing eventually. On the other hand, if a solution of Eq.* (1) *is bounded, then it is strictly decreasing to zero; such solutions do exist.*

Proof

(a) First, let $\{x_n\}$ be a positive solution of Eq. (1), i.e. $x_n > 0$ for $n \geq -1$. Then the sequence $\{r_n\}$ where $r_n = \phi(r_{n-1})$ and $r_0 = x_0/x_{-1}$ is well defined for all n. If $x_n < r_1^* x_{n-1}$ for all $n \geq 0$ then $x_n \to 0$ as $n \to \infty$ because $r_1^* < 1$ by Lemma 3. Otherwise, $x_k \geq r_1^* x_{k-1}$ for some $k \geq 0$ so by Lemma 4, Eq. (6) or Eq. (7) holds as appropriate, and since $r_2^* < 1$ the proof is completed. Next, suppose that $x_m = 0$ for some $m \geq -1$. Then $x_{m+2} = ax_{m+1}$ and therefore,

$$r_{m+2} = \frac{x_{m+2}}{x_{m+1}} = a.$$

By Lemma 3, $r_2^* \leq a$ so $r_{m+2} \geq r_2^* > r_1^*$. Letting $k = m + 2$ and applying the above argument once more we obtain Eq. (6) or Eq. (7) and complete the proof.
(b) Here $r_2^* = 1$ and $r_1^* < 1$; the proof is similar to that for Part (a).
(c) With $r_2^* > 1 > r_1^*$, if $\{x_n\}$ is a solution with $r_k = x_k/x_{k-1} > r_1^*$ for some k, then apply Lemma 4 with Eq. (6) to show that $\{x_n\}$ is unbounded and eventually increasing. This argument also shows that if $\{x_n\}$ is bounded, then it has to be strictly decreasing with $x_n \leq r_1^* x_{n-1}$ for all $n \geq 0$. It is clear in this case that $x_n \to 0$ as $n \to \infty$. Finally, to verify that the latter type of solution does exist, let $x_0/x_{-1} = r_0$ belong to either of the sets

$$B = \bigcup_{i=0}^{\infty} \phi^{-i}(\bar{r}), \quad B_1 = \bigcup_{i=0}^{\infty} \phi^{-i}(r_1^*)$$

of backward iterates of $\bar{r}$ or r_1^* respectively under the inverse map ϕ^{-1}. Then $r_n = \bar{r} < 1$ or $r_n = r_1^* < 1$ for all large n and x_n converges to zero exponentially. □

Remark 1 The region in the parameter plane that is mentioned in Theorem 2(a) contains points *outside* the unit square. In addition to being globally attracting in this region, it is also not hard to see that the origin is stable as well, using the facts that $r_2^* < 1$ and that the iterates ϕ take a fixed number (i.e. independent of x_0, x_{-1}) to drop below 1 on their way to r_2^*.

Relating to Theorem 2(b), if $b = a \pm 1$ then as might be expected the origin is not the unique fixed point of Eq. (1).

LEMMA 5

(a) $\bar{r} \geq 1$ *if and only if* $b \geq a + 1$. *Also* $\bar{r} = 1$ *if* $b = a + 1$.
(b) *If* $\bar{r} \geq 1$ *and* $b \leq a^2/4$ *then* $a \geq 2 + \sqrt{8} \approx 4.828$.

Note that the line $b = a + 1$ intersects the parabola at $a = 2 + \sqrt{8}$. Further, on both lines $b = a \pm 1$, (1) is degenerate in the sense that if $x_{-1} = x_0$ then $x_n = x_0$ for all $n \geq 0$. In particular, the origin is not the only fixed point of Eq. (1).

THEOREM 3 *Let* $a \geq 2$ *and* $b \leq a^2/4$.

(a) *Any solution* $\{x_n\}$ *of Eq.* (1) *for which* $x_k > r_1^* x_{k-1}$ *for some integer* $k \geq 0$ *is unbounded and strictly increasing eventually.*
(b) *If* $b < a + 1$, *then there are solutions of Eq.* (1) *that converge to zero eventually monotonically, and if* $b = a \pm 1$, *then there are also solutions that are eventually constant and positive.*

Proof

(a) This is proved similarly to Theorem 2(c), the only difference being that here it is possible that $r_1^* \geq 1$.
(b) If $b < a + 1$, then by Lemma 5, $\bar{r} < 1$ so if $x_0/x_{-1} = r_0 \in B$ where B is the set of backward iterates defined in the proof of Theorem 2(c), then $x_n \longrightarrow 0$ eventually monotonically. If $b = a + 1$, then $\bar{r} = 1$ so $r_0 \in B$ implies that $r_n = 1$ for all large n and thus x_n is eventually constant. For $b = a - 1$ it is the case that $r_1^* = 1$ so letting $r_0 \in B_1$ we obtain an eventually constant solution. □

THE REGION $b > a^2/4$

In this region, the absence of fixed points such as r_1^* and r_2^* mandates the use of different tools and methods. In particular, periodic solutions present themselves as interesting substitutes. We observe that a positive, p-periodic solution $\{x_n\}$ of Eq. (1) induces a periodic ratio sequence $\{x_n/x_{n-1}\}$ of the same period that satisfies the identity $\prod_{i=1}^{p} x_i/x_{i-1} = 1$. It follows that positive, period-p solutions of Eq. (1) correspond in a one-to-one fashion to the p-cycles $\{r_n\}$ of Eq. (2) with the property that $\prod_{i=1}^{p} r_i = 1$. We begin with the next lemma, which shows in particular that unlike the fixed points r_1^* and r_2^*, we can always expect to find two-cycles for Eq. (2), even when $b > a^2/4$.

LEMMA 6

(a) *For fixed* $a, b > 0$, *Eq.* (2) *has a unique two-cycle* $\{r_1, r_2\}$ *where*

$$r_1 = \frac{b}{a} - \frac{\sqrt{a^4 + 4b^2} - a^2}{2a}, \quad r_2 = \frac{b}{a} + \frac{\sqrt{a^4 + 4b^2} - a^2}{2a}. \tag{8}$$

(b) *For fixed* $a^2 > b > 0$, *Eq.* (2) *has a unique three-cycle* $\{r_1, r_2, r_3\}$ *where*

$$r_2 = \phi_{\mathrm{L}}(r_1) < \frac{b}{a}, \quad r_3 = \phi_{\mathrm{L}}(r_2) > \frac{b}{a}, \quad r_1 = \phi_R(r_3) < \frac{b}{a}$$

and r_1 *is given by*

$$r_1 = \frac{a(a^2 + 3b) - \sqrt{a^2(a^2 + b)^2 - 4b^3}}{2(a^2 + b)}. \tag{9}$$

Proof

(a) If $\{r_1, r_2\}$ is a two-cycle of Eq. (2) with $r_1 < r_2$, then it is the case that $r_1 < b/a < r_2$. This is true because ϕ_{R} is increasing and also $\phi_{\mathrm{L}}^2(r) = r$ implies that $r = \bar{r}$. Hence both of r_1 and r_2 cannot be on one side of b/a. Now setting $r_2 = \phi_{\mathrm{L}}(r_1)$ and $r_1 = \phi_{\mathrm{R}}(r_2)$ gives

$$r_1 = a - \frac{b}{\phi_{\mathrm{L}}(r_1)} = a - \frac{br_1}{b - ar_1} = \frac{ab - (a^2 + b)r_1}{b - ar_1}.$$

Solving the above equation for r_1 we obtain the value in Eq. (8) as the only solution in the interval $(0, b/a)$. Then $r_2 = \phi_{\mathrm{L}}(r_1)$ is the other value Eq. (8).

(b) We calculate the three-cycle indicated in the statement of the lemma similarly to Part(a), although the algebraic manipulations are more extensive. The unique value for $r_1 \in (0, b/a)$ in the form (9) is defined only when the expression under the square root is non-negative; i.e. r_1 exists if and only if

$$a(a^2 + b) \geq 2b^{3/2}.$$

This inequality can be written as a cubic polynomial inequality in a

$$a^3 + ab - 2b\sqrt{b} \geq 0.$$

Noting that $a = \sqrt{b}$ is a root of the polynomial on the left hand side, we have a factorisation

$$(a - \sqrt{b})(a^2 + a\sqrt{b} + 2b) \geq 0.$$

Since the quadratic factor is positive for $a, b > 0$ we conclude that r_1 (hence also the three-cycle) exist when $a \geq \sqrt{b}$ or equivalently, $a^2 \geq b$. If the equality holds, then we calculate from Eq. (9)

$$r_1 = a = \frac{b}{a},$$

where the last equality is equivalent to $b = a^2$. However, b/a maps to zero and cannot be a periodic point of ϕ; therefore, the strict inequality $a^2 > b$ must hold. □

In Lemma 6, we point out that $b > 0$ because it is necessary that $r_1 > 0$ in both Eqs. (8) and (9). We need one more lemma before presenting our main theorem on periodicity. The next lemma concerns solutions that contain zeros, i.e. they "pass through the origin" repeatedly. Note that such solutions cannot be represented by the mapping ϕ.

LEMMA 7 *Assume that $b > 0$ and let $\{x_n\}$ be a solution of Eq.* (1) *such that $x_k = 0$ for some $k \geq 0$ and $x_{k-1} > 0$.*

(a) *If $x_{k+2} = 0$, then $a = 0$ and $x_{k+2n-1} = b^n x_{k-1}$ for $n = 1, 2, 3, \ldots$*
(b) *If $x_{k+3} = 0$, then $a^2 = b$ and $x_{k+n} = a^{n+1} x_{k-1}$ for $n \neq 3j$, $j = 1, 2, 3, \ldots$*

Proof

(a) We have $x_{k+1} = bx_{k-1} > 0$ and $0 = x_{k+2} = ax_{k+1}$. Thus $a = 0$ and the stated formula is easily established by induction on n.
(b) Since $x_{k+3} = |ax_{k+2} - bx_{k+1}| = 0$, it follows that $ax_{k+2} = bx_{k+1} \neq 0$. In particular, $a \neq 0$ and $x_{k+2} \neq 0$. In fact,

$$x_{k+2} = ax_{k+1} = abx_{k-1},$$

so

$$0 = x_{k+3} = b|a^2 - b|x_{k-1}$$

and it follows that $b = a^2$. We may now write $x_{k+1} = a^2 x_{k-1}$ and $x_{k+2} = a^3 x_{k-1}$. Further,

$$x_{k+4} = bx_{k+2} = a^5 x_{k-1}, \quad x_{k+5} = ax_{k+4} = a^6 x_{k-1}.$$

Induction on n now completes the proof. □

THEOREM 4

(a) *Equation* (1) *has a period-2 solution if and only if*

$$b^2 - a^2 = 1 \quad \text{or equivalently} \quad b = \sqrt{a^2 + 1} \quad \text{for } a \geq 0. \tag{10}$$

Further, if $a > 0$, then the period-2 solutions are positive and confined to the pair of lines $y = r_1 x$ and $y = r_2 x$ in phase space, where r_1, r_2 are given by

$$r_1 = \frac{b-1}{a}, \quad r_2 = \frac{b+1}{a}. \tag{11}$$

On the other hand, the only period-2 solutions of Eq. (1) *that pass through the origin occur at $a = 0$ where $b = 1$.*
(b) *Equation* (1) *has a period-3 solution if and only if*

$$a^3 + ab - b^3 = 1, \quad a \geq 1. \tag{12}$$

Further, if $a > 1$, then the period-3 solutions are positive and confined to the three lines $y = r_i x$ in phase space where for $i = 1, 2, 3$, r_i are given by

$$r_1 = \frac{ab+1}{a^2+b}, \quad r_2 = \frac{b^2-a}{ab+1}, \quad r_3 = \frac{b+a^2}{b^2-a}. \tag{13}$$

*On the other hand, the only period-*3 *solutions of Eq.* (1) *that pass through the origin occur at* $a = 1$ *where* $b = 1$.

Proof

(a) First, let us assume that $a > 0$ and use Eq. (8) to compute

$$r_1 r_2 = \frac{1}{2}\left[\sqrt{a^4 + 4b^2} - a^2\right]. \tag{14}$$

Setting $r_1 r_2 = 1$ and simplifying, we obtain the quadratic curve in Eq. (10). Next, using Eq. (10) we may simplify the formulas in Eq. (8) to obtain the values in Eq. (11). For parameter values on the quadratic curve (10), if x_0/x_{-1} equals either r_1 or r_2 then $\phi(x_0/x_{-1}) = r_2$ and $\phi(x_1/x_0) = r_1$ or conversely, so the corresponding solution $\{x_n\}$ of Eq. (1) with period 2 is confined to the lines $y = r_1 x$ and $y = r_2 x$ in phase space.

Now consider the case $a = 0$ in Eq. (10). Although r_1, r_2 are not defined, it is clear that for $(a, b) = (0, 1)$ every solution of Eq. (1) has period 2 and by Lemma 7(a) these include all the period-2 solutions that pass through the origin. Since every positive period-2 solution of Eq. (1) induces a two-cycle in the ratios, it follows from Lemmas 6 and 7 that there are no other period-2 solutions of Eq. (1) than the ones already mentioned.

(b) With r_1 given by Eq. (9) and

$$r_2 = \phi_{\mathrm{L}}(r_1) = \frac{b - ar_1}{r_1}, \quad r_3 = \phi_{\mathrm{L}}(r_2) = \frac{(a^2 + b)r_1 - ab}{b - ar_1}$$

we see that

$$r_1 r_2 r_3 = (a^2 + b)r_1 - ab = \frac{1}{2}\left[a(a^2 + b) - \sqrt{a^2(a^2 + b)^2 - 4b^3}\right]. \tag{15}$$

Setting $r_1 r_2 r_3 = 1$ and re-arranging terms, gives the equation

$$\sqrt{a^2(a^2 + b)^2 - 4b^3} = a(a^2 + b) - 2, \tag{16}$$

which can hold only when (i) $a(a^2 + b) - 2 \geq 0$ or equivalently, $b \geq 2/a - a^2$ and (ii) $b < a^2$ so that the square root is real and $r_1 > 0$. The curve $2/a - a^2$ is strictly decreasing and intersects the parabola $b = a^2$ at $a = 1$. It follows that Eq. (16) holds for $a > 1$. Now, if we square both sides of Eq. (16) and simplify, we get the cubic equation in (12). This equation in turn may be used to simplify the values r_1, r_2, r_3 in Lemma 6(b) to get the values in Eq. (13). It follows from Lemma 6 that positive period-3 solutions of Eq. (1) exist if and only if the parameters a, b satisfy Eq. (12) with $a > 1$ and in this case the solution in the phase space is confined to the three straight lines mentioned in the statement of the theorem.

Next, consider $a = 1$ in Eq. (12). Here, also $b = 1$, so Lemma 7(b) shows that all period-3 solutions that pass through the origin are accounted for. The proof is now complete. □

Remarks 2

(a) As seen earlier, for $(a, b) = (1, 1)$ the period 3 solutions of Eq. (1) pass through the origin and are not positive. Hence, they cannot correspond to three-cycles of Eq. (2), a fact that is consistent with Theorem A(c). On the other hand, note that as (a, b)

approaches $(1,1)$ along the curve (12) the quantities r_1, r_2, r_3 in Eq. (13) have the following limits

$$\lim_{(a,b)\to(1,1)} r_1 = 1, \quad \lim_{(a,b)\to(1,1)} r_2 = 0, \quad \lim_{(a,b)\to(1,1)} r_3 = \infty.$$

These correspond to, respectively, the line $y = x$, the x-axis and the y-axis in the phase plane of Eq. (1) which are indeed lines containing the period-3 solutions of Theorem A(b) in the phase plane. Therefore, the non-positive period-3 solutions of Eq. (1) may be interpreted as limiting values of the positive period-3 solutions as (a,b) approaches $(1,1)$ along the curve (14). Also see Corollary 3 below.

(b) No period-2 or period-3 solution of Eq. (1) is stable, whether asymptotically or structurally. Theorem 4 shows the structural instability; as for asymptotic instability, recall that since a p-cycle $\{r_1, \ldots, r_p\}$ of Eq. (2) does not contain the minimum point of ϕ, it is *unstable* if

$$1 < \prod_{i=1}^{p} |\phi'(r_i)| = \prod_{i=1}^{p} \frac{b}{r_i^2} = \frac{b^p}{\prod_{i=1}^{p} r_i^2}$$

that is, if

$$\prod_{i=1}^{p} r_i < b^{p/2}.$$

If $p = 2$ and $a > 0$, then $r_1 r_2 < (a^2 + 2b - a^2)/2 = b$ by Eq. (14), so the positive two-periodic solutions are not asymptotically stable. The same conclusion clearly holds when $a = 0$ and $b = 1$. For $p = 3$ and $a > 1$ we have from Eq. (15) that $r_1 r_2 r_3 < b^{3/2}$ if and only if

$$a(a^2 + b) - \sqrt{a^2(a^2 + b)^2 - 4b^3} < 2b^{3/2}.$$

The preceding inequality reduces to $2b^{3/2} < a(a^2 + b)$ which is true if r_1 is real. Hence, positive period-3 solutions of Eq. (1) are not asymptotically stable. Also, Theorem A shows that the period-3 solutions passing through the origin are unstable, although rather ironically, these are the only solutions that appear in computer simulations because of their rationality!

We can now state the following result concerning the bifurcations of solutions of Eq. (1) at the point $(a,b) = (1,1)$.

COROLLARY 2 *Every neighborhood of* $(1,1)$ *in the parameter plane contains a pair* (a,b) *for which one of the following is true:*

(a) *Every solution of Eq.* (1) *converges to zero;*
(b) *There are unbounded solutions of Eq.* (1) *that are equal to zero infinitely often;*
(c) *There are positive solutions of Eq.* (1) *that are 3-periodic* (*hence also bounded but not convergent*).

Proof

(a) For each (a,b) in the open unit square every solution converges to zero by Theorem 1.
(b) On the parabola $b = a^2$ which contains $(1,1)$, Lemma 7 shows that the desired solutions exist for every value $a > 1$. In fact, Lemma 7 shows that as (a,b) passes $(1,1)$ on

the curve $b = a^2$, solutions of Eq. (1) change qualitatively from bounded and convergent to unbounded and non-convergent, with an "uneasy compromise" taking place at $(1, 1)$.

(c) For each (a, b) on the cubic curve $a^3 + ab - b^3 = 1$ which contains $(1, 1)$, Theorem 4 establishes the existence of positive period-3 solutions for every value $a > 1$. □

As the preceding results show, periodic ratios are useful in answering certain questions about the dynamics of Eq. (1). The next result is another example of this usage.

Theorem 5 *Let* $b > a^2/4$.

(a) *If* $b \leq (a^2 + a)/(a + 2)$ *then there are solutions of Eq.* (1) *that converge to zero eventually monotonically.*
(b) *If* $(a^2 + a)/(a + 2) < b < \sqrt{a^2 + 1}$ *then for all* $a \geq 0$ *there are solutions of Eq.* (1) *that converge to zero in an oscillatory fashion.*
(c) *If* $b > \sqrt{a^2 + 1}$ *then for all* $a \geq 0$ *there are solutions of Eq.* (1) *that are oscillatory and unbounded. However, if* $b < a + 1$, *then there are also solutions that converge to zero monotonically* (*eventually monotonically if* $b < 2a^2$).
(d) *If* $b = a$, *then for all* $a > 0$ *there are solutions of Eq.* (1) *that converge to zero in an oscillatory fashion as well as solutions that converge to zero eventually monotonically.*

Proof (a) and (b) Setting $r_2 \leq 1$ in Eq. (8) and simplifying gives

$$b \leq \frac{a^2 + a}{a + 2} \tag{17}$$

with equality holding if and only if $r_2 = 1$. Using Eq. (10) we find that $r_1 r_2 < 1$ if and only if $b < \sqrt{a^2 + 1}$. Further, for $a > 0$, we have $\sqrt{a^2 + 1} > a$ whereas $(a^2 + a)/(a + 2) < a$. It follows that the region in Part (b) is nonempty and in there, $r_1 r_2 < 1$ but $r_2 > 1$. Hence, if e.g. $x_0/x_{-1} = r_1$ then the ratio sequence is 2-periodic. In this case, $x_1 = r_2 x_0 > x_0$ while $x_2 = r_1 r_2 x_0 < x_0$. Inductively, for all $n \geq 1$, it follows that

$$x_{2n-1} > r_2 r_1 x_{2n-1} = r_2 x_{2n} = x_{2n+1} > x_{2n} > r_1 r_2 x_{2n} = x_{2n+2}.$$

It is clear that $\{x_n\}$ in this case converges to zero in an oscillatory fashion. This proves statement (b); we also note that the region of the parameter plane in this case contains a part of the unit square. On the other hand, if Eq. (17) holds, then both r_2 and $r_1 r_2$ are less than unity and the solution $\{x_n\}$ with $x_0/x_{-1} = r_1$ converges to zero monotonically. This concludes the proof of Part (a).

(c) In this case $r_1 r_2 > 1$, so solutions of Eq. (1) with $x_0/x_{-1} = r_1$ are unbounded and they oscillate because $r_1 < 1$. In fact, this last inequality holds provided that $b > (a^2 - a)/(2a - 1)$ which is true because

$$(a^2 - a)/(2a - 1) = a[(a - 1)/(2a - 1)] < a < \sqrt{a^2 + 1} < b.$$

If $b < a + 1$, then by Lemma 5, $\bar{r} < 1$ so if $x_0 = \bar{r} x_{-1}$ then $x_n = \bar{r} x_{n-1}$ for all n and $\{x_n\}$ converges to zero monotonically. If also $b < 2a^2$ (possible for $a > [(1 + \sqrt{17})/8]^{1/2}$) then we find that $\bar{r} < a$ so that $\phi^{-1}(\bar{r})$ is not empty. In this case, we can find solutions $\{x_n\}$ of Eq. (1) where there is $k \geq 0$ such that $x_k = \bar{r} x_{k-1}$ and thus $\{x_n\}$ converges to zero eventually monotonically. Note that these conclusions extend similar ones made in Theorem 3(b) for the region $b \leq a^2/4$.

(d) The line $b = a$ in the parameter space lies within the region in Part (b) so there are solutions of Eq. (1) in this case that converge to zero in an oscillatory fashion. Further, $b < a + 1$ so $\bar{r} < 1$ and if $\{x_n\}$ is a solution of Eq. (1) with x_0/x_{-1} in the set B of backward iterates of $\bar{r}$ under ϕ, then $x_n = \bar{r}x_{n-1}$ for all sufficiently large n. Clearly, such $\{x_n\}$ converges to zero eventually monotonically. □

If we raise the lower limit in Theorem 5(c) to $b > a + 1$ the following stronger conclusion is obtained.

THEOREM 6 *Let $b > a^2/4$. If $b > a + 1$, then every solution of Eq.* (1) *is unbounded. If a solution $\{x_n\}$ has the property that $x_k \leq x_{k-1}$ for some $k \geq 0$ then $\{x_n\}$ is unbounded and oscillatory. There are also unbounded solutions of Eq.* (1) *that are monotonically increasing.*

Proof Since the case $a = 0$ is discussed in Lemma 7, we assume from here on that $a > 0$. Define $c = b - a$ and note that $c > 1$. Let $\{x_n\}$ be a non-zero solution of Eq. (1) where there is $k \geq 0$ such that $x_k \leq x_{k-1}$. Then

$$x_{k+1} = |bx_{k-1} - ax_k| = |cx_{k-1} - a(x_k - x_{k-1})| \geq cx_{k-1} \geq cx_k > x_k. \tag{18}$$

Notice that a decrease is always followed by an increase in such a way that the post-decrease high value $x_{k+1} \geq cx_{k-1}$ with x_{k-1} representing the pre-decrease high.

Next, $x_{k+2} = |cx_k - a(x_{k+1} - x_k)|$ and here are two possible cases: If

$$x_{k+2} = cx_k - a(x_{k+1} - x_k) \tag{19}$$

then clearly $x_{k+2} < cx_k \leq cx_{k-1} \leq x_{k+1}$ which implies that

$$x_{k+3} = |cx_{k+1} - a(x_{k+2} - x_{k+1})| > cx_{k+1} > x_{k+1} > x_{k+2}.$$

Therefore, the situation in Eq. (18) is repeated with the new low and high values x_{k+2} and x_{k+3}. If Eq. (19) does not hold, then its negative holds for possibly more than one index value:

$$x_{k+j} = a(x_{k+j-1} - x_{k+j-2}) - cx_{k+j-2} = ax_{k+j-1} - bx_{k+j-2}, \quad j = 2, 3, 4.... \tag{20}$$

Equation (20) is linear with complex eigenvalues since $b > a^2/4$. If $\lambda^{\pm}$ are these eigenvalues, then $|\lambda^{\pm}| = \sqrt{b} > 1$ if $b > a + 1$. It follows that there is an integer $j_{\max}(a, b) \geq 2$ such that

$$x_{k+j_{\max}+1} \leq x_{k+j_{\max}}$$

and once again a situation analogous to that in Eq. (19) is obtained. We have shown that after at most a finite number of terms (determined only by the values of a, b) there will be a drop in the value of x_n for all n, i.e. $\{x_n\}$ is oscillatory. Further, if $\{n_m\}$ where $m = 0, 1, 2, \ldots$ is the sequence of indicies at which drops in x_n occur, then $x_{-1+n_m} \geq x_{1+n_{m-1}}$, so from the preceding arguments and Eq. (18) it may be concluded that

$$x_{1+n_m} \geq cx_{-1+n_m} \geq cx_{1+n_{m-1}}.$$

Therefore, inductively

$$x_{1+n_m} \geq cx_{1+n_{m-1}} > c^2 x_{1+n_{m-2}} > \cdots > c^m x_{1+n_0}$$

from which we may conclude that $\{x_n\}$ is unbounded.

Finally, if $b > a + 1$ then any strictly increasing solution of Eq. (1) must approach infinity, since the origin is the only possible fixed point of Eq. (1). Such strictly increasing solutions do exist; for example, since $\bar{r} > 1$ by Lemma 5, a solution $\{x_n\}$ with $x_0/x_{-1} = \bar{r}$ converges to infinity monotonically.

We note here that if $a+1 < b \leq a^2/4$, then by Theorem 3 the solutions of Eq. (1) are generally unbounded, approaching infinity eventually monotonically. □

The special role of period 3 in the Li-Yorke Theorem in Ref. [4] gives the following result on the complex nature of the mapping ϕ. The significant consequence of the following corollary and its extensions is that regardless of whether a solution $\{x_n\}$ of Eq. (1) converges or not, it may oscillate in a complicated way.

COROLLARY 3 *Let $b \leq a^2$. Then the mapping ϕ is chaotic in the sense of* [4]*; i.e. it has an uncountable set S called a scrambled set with the following properties:*

(i) *S contains no periodic points of ϕ and $\phi(S) \subset S$;*
(ii) *For every $x, y \in S$ and $x \neq y$,*

$$\limsup_{k\to\infty} \|\phi^k(x) - \phi^k(y)\| > 0, \quad \liminf_{k\to\infty} \|\phi^k(x) - \phi^k(y)\| = 0.$$

(iii) *For every $x \in S$ and periodic y,*

$$\limsup_{k\to\infty} \|\phi^k(x) - \phi^k(y)\| > 0.$$

Proof Let $[0,\infty]$ be the one-point compactification of the closed half-line $[0,\infty)$ and extend ϕ continuously to $[0,\infty]$ as follows:

$$\phi^*(0) = \infty, \quad \phi^*(\infty) = a, \quad \phi^*(r) = \phi(r) \quad \text{if } r \in (0,\infty).$$

Then ϕ^* defines a continuous dynamical system on the compact interval $[0,\infty]$. Now, if $b < a^2$ then by Lemma 6(b) ϕ has a three-cycle, which is also clearly a three-cycle for ϕ^*. Also if $b = a^2$ then since

$$a = \frac{b}{a} \xrightarrow{\phi} 0 \xrightarrow{\phi^*} \infty \xrightarrow{\phi^*} a$$

it follows that ϕ^* again has a three-cycle, namely, $\{a, 0, \infty\}$. In either case, since $[0,\infty]$ is homeomorphic to $[0,1]$, by the Li-Yorke theorem ϕ^* has a scrambled set $S^* \subset [0,\infty]$. Define

$$S = S^* - \left[\{\infty\} \cup \bigcup_{n=0}^{\infty} \phi^{-n}(0)\right].$$

Since $\cup_{n=0}^{\infty}\phi^{-n}(0)$ is countable for each n, it follows that S is an uncountable subset of $(0,\infty) \cap S^*$ which satisfies conditions (i)–(iii) above because

$$\phi|_S = \phi^*|_S.$$

Hence, S is a scrambled set for ϕ as required. □

With the aid of a more general result on chaos from Ref. [5] Corollary 3 can be extended to the region $b \leq 2a^2$ where ϕ may not have a period-3 point. Indeed, it can be shown that ϕ has a snap-back repeller (in the weak or non-smooth sense—see Refs. [5,7]) if and only if $b < 2a^2$ and also when $b = 2a^2$ then ϕ^* has a snap-back repeller. We do not work out the routine details of this extension here, but refer to [7,9] for similar problems.

THE LINE *b* = 1 : PERIODICITY

By Lemma 7 and Theorem 4 in the preceding section, periodic solutions of Eq. (1) occur when $(a,b) = (0,1), (1,1)$. These points are both located on the part of the line $b = 1$ that lies inside the region $b > a^2/4$ in the parameter plane. In this section, we find that this is not a coincidence, but part of a larger picture. Let us begin with the next result, which extends Lemma 7 to periods 4,5 and provides additional information with respect to periodic solutions that pass through the origin.

LEMMA 8 *Assume that* $a, b > 0$ *and let* $\{x_n\}$ *be a solution of Eq.* (1) *such that* $x_k = 0$ *and* $x_{k-1} > 0$ *for some* $k \geq 0$.

(a) *If* $x_{k+4} = 0$ *then* $a^2 = 2b$ *and*

$$x_{k+4n+j} = b^2 x_{k+4(n-1)+j}, \quad j = 0,1,2,3, \quad n \geq 1. \tag{21}$$

In particular, $\{x_n\}$ *is* 4*-periodic if and only if* $b = 1$ *and* $a = \sqrt{2}$.

(b) *If* $x_{k+5} = 0$ *then* $a^2 = \gamma b$, $a^2 = b/\gamma$ *or* $a^2 = \gamma^2 b$ *where* $\gamma = (\sqrt{5}+1)/2$ *is the golden mean. If* $a^2 = \gamma b$ *then*

$$x_{k+5n+j} = \frac{ab^2}{\gamma^2} x_{k+5(n-1)+j}, \quad j = 0,1,2,3,4, \quad n \geq 1. \tag{22}$$

In particular, $\{x_n\}$ *is* 5*-periodic if and only if* $ab^2/\gamma^2 = 1$*; i.e.* $b = \gamma^{3/5}$ *and* $a = \gamma^{4/5}$. *If* $a^2 = b/\gamma$ *then*

$$x_{k+5n+j} = \frac{ab^2}{\gamma} x_{k+5(n-1)+j}, \quad j = 0,1,2,3,4, \quad n \geq 1. \tag{23}$$

In particular, $\{x_n\}$ *is* 5*-periodic if and only if* $ab^2/\gamma = 1$; *i.e.* $b = \gamma^{3/5}$ *and* $a = \gamma^{-1/5}$. *If* $a^2 = \gamma^2 b$ *then*

$$x_{k+5n+j} = \frac{ab^2}{\gamma} x_{k+5(n-1)+j}, \quad j = 0,1,2,3,4, \quad n \geq 1. \tag{24}$$

In particular, $\{x_n\}$ *is* 5*-periodic if and only if* $ab^2/\gamma = 1$*; i.e.* $b = 1$ *and* $a = \gamma$.

Proof In general, $x_{k+1} = bx_{k-1} > 0$ and $x_{k+2} = ax_{k+1} > 0$.

(a) From $0 = x_{k+4} = |ax_{k+3} - bx_{k+2}|$ we get

$$ax_{k+3} = bx_{k+2} = abx_{k+1}$$

i.e. $x_{k+3} = bx_{k+1}$. But $x_{k+3} = |ax_{k+2} - bx_{k+1}| = |a^2 - b|x_{k+1}$ so

$$|a^2 - b| = b.$$

For positive a, b this equation is equivalent to $a^2 = 2b$. Further, we calculate

$$x_{k+5} = bx_{k+3} = b^2 x_{k+1}, \quad x_{k+6} = ax_{k+5} = abx_{k+3} = b^2 x_{k+2}.$$

These in turn imply that

$$x_{k+7} = |ab^2 x_{k+2} - b^3 x_{k+1}| = b^2 x_{k+3},$$
$$x_{k+8} = |ab^2 x_{k+3} - b^3 x_{k+2}| = b^2 x_{k+4} = 0.$$

Now simple induction on n proves Eq. (21). The statement about 4-periodic solutions is now obvious.

(b) $x_{k+3} = |ax_{k+2} - bx_{k+1}| = |a^2 - b|x_{k+1}$ and

$$x_{k+4} = |ax_{k+3} - bx_{k+2}| = |a|a^2 - b| - ab|x_{k+1}.$$

Since $0 = x_{k+5} = |ax_{k+4} - bx_{k+3}|$ we have $ax_{k+4} = bx_{k+3}$, i.e.

$$a^2|\,|a^2 - b| - b| = b|a^2 - b|.$$

For positive a, b this equation is equivalent to one of three possible forms:

$$b^2 - a^2 b - a^4 = 0, \quad a^2 < b$$
$$b^2 + a^2 b - a^4 = 0, \quad b < a^2 < 2b$$
$$b^2 - 3a^2 b + a^4 = 0, \quad a^2 > 2b.$$

For $a^2 < b$ solving the corresponding quadratic equation for b in terms of a gives a single positive solution

$$b = \frac{\sqrt{5}+1}{2} a^2 = \gamma a^2 \quad \text{or} \quad a^2 = \frac{1}{\gamma} b.$$

For the range $b < a^2 < 2b$ solving the corresponding quadratic equation also yields a single positive solution

$$b = \frac{\sqrt{5}-1}{2} a^2 = \frac{a^2}{\gamma} \quad \text{or} \quad a^2 = \gamma b$$

and for the last range, we find two positive solutions only one of which is acceptable for $a^2 > 2b$, namely,

$$b = \frac{3-\sqrt{5}}{2} a^2 = \frac{a^2}{\gamma^2} \quad \text{or} \quad a^2 = \gamma^2 b.$$

First, let $a^2 = \gamma b$. Then

$$x_{k+6} = bx_{k+4} = ab|\,|a^2 - b| - b|x_{k+1} = \frac{ab^2}{\gamma^2} x_{k+1},$$

$$x_{k+7} = ax_{k+6} = \frac{a^2 b^2}{\gamma^2} x_{k+1} = \frac{ab^2}{\gamma^2} x_{k+2},$$

$$x_{k+8} = |ax_{k+7} - bx_{k+6}| = \frac{ab^2}{\gamma^2} |ax_{k+2} - bx_{k+1}| = \frac{ab^2}{\gamma^2} x_{k+3},$$

$$x_{k+9} = |ax_{k+8} - bx_{k+7}| = \frac{ab^2}{\gamma^2} |ax_{k+3} - bx_{k+2}| = \frac{ab^2}{\gamma^2} x_{k+4},$$

$$x_{k+10} = |ax_{k+9} - bx_{k+8}| = \frac{ab^2}{\gamma^2} |ax_{k+4} - bx_{k+3}| = \frac{ab^2}{\gamma^2} x_{k+5} = 0.$$

Now induction on n proves Eq. (22). The assertion about periodicity follows immediately. The proofs of Eqs. (23) and (24) being similar to the above, are omitted. □

For period-5 solutions of Eq. (1) that pass through the origin, Lemma 8 shows that they do not all occur at the parameter value $b = 1$; 5 is evidently the least period with this property. However, for $b = 1$ Lemmas 7 and 8 do show that eventually periodic solutions of Eq. (1) of periods 2–5 exist that pass through the origin. In fact, if a_p denotes the value of a at which a period-p solution of Eq. (1) occurs with $b = 1$, then

$$0 = a_2 < a_3 = 1 < a_4 < a_5 < 2.$$

In particular, a_p seems to be an increasing function of p and periods greater than 3 occur when $1 < a < 2$. In the remainder of this section, we see why this happens on the line $b = 1$ and further, we establish the remarkable fact that periodic solutions of Eq. (1) for all $p > 3$ exist when $1 < a < 2$ and $b = 1$. We proceed by introducing a linear equation corresponding to Eq. (1) that plays an important role in the proof of the existence of periodic solutions

$$\begin{cases} z_{n+1} = az_n - z_{n-1}, & \text{for } n = 0, 1, \ldots \\ z_{-1}, z_0 \in (0, \infty), & a \in (1, 2) \end{cases} \tag{25}$$

In particular, in the sequel we consider Eq. (25) with initial conditions

$$z_{-1} = 1 \quad \text{and} \quad z_0 = a. \tag{26}$$

Next, let $g_1, g_2, \ldots : [1, 2] \longrightarrow \mathbb{R}$ be recursively defined by

$$g_{n+1}(x) = xg_n(x) - g_{n-1}(x), \quad n = 0, 1, \ldots, \tag{27}$$

where

$$g_{-1}(x) = 1 \quad \text{and} \quad g_0(x) = x. \tag{28}$$

Then, we define two more collections of functions.

For $n = 0, 1, \ldots$, let $G_n : \mathscr{D}_n \longrightarrow \mathbb{R}$ be defined by

$$G_n(x) = \frac{g_n(x)}{g_{n-1}(x)}, \tag{29}$$

where

$$\mathscr{D}_n = \{x \in [1, 2] : \quad g_k(x) \neq 0, 0 \leq k \leq n\}. \tag{30}$$

For $n = 0, 1, \ldots$ and $\mathscr{D}_n(x)$ as defined in Eq. (30), let $H_n : \mathscr{D}_n \longrightarrow \mathbb{R}$ be defined by

$$H_n(x) = G_n(x) - \frac{1}{x}, \quad n = 0, 1, \ldots. \tag{31}$$

Observe that the particular solution $\{z_n\}_{n=-1}^{\infty}$ of Eqs. (25) and (26) is also the sequence of values $\{g_n(a)\}_{n=-1}^{\infty}$.

Lemma 9 *Consider the sequence of polynomials* $\{g_n(x)\}_{n=-1}^{\infty}$, *as defined by Eqs.* (27) *and* (28), *and the sequence of rational functions,* $\{G_n(x)\}_{n=-1}^{\infty}$, *together with their sequence of domains* $\{\mathscr{D}_n\}_{n=-1}^{\infty}$ *as defined by Eqs.* (29) *and* (30). *Then* $\{g_n(x)\}_{n=-1}^{\infty}$ *and*

$\{G_n(x)\}_{n=-1}^{\infty}$ *satisfy the following properties:*

(P1) *For all* $n \geq 0$, G_n *is strictly increasing on its domain,* $\mathscr{D}_n$.
(P2) *If, for some* $N \in \{1, 2, \ldots\}$, *there exists* $a_0^{(N)} \in [1, 2]$ *such that* $g_N(a_0^{(N)}) = 0$, *then* $a_0^{(N)}$ *is the unique zero of* $g_N(x)$ *in the interval* $[1, 2]$ *and* $g_N(x) > 0$ *for all* $x \in \left(a_0^{(N)}, 2\right]$.
(P3) *For every* $n \geq 1$, *there exists* $a_0^{(n)} \in [1, 2]$ *such that* $g_n(a_0^{(N)}) = 0$. *Furthermore,*

$$1 = a_0^{(1)} < a_0^{(2)} < \cdots < 2.$$

Proof We first establish Property (P1). For each integer $n \geq 0$, let $\mathscr{P}(n)$ be the proposition that

$$G_n(x) < G_n(y), \quad \text{for}\, x, y \in \mathscr{D}_n \,\text{and}\, x < y.$$

We will show by induction that $\mathscr{P}(n)$ is true for all $n \geq 0$. Clearly, $\mathscr{P}(0)$ is true, where, for $x, y \in \mathscr{D}_0 = [1, 2]$ and $x < y$,

$$G_0(x) = x < y = G_0(y).$$

Next suppose that $n \geq 0$ is an integer such that $\mathscr{P}(n)$ is true. We will show that $\mathscr{P}(n+1)$ is true. Now, let $x, y \in \mathscr{D}_{n+1}$ and $x < y$. Then $x, y \in \mathscr{D}_n$, since $\mathscr{D}_{n+1} \subset \mathscr{D}_n$ by Eq. (30). Thus, we have $G_n(x)$, $G_n(y) \neq 0$ and by assumption, $G_n(x) < G_n(y)$. We then can write

$$x - y < 0 < \frac{1}{G_n(x)} < y - \frac{1}{G_n(y)}.$$

It then follows that

$$x - \frac{1}{G_n(x)} < y - \frac{1}{G_n(y)},$$

which, in turn, implies that

$$\frac{x g_n(x) - g_{n-1}(x)}{g_n(x)} < \frac{y g_n(y) - g_{n-1}(y)}{g_n(y)}$$

by Eq. (29). From Eq. (27), we then have

$$\frac{g_{n+1}(x)}{g_n(x)} < \frac{g_{n+1}(y)}{g_n(y)},$$

and, so, from Eq. (29) $G_{n+1}(x) < G_{n+1}(y)$. Therefore, $\mathscr{P}(n+1)$ is true and Property (P1) is established.

To prove Property (P2), suppose that for some $N \in \{1, 2, \ldots\}$, there exists $a_0^{(N)}$, $b_0^{(N)} \in [1, 2]$ such that $g_N(a_0^{(N)}) = 0$ and $g_N(b_0^{(N)}) = 0$. Without loss of generality, we assume that $a_0^{(N)} \leq b_0^{(N)}$ Then, from Eq. (27), we have

$$g_N\left(a_0^{(N)}\right) = a_0^{(N)} g_{N-1}\left(a_0^{(N)}\right) - g_{N-2}\left(a_0^{(N)}\right) = 0$$

and

$$g_N\left(b_0^{(N)}\right) = b_0^{(N)} g_{N-1}\left(b_0^{(N)}\right) - g_{N-2}\left(b_0^{(N)}\right) = 0.$$

From these and the fact that $a_0^{(N)} \leq b_0^{(N)}$ it follows that

$$\frac{g_{N-1}\left(a_0^{(N)}\right)}{g_{N-2}\left(a_0^{(N)}\right)} \geq \frac{g_{N-1}\left(b_0^{(N)}\right)}{g_{N-2}\left(b_0^{(N)}\right)}, \quad \text{while } a_0^{(N)} \leq b_0^{(N)}. \tag{32}$$

On the other hand, note that

$$\frac{g_{N-1}(x)}{g_{N-2}(x)} = G_{N-1}(x),$$

and G_{N-1} is strictly increasing on $\mathscr{D}_{N-1}$, by Property (P1). Hence, in Eq. (32), we must have

$$a_0^{(N)} = b_0^{(N)},$$

and, so, $g_N(x)$ has a unique zero in the interval $[1, 2]$.

Moreover, we claim that $g_N(x) > 0$ for all $x \in (a_0^{(N)}, 2]$. For we have $g_N(2) > 0$ by the observation before this lemma and the fact that the particular solution of Eqs. (25) and (26) with $a = 2$ is

$$z_n = 2 + n, \quad n = -1, 0, 2, \ldots. \tag{33}$$

Hence, Property (P2) is established.

Given Properties (P1) and (P2), we can now show that Property (P3) is true. For each integer $n \geq 1$, let $\mathscr{P}(n)$ be the proposition that there exists $a_0^{(n)}, a_0^{(n+1)} \in [1, 2]$ such that for $i = n, n+1$, $g_i(a_0^{(i)}) = 0$ and $a_0^{(i)}$ is a unique zero of $g_i(x)$ in the interval [1,2] and

$$1 \leq a_0^{(n)} < a_0^{(n+1)} < 2.$$

We show by induction that $\mathscr{P}(n)$ is true for all $n \geq 1$.

Clearly, $\mathscr{P}(1)$ is true, where, in the interval [1,2], $g_1(x) = x^2 - 1$ has the unique zero $a_0^{(1)} = 1$, $g_2(x) = x^3 - 2x$ has the unique zero $a_0^{(2)} = \sqrt{2}$, and $a_0^{(1)} = 1 < a_0^{(2)} = \sqrt{2}$.

Next suppose that $n \geq 1$ is an integer such that $\mathscr{P}(n)$ is true. We will show that $\mathscr{P}(n+1)$ is true. By assumption, we have that, in the interval [1,2], $g_n(x)$ has the unique zero $a_0^{(n)}$, $g_{n+1}(x)$ has the unique zero $a_0^{(n+1)}$, and $a_0^{(n)} < a_0^{(n+1)}$. From Property (P2), we also have that $g_n(x)$, $g_{n+1}(x) > 0$ for all $x \in (a_0^{(n+1)}, 2]$. Thus, for some $a_0^{(n+1)} < \alpha \leq 2$, $g_n(x) > g_{n+1}(x)$ for all $x \in (a_0^{(n+1)}, \alpha]$. On the other hand, by the observation before this lemma and Eq. (33), we have $g_n(2) < g_{n+1}(2)$. Therefore, since $g_n(x)$ and $g_{n+1}(x)$ are continuous, there exists $\tilde{a} \in (a_0^{(n+1)}, 2)$ such that

$$g_n(\tilde{a}) = g_{n+1}(\tilde{a}). \tag{34}$$

So, we have the following:

(i) $G_{n+1}(a_0^{(n+1)}) = 0$, by Eq. (29).
(ii) $G_{n+1}(\tilde{a}) = 1$, by Eqs. (29) and (34).
(iii) $G_{n+1}(x)$ is defined and continuous on the interval $[a_0^{(n+1)}, 2]$, by Eq. (30) and Property (P2), where $a_0^{(n)}$ is the unique zero of $g_n(x)$ and $a_0^{(n)} < a_0^{(n+1)}$.

Therefore, from Statements (i), (ii) and (iii), Eqs. (30) and (31), and the fact that $1 < a_0^{(n+1)} < \tilde{a} < 2$, we have the following:

(i) $H_{n+1}\left(a_0^{(n+1)}\right) < 0$.
(ii) $H_{n+1}(\tilde{a}) > 0$, by Eqs. (29) and (34).
(iii) $H_{n+1}(x)$ is defined and continuous on the interval $[a_0^{(n+1)}, 2]$.

Hence, by the intermediate value theorem, there exists $\bar{a} \in (a_0^{(n+1)}, \tilde{a})$ such that

$$H_{n+1}(\bar{a}) = 0$$

which, in turn, implies that

$$g_{n+2}(\bar{a}) = 0 \tag{35}$$

by Eq. (27). Let $a_0^{(n+2)} = \bar{a}$. Then, by Eq. (35) and Property (P2), $a_0^{(n+2)}$ is the unique zero of $g_{n+2}(\bar{a}) = 0$ and $1 \leq a_0^{(n+1)} < a_0^{(n+2)} < 2$. So, $\mathscr{P}(n+1)$ is true and Property (P3) is established. □

LEMMA 10 *Consider the sequence of polynomials* $\{g_n(x)\}_{n=1}^{\infty}$, *as defined by Eqs.* (27) *and* (28)*, together with the sequence of their unique zeros in the interval [1,2],* $\{a_0^{(n)}\}_{n=1}^{\infty}$. *Then*

$$\lim_{n\to\infty} a_0^{(n)} = 2.$$

Proof From Lemma 9, Property (P3), and the fact that $a_0^{(2)} = \sqrt{2}$ for $g_2(x) = x^3 - 2x$, we infer that

(i) $\lim_{n\to\infty} a_0^{(n)}$ exists;
(ii) $a_0^{(n)} \leq \lim_{n\to\infty} a_0^{(n)}$ for all $n \geq -1$;
(iii) $\sqrt{2} < \lim_{n\to\infty} a_0^{(n)} \leq 2$.

Let

$$\lim_{n\to\infty} a_0^{(n)} = \mathrm{L}$$

and for the sake of contradiction, assume that $\mathrm{L} \in (\sqrt{2}, 2)$. Observe that the particular solution $\{u_n\}_{n=-1}^{\infty}$ of the initial value problem

$$\begin{cases} u_{n+1} = Lu_n - u_{n-1}, & \text{for } n = 0, 1, \ldots, \\ u_{-1} = 1, & u_0 = \mathrm{L}, \end{cases} \tag{36}$$

is also the sequence of values $\{g_n(\mathrm{L})\}_{n=-1}^{\infty}$. Also observe that the particular solution of Eq. (36) is given by

$$u_n = \mathrm{L}\cos n\theta + \frac{\mathrm{L}^2 - 2}{\sqrt{4 - \mathrm{L}^2}} \sin n\theta, \quad n \geq -1,$$

where

(i) $\frac{\sqrt{4-\mathrm{L}^2}}{\mathrm{L}} \in (0, \sqrt{3})$ for $\mathrm{L} \in (\sqrt{2}, 2) \subset (1, 2)$, so that $\theta = \arctan\left(\frac{\sqrt{4-\mathrm{L}^2}}{\mathrm{L}}\right) \in \left(0, \frac{\pi}{3}\right) \bigcup \left(\pi, \frac{4\pi}{3}\right)$
(ii) $\mathrm{L}, \frac{\mathrm{L}^2-2}{\sqrt{4-\mathrm{L}^2}} > 0$
since $\mathrm{L} \in (\sqrt{2}, 2)$. Therefore,

$$g_n(\mathrm{L}) = \mathrm{L}\cos n\theta + \frac{\mathrm{L}^2 - 2}{\sqrt{4 - \mathrm{L}^2}} \sin n\theta, \quad n \geq -1$$

and, if for some $N \geq 1$, we have $N\theta \in [\pi, 3\pi/2]$, then $g_N(\mathrm{L}) < 0$. Indeed, we have the following, given that $\theta \in (0, \pi/3) \cup (\pi, 4\pi/3)$:

(i) If $\theta \in \left(\pi, \frac{4\pi}{3}\right)$, then for $N = 1$, we have $g_N(\mathrm{L}) < 0$.
(ii) If $0 < \theta < \frac{\pi}{6}$, then there exists $k \geq 2$ such that $\frac{\pi}{3(k+1)} < \theta < \frac{\pi}{3k}$, which, in turn, implies that for $N = 3(k+1)$, we have $g_N(\mathrm{L}) < 0$.
(iii) If $\frac{\pi}{6} \leq \theta < \frac{2\pi}{9}$, then for $N = 6$, we have $g_N(\mathrm{L}) < 0$.

(iv) If $\frac{2\pi}{9} \le \theta < \frac{5\pi}{18}$, then for $N = 5$, we have $g_N(\mathrm{L}) < 0$.
(v) If $\frac{5\pi}{18} \le \theta < \frac{\pi}{3}$, then for $N = 4$, we have $g_N(\mathrm{L}) < 0$.
Hence, there exists $N \ge 1$ such that

$$g_N(\mathrm{L}) < 0,$$

a result which contradicts Property (P2) of Lemma 9 since

$$a_0^{(n)} \le \mathrm{L} \quad \text{for all } n \ge -1.$$

Therefore, our original assumption that $\mathrm{L} \in (\sqrt{2}, 2)$ is false and we have $\mathrm{L} = 2$. □

Note that Eq. (25) can be written as

$$z_{n-1} = az_n - z_{n+1} \tag{37}$$

and that with initial conditions z_{-1}, z_0, we can determine not only the "future terms", $z_1, z_2, \ldots$, but the "past terms", $z_{-2}, z_{-3}, \ldots$. We will refer to Eq. (37) as the backwards-in-time version of Eq. (25). We also note that for a difference equation

$$x_{n+1} = f(x_n, x_{n-1}), \quad n = 0, 1, \ldots, \tag{38}$$

and each integer $N \ge -1$, we refer to $\{x_n\}_{n=-1}^{N}$ as the partial solution of Eq. (38).

We now make several observations with respect to Eq. (25) and its backwards-in-time version. Although these observations might be considered trivial in nature, we list them explicitly in order to facilitate any references made to them in the proof of the main result of this paper in the next section.

Remark 3 Let $\{z_n\}_{n=-1}^{\infty}$ be a solution of Eq. (25). Let $N \ge 0$, and consider the equation

$$\begin{cases} w_{m+1} = aw_m - w_{m-1}, & \text{for } m = 0, 1, \ldots, \\ w_{-1} = z_N, w_0 = z_{N-1}, & a \in (1, 2) \end{cases} \tag{39}$$

Then the solution, $\{w_m\}_{m=-1}^{\infty}$, of Eq. (39) is equivalent to the solution, $\{z_n\}_{n=N}^{-\infty}$, of the backwards-in-time version of Eq. (25),

$$\begin{cases} z_{n-2} = az_{n-1} - z_n, & \text{for } n = N, N-1, \ldots \\ z_N = w_{-1}, z_{N-1} = w_0, & a \in (1, 2) \end{cases}$$

In particular, the partial solution $\{w_m\}_{m=-1}^{N}$ of Eq. (39) is such that

$$\{w_m\}_{m=-1}^{N} = z_N, z_{N-1}, \ldots, z_0, z_{-1}, \tag{40}$$

where

$$w_m = z_{N-m-1}, \quad \text{for } m = -1, 0, \ldots, N-1, N. \tag{41}$$

Remark 4 Let $\{\tilde{z}_n\}_{n=-1}^{\infty}$ be a particular solution of Eq. (25) with initial conditions $\tilde{z}_{-1}$ and $\tilde{z}_0$. Let $\{\bar{z}_n\}_{n=-1}^{\infty}$ be another particular solution of Eq. (25) with

$$\bar{z}_{-1} = \lambda\tilde{z}_{-1} \quad \text{and} \quad \bar{z}_0 = \lambda\tilde{z}_0.$$

Then it is easy to show that $\bar{z}_n = \lambda\tilde{z}_n$, for all $n \ge -1$.

Remark 5 Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq. (1), and suppose there exists $N \geq 1$ such that

$$ax_n - x_{n-1} \geq 0, \quad \text{for}\, n = -1, 0, \ldots, N.$$

Let $\{z_n\}_{n=-1}^{\infty}$ be a solution of Eq. (25) with

$$z_{-1} = x_{-1} \quad \text{and} \quad z_0 = x_0$$

and let $\{w_m\}_{m=-1}^{\infty}$ be a solution of Eq. (39) in which we have

$$w_{-1} = z_N \quad \text{and} \quad w_0 = z_{N-1}.$$

Then $x_n = z_n = w_{N-n-1}$, for $n = -1, 0, \ldots, N$.

Remark 6 Let $\{z_n\}_{n=-1}^{\infty}$ be a solution of Eq. (25) and let $\{w_m\}_{m=-1}^{\infty}$ be a solution of

$$w_{m+1} = aw_m - w_{m-1}, \quad \text{for}\, m = 0, 1, \ldots$$

with $w_{-1} = z_{-1}$ and $w_0 = z_0$. Then $\{w_m\}_{m=-1}^{\infty} = \{z_n\}_{n=-1}^{\infty}$.

We now state the main result of this section.

THEOREM 7 *Let $b = 1$ in Eq. (1). There exists a strictly increasing sequence of values $\{a_p\}_{p=3}^{\infty}$ with*

$$a_3 = 1 \quad \textit{and} \quad \lim_{p\to\infty} a_p = 2$$

such that for each $p = 3, 4, \ldots$, the particular solution $\{x_n\}_{n=-1}^{\infty}$ of Eq. (1) with $b = 1$, $a = a_p$ and initial values $x_{-1} = 1$, $x_0 = a_p$, is periodic with period p.

Proof The proof is a consequence of the following two lemmas. □

LEMMA 11 *Let $\tilde{a} \in [1, 2)$ and let $\{x_n\}_{n=-1}^{\infty}$ be a particular solution of Eq. (1) with $a = \tilde{a}$ and*

$$x_{-1} = 1 \quad \textit{and} \quad x_0 = \tilde{a}.$$

Suppose there exists $N \geq 0$ such that

$$x_{N+1} = 0 \quad \textit{and} \quad x_n \neq 0 \quad \text{for}\, n = -1, 0, 1, \ldots, N.$$

Then $\{x_n\}_{n=-1}^{\infty}$ is periodic with period $N + 3$.

Proof Let $\tilde{a} \in [1, 2)$ and let $\{x_n\}_{n=-1}^{\infty}$ be a particular solution of Eq. (1) with $a = \tilde{a}$ and

$$x_{-1} = 1 \quad \text{and} \quad x_0 = \tilde{a}\,.$$

Suppose there exists $N \geq 0$ such that

$$x_{N+1} = 0 \quad \text{and} \quad x_n \neq 0 \quad \text{for} \;\; n = -1, 0, 1, \ldots, N.$$

Then, from Lemma 9 and the observation preceding it, and from Property (P2), we have

$$\tilde{a}x_n - x_{n-1} > 0, \quad \text{for}\, n = 0, 1, \ldots, N - 1. \tag{42}$$

Therefore, if we let $\{z_n\}_{n=-1}^{\infty}$ be a particular solution of Eqs. (25) and (26), then it follows from Remark 6 that

$$x_{-1} = z_{-1} = 1, x_0 = z_0 = \tilde{a}, x_1 = z_1, \ldots, x_N = z_N, x_{N+1} = z_{N+1} = 0. \tag{43}$$

From Eq. (43) and the fact that $x_{N+1} = 0$, we have $x_{N-1} = \tilde{a}x_N - x_{N+1} = \tilde{a}x_N$. Let $x_N = \lambda$, where $\lambda > 0$ since $x_N > 0$. Then

$$x_N = \lambda \quad \text{and} \quad x_{N-1} = \lambda\tilde{a}, \quad \lambda > 0. \tag{44}$$

Thus, from Eqs. (43) and (44), we have

$$z_N = \lambda \quad \text{and} \quad z_{N-1} = \lambda\tilde{a}, \quad \lambda > 0. \tag{45}$$

Now, let $\{w_n\}_{n=-1}^{\infty}$ be the particular solution of

$$w_{m+1} = aw_m - w_{m-1}, \quad \text{for } m = 0, 1, \ldots$$

with $w_{-1} = \lambda$ and $w_0 = \lambda\tilde{a}$. Then we have the following:

1. From Eq. (45) and Remark 3,

$$w_{-1} = z_N, w_0 = z_{N-1}, \ldots, w_{N-1} = z_0, w_N = z_{-1}. \tag{46}$$

2. On the other hand, from Eq. (45) and Remarks 4 and 6 we have

$$w_{-1} = \lambda z_{-1}, w_0 = \lambda z_0, \ldots, w_{N-1} = \lambda z_{N-1}, w_N = \lambda z_N. \tag{47}$$

Therefore, from Eqs. (43), (45), (46) and (47), we have

$$1 = x_{-1} = z_{-1} = w_N = \lambda z_N = \lambda^2.$$

Hence, $\lambda = 1$, and this, in turn, implies that

$$x_{N-1} = \tilde{a} \quad \text{and} \quad x_N = 1.$$

Therefore,

$$x_{N+2} = |\tilde{a}x_{N+1} - x_N| = |\tilde{a} \cdot 0 - 1| = 1,$$
$$x_{N+3} = |\tilde{a}x_{N+2} - x_{N+1}| = |\tilde{a} \cdot 1 - 0| = \tilde{a}.$$

Clearly, then, $\{x_n\}_{n=-1}^{\infty}$ is periodic with period $N + 3$. □

LEMMA 12 *Let $b = 1$ in Eq.* (1). *There exists a strictly increasing sequence of values $\{a_p\}_{p=3}^{\infty}$ with*

$$a_3 = 1 \quad \textit{and} \quad \lim_{p\to\infty} a_p = 2,$$

such that for each $p = 3, 4, \ldots$, the particular solution $\{x_n\}_{n=-1}^{\infty}$ of Eq. (1) *with $b = 1$ and initial values $x_{-1} = 1$, $x_0 = a_p$, is characterized by the following:*

(i) $x_p = 0$.
(ii) $x_n \neq 0$ *for* $n = -1, 0, 1, \ldots, p - 1$.
(iii) $a_p x_n - x_{n-1} > 0$ *for* $n = -1, 0, 1, \ldots, p - 2$.

Proof The result follows from Lemma 9, Properties (P2) and (P3) and Lemma 10. □

OPEN PROBLEMS AND CONJECTURES

It is fitting to close this paper with a section on what remains to be done, which is considerable. The preceding results clearly indicate that the two-parameter absolute difference equation (1) is a simple equation exhibiting complex behavior over ranges that cover much of its parameter plane. The following is a partial list of open questions and problems:

CONJECTURE 1 *For every integer $p \geq 2$ there is an open subset* $\mathrm{U_p}$ *of the parameter plane such that for all $(a, b) \in U_p$ there are p-periodic points of ϕ.*

Lemma 6 shows Conjecture 1 to be true for $p = 2, 3$ and gives

$$U_2 = \{(a, b) : a, b > 0\}, \quad U_3 = \{(a, b) : a^2 > b > 0\}.$$

For $p = 2, 3$ the points $(a_p, 1)$ where a_p is as defined in the previous section, are on the boundaries ∂U_2, ∂U_3 respectively. Such boundaries are obviously bifurcation curves in the parameter plane, and Lemmas 6, 7 and 8 show that these boundaries are related to the existence of periodic solutions of Eq. (1) that pass through the origin. Since in the preceding section we found that *p*-periodic solutions of Eq. (1) for all values of $p \geq 2$ do occur when $b = 1$, the following problem presents itself.

Open Problem 1 *Determine the sets* $\mathrm{U_p}$ *and their boundary curves in the parameter plane for periods $p \geq 4$.*

CONJECTURE 2 *For each $p \geq 4$ the point $(a_p, 1)$ is on the boundary of U_p.*

Open Problem 2 *For $p = 6$, computations similar to those in Theorem 4 give $a_6 = \sqrt{3}$. Find the values of* $\mathrm{a_p}$ *for all $p \geq 7$, or alternatively, give a formula for* $\mathrm{a_p}$ *in terms of* p.

In general, the instability of solutions of Eq. (1) is a matter that is related to the non-smoothness of ϕ at its critical or minimum point $r = b/a$. This makes the following problem interesting.

Open Problem 3 *Determine the backward sets of* b/a *under ϕ^{-1} for $a, b > 0$; i.e. determine the sets*

$$B_{a,b} = \bigcup_{n=0}^{\infty} \phi^{-n}\left(\frac{b}{a}\right)$$

and specify parameter values a, b *for which* $\mathrm{B_{a,b}}$ *is dense in the half line* $[0, \infty)$.

By Theorem A, $B_{1,1}$ is the set of all non-negative rationals; see Ref. [9]. For additional related ideas, see Ref. [8]. It is possible to determine the backward sets for the simpler monotonic maps ϕ_{L} and ϕ_R by solving a linear second order difference equation, similarly to what was done in the proof of Lemma 4 and Theorem 7; see Ref. [2] for similar ideas. Recall also that since $\phi(b/a) = 0$, it follows that if $\{x_n\}$ is a nonzero solution of Eq. (1) that hits zero, i.e. if $x_k = 0$ for some $k \geq 1$ then the ratio $x_k/x_{k-1} \in B_{a,b}$ because $x_{k-1}/x_{k-2} = b/a$. Therefore, the existence and orbit-density of the set of solutions of Eq. (1), periodic or otherwise, that pass through the origin are issues that are closely related to the nature of the sets $B_{a,b}$.

Next, we consider chaotic solutions. In Corollary 3 a solution of Eq. (1) may have chaotic *ratios* of consecutive terms without itself being chaotic (the solution may converge or be unbounded). Hence the following problem is of evident interest.

Open Problem 4 *Find parameter values/ranges for which all solutions of Eq. (1) are* bounded. *These parameter values evidently include the unit square* $[0, 1]^2$. *Among parameter values that generate bounded solutions, determine those for which Eq. (1) has chaotic solutions or show that there are no chaotic solutions for Eq. (1).*

There are other regions and curves of interest in the parameter plane. The following are samples.

CONJECTURE 3 *If* $a = b > 1$, *then except for a countable set of initial conditions, all solutions of Eq.* (1) *are unbounded and oscillatory.*

CONJECTURE 4 *If* $1 < a < 2$ *and* $a^2/4 < b < 1$ *then every solution of Eq.* (1) *converges to zero.*

References

[1] S. N. Elaydi, An Introduction to Difference Equations, 2nd Ed., Springer, New York, 1999.
[2] E. Grove, G. Ladas, L. C. McGrath and C. T. Teixeira, Existence and behavior of solutions of a rational system, *Comm. Appl. Nonlinear Anal.* **8** (2001), 1–25.
[3] M. R. S. Kulenovic and G. Ladas, Dynamics of Second Order Rational Difference Equations with Open Problems and Conjectures, CRC Press, 2001.
[4] T.-Y. Li and J. A. Yorke, Period three implies chaos, *Am. Math. Monthly* **82** (1975), 985–992.
[5] F. R. Marotto, Snap-back repellers imply chaos in $\mathbb{R}^n$, *J. Math. Anal. Appl.* **63** (1978), 199–223.
[6] H. Sedaghat, Nonlinear Difference Equations: Theory with Applications to Social Science Models, Kluwer Academic Publishers, Dordrecht, 2003.
[7] H. Sedaghat, The global stability of equilibrium in a nonlinear second-order difference equation, *Int. J. Pure Appl. Math.* **8** (2003), 209–223.
[8] H. Sedaghat, Existence of solutions for certain singular difference equations, *J. Differ. Eqs. Appl.* **6** (2000), 535–562.
[9] H. Sedaghat, Periodicity and convergence in $x_{n+1} = |x_n - x_{n-1}|$, *J. Math. Anal. Appl.* **291** (2004), 31–39.