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Journal of Difference Equations and Applications

ISSN: 1023-6198 (Print) 1563-5120 (Online) Journal homepage: http://www.tandfonline.com/loi/gdea20

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Journal of Difference Equations and Applications,
Vol. 12, No. 2, February 2006, 109–123

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Asymptotic stability for difference equations with decreasing arguments

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(Received 12 September 2005; in final form 26 October 2005)

We consider general, higher order difference equations of type

$$x_n = f(x_{n-1}, \ldots, x_{n-m})$$

in which the function f is non-increasing in each coordinate. We obtain sufficient conditions for the asymptotic stability of a unique fixed point relative to an invariant interval. We also discuss various applications of our main results.

Keywords: Asymptotic stability; Decreasing arguments; Invariant interval; Monotonic

AMS Subject Classification: 39A10; 39A11

1. Introduction

Consider the higher order difference equation

$$x_n = f(x_{n-1}, \ldots, x_{n-m}), \quad n = 1, 2, \ldots \tag{1}$$

where m is a non-negative integer and $f : \mathbb{R}^m \to \mathbb{R}$ is a given function. In the literature on difference equations and their applications, problems involving the asymptotic stability of fixed points of equation (1) in the case in which f is monotonic (non-increasing or non-decreasing) in each of its arguments or coordinates arise frequently. In the sections that follow our main results, we discuss several examples along with appropriate references.

For the non-decreasing case the following was established in [10] (also see [13] or [18] for proofs).

Theorem A *In equation* (1) *assume that* $f : I^m \to I$ *is continuous and non-decreasing in each coordinate* (*with all other coordinates kept fixed*), *where* I *is a nontrivial interval in* $\mathbb{R}$. *If the function* $g(u) = f(u, \ldots, u)$ *has a fixed point* $x^* \in I$ *and*

$$g(u) > u \quad if \quad u < x^*, \quad g(u) < u \quad if \quad u > x^*, u \in I \tag{2}$$

then x^* *attracts all solutions of equation* (1) *with initial values in I.*

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Journal of Difference Equations and Applications
ISSN 1023-6198 print/ISSN 1563-5120 online © 2006 Taylor & Francis
http://www.tandf.co.uk/journals
DOI: 10.1080/10236190500438357

There is also the following general result involving mixed monotonicity in coordinates from [9]. Though a proof of this result is not given in [9], the argument seems to be similar to those given for the second-order case ($m = 2$) in [14].

THEOREM B *In equation* (1) *assume that* $f : [a,b]^m \to [a,b]$ *is continuous and satisfies the following conditions:*

(i) *For each* $i \in \{1.,\ldots,m\}$ *the function* $f(u_1,\ldots,u_m)$ *is monotone in the coordinate* u_i (*with all other coordinates kept fixed*);

(ii) *If* (μ,ν) *is a solution of the system*

$$f(\mu_1,\mu_2,\ldots,\mu_m) = \mu$$
$$f(\nu_1,\nu_2,\ldots,\nu_m) = \nu$$

then

$$\mu = \nu$$

where for $i \in \{1,\ldots,m\}$ *we set*

$$\mu_i = \begin{cases} \mu & \text{if } f \text{ is non-decreasing in } u_i \\ \nu & \text{if } f \text{ is non-increasing in } u_i \end{cases}$$

and

$$\nu_i = \begin{cases} \nu & \text{if } f \text{ is non-decreasing in } u_i \\ \mu & \text{if } f \text{ is non-increasing in } u_i \end{cases}$$

Then there is a unique fixed point $x^* \in [a,b]$ *for equation* (1) *that attracts every solution of equation* (1) *with initial values in* $[a,b]$.

In this paper we consider the case where f is non-increasing in each of its arguments or coordinates over some interval. Our main result, Theorem 1, complements Theorem A and extends or complements previously known results about the non-increasing case. Theorem 1 also extends a special case of Theorem B (where f is non-increasing in all coordinates) without assuming that f is continuous, and without requiring the domain of f be a compact interval; see the Remarks following Corollary 1. Further, Theorem 1 establishes the stability of the fixed point without requiring differentiability. A practical advantage of this is that we do not need to examine the eigenvalues of possibly very large matrices.

2. The main results

We assume the following:

There exist r_0, s_0 with $-\infty \le r_0 < s_0 \le \infty$ such that:

(H1) $f(u_1,\ldots,u_m)$ is non-increasing in each $u_1,\ldots,u_m \in I_0$ where $I_0 = (r_0, s_0]$ if $s_0 < \infty$ and $I_0 = (r_0,\infty)$ otherwise;

(H2) $g(u) = f(u,\ldots,u)$ is continuous and strictly decreasing for $u \in I_0$;

(**H3**) There is $r \in [r_0, s_0)$ such that $r < g(r) \leq s_0$. If $r_0 = -\infty$ or $\lim_{t \to r_0^+} g(t) = \infty$ then we assume that $r \in (r_0, s_0)$.

LEMMA 1 *If* (**H2**) *and* (**H3**) *hold then Equation* (1) *has a unique fixed point* x^* *in the open interval* $(r, g(r))$.

Proof. Each fixed point of equation (1) is a solution of the equation $u = g(u)$. Define $h(u) = g(u) - u$ so that the fixed points of equation (1) are zeros of h. Note that $h(r) > 0$ by (**H3**) and

$$h(g(r)) = g(g(r)) - g(r) < 0$$

because g is decreasing by (**H2**). Since h is continuous and decreasing by (**H2**) it follows that h has a unique zero $x* \in (r, g(r))$. □

Next, consider the additional hypothesis:

(**H4**) There is $s \in [r, x*)$ such that $g^2(s) \geq s$, where $g^2(s) = g(g(s))$.

LEMMA 2 *Let* $I = [s, g(s)]$. *If* (**H1**)–(**H4**) *hold then* I *is an invariant interval for equation* (1) *and* $x* \in I$.

Proof. Note that by (**H2**)–(**H4**)

$$g(s) > g(x^*) = x^* > s$$

so that $x* \in I$. Let $x_0, x_{-1}, \ldots, x_{-m+1} \in I$. Then by (**H1**)

$$x_1 = f(x_0, x_{-1}, \ldots, x_{-m+1}) \leq f(s, s, \ldots, s) = g(s)$$

and also by (**H1**) and (**H4**)

$$x_1 = f(x_0, x_{-1}, \ldots, x_{-m+1}) \geq f(g(s), g(s), \ldots, g(s)) = g^2(s) \geq s.$$

Thus $x_1 \in I$. Now inductively assume that $x_1, \ldots, x_k \in I$ for some $k \geq 1$. Then an argument similar to the one above for x_1 shows that $x_{k+1} \in I$. Hence I is invariant. □

We now introduce the following stronger version of (**H4**).

(**H5**) There is $s \in [r, x*)$ such that $g^2(u) > u$ for all $u \in (s, x*)$.

THEOREM 1 *If* (**H1**)–(**H3**) *and* (**H5**) *hold then* $x*$ *is stable and attracts all solutions of equation* (1) *with initial values in* $(s, g(s))$.

Proof. First we establish the attracting nature of $x*$. Let $x_0, \ldots, x_{-m+1}$ be in the interval $(s, g(s))$, and define

$$\mu_1 = \min\{x^*, x_0, \ldots, x_{-m+1}\}, \quad \mu_2 = \max\{x^*, x_0, \ldots, x_{-m+1}\}.$$

Since g is continuous, we have $g(u) \to g(s)$ as $u \to s$. Thus we can find $q \in (s, \mu_1)$ sufficiently close to s that $g(q) \in (\mu_2, g(s))$. Next, observe that since $x_0, \ldots, x_{-m+1} > q$,

$$x_1 = f(x_0, x_{-1}, \ldots, x_{-m+1}) < f(q, q, \ldots, q) = g(q)$$

Similarly, $x_0, \ldots, x_{-m+1} < g(q)$ implies

$$x_1 = f(x_0, x_{-1}, \ldots, x_{-m+1}) \geq f(g(q), g(q), \ldots, g(q)) = g^2(q).$$

If (**H5**) holds, then $g^2(q) > q$ so that $x_1 \in (g^2(q), g(q)) \subset (q, g(q))$. Repeating a similar calculation for $x_2, \ldots, x_m$ we conclude that

$$x_k \in (g^2(q), g(q)) \subset (q, g(q)), k = 1, \ldots, m. \tag{3}$$

Next, we move on to the next cycle and look at x_{m+1}. Since by equation (3) $x_1, \ldots, x_m > g^2(q)$,

$$x_{m+1} = f(x_m, \ldots, x_1) < f(g^2(q), \ldots, g^2(q)) = g^3(q);$$

further, $x_1, \ldots, x_m < g(q)$ gives

$$x_{m+1} = f(x_m, \ldots, x_1) > f(g(q), g(q), \ldots, g(q)) = g^2(q).$$

Since by (**H5**) $g^3(q) < g(q)$, this argument can be repeated for $x_{m+2}, \ldots, x_{2m}$ to yield

$$x_k \in (g^2(q), g^3(q)) \subset (g^2(q), g(q)), \quad k = m+1, \ldots, 2m.$$

Continuing this argument inductively leads to the conclusion that

$$\begin{aligned} &x_k \in (g^{2n}(q), g^{2n-1}(q)), \quad k = m(2n-2)+1, \ldots, m(2n-1) \\ &x_k \in (g^{2n}(q), g^{2n+1}(q)), \quad k = m(2n-1)+1, \ldots, 2mn. \end{aligned} \tag{4}$$

From these relations and the following claim it is easy to see that x_k converges to x^* as $k \to \infty$.

Claim *For every $x_0 \in (s, x^*)$*

$$s < x_0 < g^2(x_0) < \cdots < x^* < \cdots < g^3(x_0) < g(x_0) < g(s) \tag{5}$$

and

$$\lim_{n\to\infty} g^{2n}(x_0) = \lim_{n\to\infty} g^{2n+1}(x_0) = x^*. \tag{6}$$

To prove the claim, note that g is decreasing, so if $x_0 \in (s, x^*)$ then $g(x_0) > g(x^*) = x^*$ and $g(x_0) < g(s)$. Thus

$$x^* < g(x_0) < g(s). \tag{7}$$

Applying g to equation (7) in the above fashion gives

$$g^2(s) < g^2(x_0) < x^*.$$

Now equation (5) follows by simple induction. Statements (6) follow from (5) because g has no fixed points in $(s, g(s))$ other than x^* to which the odd and even iterates of g can converge. The claim is proved.

It remains to show that $x*$ is stable (dynamically in the sense of Liapunov). Let $\varepsilon > 0$ be such that $(x* - \varepsilon, x* + \varepsilon) \subset (s, g(s))$ and use the continuity of g to pick $\delta \in (0, \varepsilon)$ small enough that $g(x* - \delta) < x* + \varepsilon$. If $x_0, \ldots, x_{-m+1} \in (x* - \delta, x* + \delta)$ then it follows from equations (5) and (4) that

$$x_k \in (x* - \delta, g(x* - \delta)) \subset (x* - \varepsilon, x* + \varepsilon), \quad k \geq 1.$$

Hence $x*$ is stable. □

Condition (**H5**) is equivalent to $x*$ being an asymptotically stable fixed point of the function g relative to the interval $(s, g(s))$; see Theorem 2.1.2 in [18]. Hence, Theorem 1 may alternatively be stated as follows.

THEOREM 2 *If* (**H1**)–(**H3**) hold then $x*$ is an asymptotically stable fixed point of equation (1) if it is an asymptotically stable fixed point of the mapping g.

An obvious corollary of Theorem 2 is the following.

COROLLARY 1 *Let* (**H1**)–(**H3**) *hold and assume that g is continuously differentiable with* $g'(x*) > -1$. *Then $x*$ is an asymptotically stable fixed point of equation* (1).

Remarks.

(i) The converse of Theorem 2 (or of Corollary 1) is not true; see, e.g. the Remarks following Corollary 2 below for an example where equation (1) is known to have an asymptotically stable fixed point $x*$ but $g'(x*) = -1$.
(ii) By way of analogy, we note that since g in Theorem A is non-decreasing on I, conditions (2) are equivalent to $x*$ being an asymptotically stable fixed point of g; see [18].
(iii) If f is continuous and (**H5**) holds, then the attractivity portion of Theorem 1 follows immediately from Theorem B. By (**H5**), the system in Theorem B(ii), namely,

$$f(\nu, \nu, \ldots, \nu) = g(\nu) = \mu,$$
$$f(\mu, \mu, \ldots, \mu) = g(\mu) = \nu$$

can have only one solution within the invariant interval $[s, g(s)]$, namely, $x*$ so that $\mu = x* = \nu$. If (**H4**) holds but not (**H5**), then the smallest invariant interval containing $x*$ is the one bounded by the points s and $g(s)$ of the smallest 2-cycle. But for this invariant interval, the system of Theorem B yields distinct numbers $\mu = s$ and $\nu = g(s)$ so Theorem B does not apply in this case, as might be expected.

It may be pointed out that (**H5**) may hold even if f is not continuous on $[s, g(s)]^m$. In that case, Theorem B is not applicable but Theorem 1 can still be applied. As an example, let g be continuous and strictly decreasing on $(-\infty, \infty)$ and suppose that g has a fixed point $x*$ such that $g(u) > x*$ if $u < x*$ (note that $x*$ is necessarily unique). Now define f as follows:

$$f(u_1, u_2) = \begin{cases} g(x*), & \text{if } u_1 > x* \text{ and } u_2 < x* \\ g(u_2), & \text{otherwise} \end{cases}$$

Then f has the following properties:

$f(u, u) = g(u)$ when $u_1 = u_2 = u$;
x^* is the only fixed point of f;
f is non-increasing on $(-\infty,\infty)^2$;
f is not continuous at points (x^*, u_2) for all $u_2 < x^*$;
f is continuous at (x^*, x^*) but it has points of discontinuity in every neighborhood of (x^*, x^*).

If (**H5**) holds for g, then Theorem 1 shows that the fixed point x^* is asymptotically stable for the difference equation

$$x_n = f(x_{n-1}, x_{n-2}).$$

However, Theorem B cannot be applied to this equation.

3. Rational difference equations

Difference equations that involve rational functions are among the most persistently studied among non-linear difference equations. Theorem 1 can be used in the study of asymptotic stability in certain difference equations of this type regardless of their order. In this section, we apply the results of the preceding section to establish the global asymptotic stability of the positive fixed points of the following difference equations:

$$x_n = \left(\sum_{i=1}^{m} \frac{a_i}{x_{n-i}^{p_i}}\right)^p, \quad n = 1, 2, \ldots \tag{8}$$

and

$$x_n = \frac{1}{\left[\sum_{i=1}^{m} a_i x_{n-i}^{p_i}\right]^p} \tag{9}$$

where for $i = 1, \ldots, m$,

$$x_{-i+1},\ p > 0, a_i,\ p_i \geq 0,\ \sum_{i=1}^{m} a_i > 0. \tag{10}$$

Note that the substitution

$$y_n = \frac{1}{x_n} \tag{11}$$

transforms (9) into (8), and conversely. The asymptotic stability of the unique positive equilibrium of a special case of equation (9) in which $p_i = 1$ and $a_i > 0$ for all i is proved in [1]; also see [12, Section 3.3]. We need only equation (8) here. Equations similar to (8) have been studied extensively in the literature. For instance, results concerning the global asymptotic stability of the positive fixed point of equations of this type (with $p = 1$) appear in [7,13,15–17,19]. In [17] a particularly detailed account of equation (8) and its non-autonomous extension is given for $p = 1$ using general methods applicable in metric spaces. In [16] the case with all powers $p_i = 1$ is studied with the aid of the "full limiting sequences" method discussed in [12]. Equation (11) in particular relates the work in [16] to that in [1] and indicates that the hypothesis $a_i > 0$ for all i in [1] is essential in this case; see the remarks following the next corollary.

In addition to extending existing published work to a more general combination of powers, our next result presents a new and completely different approach to the study of asymptotic stability in equation (8) that is based on Theorem 1 above.

COROLLARY 2 *Assume that equation* (10) *holds and further, we have*

(a) $a_q > 0$ *and* $0 < p_q p < 1$ *for at least one* $q \in \{1, 2, \ldots, m\}$,
(b) $p_i p \leq 1$ *for* $i = 1, \ldots, m$.

Then equation (8) *has a unique positive fixed point* $x*$ *which is stable and attracts all positive solutions of equation* (8).

Proof. Define the functions

$$f\left(u_1, u_2, \ldots, u_m\right) \stackrel{\text{def}}{=} \left(\sum_{i=1}^{m} \frac{a_i}{u_i^{p_i}}\right)^p$$

and

$$g(u) \stackrel{\text{def}}{=} f(u, u, \ldots, u) = \left(\sum_{i=1}^{m} \frac{a_i}{u^{p_i}}\right)^p.$$

We show that f and g defined here satisfy Hypotheses (**H1**)–(**H5**). Clearly $f\left(u_1, u_2, \ldots, u_m\right)$ is non-increasing in each of its coordinates with positive values, and because of a_q, p_q, the function $g(u)$ is continuous and decreasing for $u > 0$. Therefore, f and g satisfy (**H1**) and (**H2**) respectively with $r_0 = 0$ and $s_0 = \infty$.

Next, since

$$\lim_{x \to 0^+} g(x) = \infty$$

we see that (**H3**) is satisfied for any value of $r = \varepsilon$ for ε sufficiently close to zero. Hence, it follows from Lemma 1 that equation (8) has a unique fixed point $x* > 0$. Note further that $g[(0, \infty)] \subset (0, \infty)$ since g is continuous and decreasing on $(0, \infty)$. It follows that g^2 is continuous and increasing on $(0, \infty)$. We now show that (**H5**) also holds. Using the definition of g above, we obtain

$$g^2(u) = \left[\sum_{i=1}^{m} \frac{a_i}{\left(\sum_{j=1}^{m} a_j u^{-p_j}\right)^{pp_i}}\right]^p.$$

To establish (**H5**) it is more convenient to consider the function G defined as

$$G(u) \stackrel{\text{def}}{=} \frac{g^2(u)}{u} = \left[\sum_{i=1}^{m} \frac{a_i}{u^{1/p}\left(\sum_{j=1}^{m} a_j u^{-p_j}\right)^{p_i p}}\right]^p.$$

We first show that G is decreasing on $(0,\infty)$. Let ϕ_i be the function in the denominator, i.e.

$$\phi_i(u) = u^{1/p}\left(\sum_{j=1}^{m} \frac{a_j}{u^{p_j}}\right)^{p_i p}, \quad i = 1,\ldots,m.$$

Then for each i, the derivative of ϕ_i after routine calculation is seen to be

$$\phi_i{}'(u) = u^{(1/p)-1}\left(\sum_{j=1}^{m} \frac{a_j}{u^{p_j}}\right)^{p_i p-1} \sum_{j=1}^{m}\left(\frac{1}{p} - p_i p_j p\right)\frac{a_j}{u^{p_j}}.$$

Condition (b) implies that $\phi_i{}'(u) \geq 0$ and by Condition (a), the qth term of the last sum above is positive for every i. Thus, $\phi_i{}'(u) > 0$ for $u > 0$ and each i. Therefore, ϕ_i is increasing for every i, which implies that G is decreasing on $(0,\infty)$. From this we may conclude that for $u \in (0, x*)$,

$$G(u) > G(x^*) = 1.$$

But for $u > 0$, $G(u) > 1$ if and only if $g^2(u) > u$ so (**H5**) is satisfied with $s = r = \varepsilon$. Thus by Theorem 1 the fixed point $x*$ is stable and attracts all solutions of equation (8) that start in $(\varepsilon, g(\varepsilon))$. Since ε was chosen arbitrarily, we may let $\varepsilon \to 0^+$ to conclude that $x*$ will attract all solutions starting in $(0,\infty)$, as required. □

Remarks

(i) Condition (a) or some other restriction is necessary in Corollary 2 for ensuring asymptotic stability. For example, consider the case where $p_m = p = 1$ and $a_i = 0$ for all $i \neq m$ (in particular, if $m = 1$) which would violate (a). In such a case (b) holds, but

$$g(u) = \frac{a_m}{u} \Rightarrow g^2(u) = u$$

and (H5) cannot hold. Indeed, in this case equation (8) reduces to

$$x_n = \frac{a_m}{x_{n-m}}$$

whose positive solutions all have period $2m$. In [15], Corollary 2 is proved (with $p = 1$) under the hypotheses $a_i > 0$ for all i and $p_i < 1$ for at least one i. They use a direct argument pertaining to equation (8) that is different from that used in the proof of Corollary 2.

(ii) Although Theorem 1 does not apply when $p_i = p = 1$ for $i = 1,\ldots,m$, it does apply when $p_i \in \{0,1\}$ for all i; i.e. in the following situation: $p = 1$ and there is a nonempty, proper subset $K \subsetneq \{1, \ldots, m\}$ such that $p_j = 0$ for all $j \in K$ with $a_k > 0$ for some $k \in K$, and $p_i = 1$ for all $i \notin K$ with $a_l > 0$ for some $l \notin K$. In this case, Condition (a) in Corollary 2 does not hold, but equation (8) reduces to

$$x_n = c + \sum_{i \notin K} \frac{a_i}{x_{n-i}} = c + \sum_{i=1}^{m} \frac{b_i}{x_{n-i}}, \quad c > 0,\ b_i = a_i \quad \text{if} \quad i \notin K \quad \text{and} \quad b_i = 0 \quad \text{if} \quad i \in K$$

to which Theorem 1 applies; see [19]. Of course, if $p_i = 1$ for all i but with $p < 1$, then Corollary 2 ensures the asymptotic stability of $x*$.

The next variation of Corollary 2 shows that under suitable hypotheses, $x*$ may be asymptotically stable if $p = 1$ and $p_i > 1$ for some i. We note the restrictions involving the coefficients a_i.

COROLLARY 3 *Assume that equation* (10) *holds and further, we have the following conditions:*

(a) $a_q, p_q > 0$ for some $q \in \{1, 2, \ldots, m\}$;
(b) $a_{\max} > 1$, where $a_{\max} \stackrel{\text{def}}{=} \max\{a_i : i = 1, 2, \ldots, m\}$
(c) If $p_{\min} \stackrel{\text{def}}{=} \min\{p_i : i = 1, 2, \ldots, m\}$ then

$$\sum_{i=1}^{m} p_i a_i < \max\left\{a_i^{\frac{1}{1+p_i}} : i = 1, 2, \ldots, m\right\}^{1+p_{\min}}.$$

Then equation (8) *has a unique positive fixed point $x*$ that is asymptotically stable.*

Proof. The existence and uniqueness of $x*$ is proved similarly to Corollary 2. The asymptotic stability follows from Corollary 1 if we show that $g'(x*) > -1$ where g is the function defined in the proof of Corollary 2 with $p = 1$. Note that

$$x* = g(x*) = \sum_{i=1}^{m} \frac{a_i}{(x*)^{p_i}}$$

so dividing by $x*$ gives

$$\sum_{i=1}^{m} \frac{a_i}{(x*)^{p_i+1}} = 1.$$

Thus each term of the above sum is bounded above by 1 which implies that

$$x* \geq a_i^{1/(p_i+1)}, i = 1, \ldots, m.$$

From this and assumption (b) it follows that

$$x^* \geq \max\left\{a_i^{\frac{1}{1+p_i}} : i = 1, 2, \ldots, m\right\} > 1.$$

For convenience, let us define

$$\mu = \max\left\{a_i^{\frac{1}{1+p_i}} : i = 1, 2, \ldots, m\right\}$$

and note that for $i = 1, \ldots, m$

$$\frac{1}{(x^*)^{p_i+1}} \leq \frac{1}{(x^*)^{p_{min}+1}} \leq \frac{1}{\mu^{p_{\min}+1}}.$$

Hence, by assumption (c),

$$g'(x^*) = -\sum_{i=1}^{m} \frac{a_i p_i}{(x^*)^{p_i+1}} \geq -\frac{1}{\mu^{p_{\min}+1}} \sum_{i=1}^{m} a_i p_i > -1$$

as required. □

For a further application of Theorems 1 and A to difference equations of rational type, see [6] where the equation

$$x_n = \frac{\alpha + \sum_{i=1}^{m} a_i x_{n-i}}{\beta + \sum_{i=1}^{m} b_i x_{n-i}}, \quad n = 1, 2, \ldots$$

is studied in some detail.

4. Models from biology and medicine

The generality of Theorem 1 makes it applicable to a wide range of applied problems. In this section, we present applications of Theorem 1 to two different equations taken from mathematical models in the fields of biology and medicine.

4.1 A single reproductive age class model

Consider a single species that has multiple age classes or stages. Assume that only one of these age classes or stages are capable of reproduction. Define x_n to be the nth age class or generation of the species. In order to predict the population of the next age class we use a Kolmogorov-type equation where the generation that is capable of reproducing is multiplied by a growth function. In particular, we will examine

$$x_n = a x_{n-k} e^{-(b_1 x_{n-1} + b_2 x_{n-2} + \ldots + b_m x_{n-m})} \tag{12}$$

where $1 \leq k \leq m$, a is a measure of the fecundity of generation $n - k$, and the coefficients b_i are a measure of how age class k consumes the available resources. It may be inferred from the context of the model that

$$a > 0, \;\; b_i \geq 0, \;\; i = 1, \ldots, m, \;\; \text{with} \quad b_k > 0. \tag{13}$$

This model limits the growth of the species by restricting the amount of resources available to the species. As the population of the species increases the growth function, g decreases toward 0. If $b_i > 0$ for all i, then each age class is competing for the available resources. Similar models have been used in [3,4,8]. We use Theorem 1 to show that equation (12) has a positive, attracting solution for certain conditions on the parameters.

COROLLARY 4 *Let $b = \sum_{i=1}^{m} b_i$ and note that $b \geq b_k > 0$ by equation* (13). *If*

$$\sum_{i=1, i\neq k}^{m} b_i < b_k \quad \text{and} \quad e^{b/b_k} < a < e^2, \tag{14}$$

then $x* = (\ln a)/b$ is a stable fixed point of equation (12) *that attracts all solutions with initial conditions in the interval $\left(1/b_k, e^{-b/b_k} a/b_k\right)$.*

Proof. Define $f(u_1, u_2, \ldots, u_m) : [0, \infty)^m \longrightarrow [0, \infty)$ as

$$f(u_1, u_2, \ldots, u_m) = a u_k e^{-(b_1 u_1 + b_2 u_2 + \ldots + b_m u_m)}$$

Then the partial derivatives of f are

$$\frac{\partial f}{\partial u_k} = (a - b_k a u_k) e^{-(b_1 u_1 + b_2 u_2 + \ldots + b_m u_m)}, \tag{15}$$

and

$$\frac{\partial f}{\partial u_i} = -b_i a u_k e^{-(b_1 u_1 + b_2 u_2 + \ldots + b_m u_m)}, \quad 1 \leq i \leq m, i \neq k. \tag{16}$$

From equations (15) and (16) it follows that f is decreasing for $u_k > 1/b_k$ and for $u_i > 0$ with $i \neq k$. In particular, Hypothesis (**H1**) is satisfied on the interval $(1/b_k, \infty)$. Next, we define

$$g(u) = a u e^{-bu}, \quad u \geq 0.$$

Solving the equation $g(u) = u$ easily gives the fixed point $x*$ of g and also of equation (12) as stated above. Note that g is not monotonic but it is decreasing beyond its maximum point at $u = 1/b$. Since $1/b_k \geq 1/b$, Hypothesis (**H2**) is satisfied on the same interval as (**H1**). Further, we may take $r_0 = r = 1/b_k$ and note that (**H3**) is satisfied because

$$g(r) = g\left(\frac{1}{b_k}\right) = \frac{a}{b_k e^{b/b_k}} > \frac{1}{b_k} = r$$

where the inequality holds by equation (14) since $a > e^{b/b_k}$. Note that this inequality also assures us that $x* > 1/b_k$. At this stage, we can apply Corollary 1. Since

$$g'(u) = a e^{-bu}(1 - bu)$$

it follows that $g'(x*) = 1 - bx* = 1 - \ln a > -1$ because $\ln a < 2$, a fact that follows from (14) where $a < e^2$. Thus by Corollary 1, $x*$ is asymptotically stable for (12). Note that the first inequality in equation (14) is necessary to assure that $b/b_k < 2$ and hence the stated range for a is not empty.

To obtain the interval of attractivity, we examine (**H5**). Let $s = 1/b_k$ and consider

$$g^2(u) = a^2 u e^{-bu(1+ae^{-bu})}, u \geq 0.$$

It is necessary to show that $g^2(u) > u$ for $u \in (s, x*)$. Let $bu = y$ so that

$$g^2\left(\frac{y}{b}\right) = \frac{a^2 y}{b e^{y(1+a/e^y)}}.$$

Now note that

$$\frac{\mathrm{d}}{\mathrm{d}y} e^{y(1+a/e^y)} = e^{y(1+a/e^y)} \left[1 + \frac{a}{e^y} - \frac{ay}{e^y}\right].$$

For $e < a < e^2$ and $1 < y < \ln(a)$, it is true that $\left[1 + a/e^y - ay/e^y\right] > 0$ since

$$\frac{\mathrm{d}}{\mathrm{d}y}\left[1 + \frac{a}{e^y} - \frac{ay}{e^y}\right] = -\frac{a(1-y)}{e^y} - \frac{a}{e^y}$$

and the right hand side above is zero when $y = 2$. This is a minimum and since $a < e^2$,

$$y < \ln(a) < 2 \text{ and} \left[1 + \frac{a}{e^y} - \frac{ay}{e^y}\right] > 0.$$

This imples that $e^{y(1+a/e^y)}$ has a positive derivative and so

$$e^{y(1+a/e^y)} < e^{\ln(a)(1+a/e^{\ln(a)})} = a^2$$

which implies that $g^2\big(y/b\big) > \big(y/b\big)a^2/a^2 = y/b$.Thus (**H5**) holds for $(s, g(s))$ where $g(s) = e^{-b/b_k} a/b_k$ and the proof is complete. □

Although the ranges on the parameters are restrictive in Corollary 4, the method is quite general and applies easily to other types of growth functions. For instance, the proof of the following result is sufficiently similar to the proof of Corollary 4 that we may omit it.

COROLLARY 5 *Assume that equation* (13) *holds and let* $b = \sum_{i=1}^m b_i$ *so that* $b \geq b_k > 0$. *If*

$$\sum_{i=1,\, i \neq k}^{m} b_i < b_k \text{ and } e^{b/2b_k} < a < e, \tag{17}$$

then $x* = \sqrt{(\ln a)/b}$ *is a stable fixed point of the difference equation*

$$x_n = a x_{n-k} e^{-(b_1 x_{n-1}^2 + b_2 x_{n-2}^2 + \ldots + b_m x_{n-m}^2)}$$

that attracts all solutions starting in the interval $\left(1/\sqrt{2b_k}, e^{-b/2b_k} a/\sqrt{2b_k}\right)$.

See [2] for additional results involving the applications of Theorems 1 and A to the more general exponential type model of population dynamics

$$x_n = (a_1 x_{n-1} + \cdots + a_m x_{n-m})\, e^{-(b_1 x_{n-1}^k + b_2 x_{n-2}^k + \ldots + b_m x_{n-m}^k)}$$

4.2 Pulse circulation in a ring of cardiac tissue

In [20] it is shown that the circulation of a reentrant action potential pulse in a homogeneous ring of cardiac tissue (or more generally, a ring of excitable media) may be modelled by the higher order non-linear difference equation

$$x_n = \sum_{i=1}^{m} C(x_{n-i}) - A(x_{n-m}) \tag{18}$$

where $A : (0, \infty) \to (0, \infty)$ is a continuous and increasing function called the "restitution of action potential" and $C : (0, \infty) \to (0, \infty)$ is a non-increasing continuous function that represents the "restitution of conduction time". The integer m represents the number of cardiac units (usually aggregates of cells) in the ring and x_n represents the "diastolic interval" or the recovery period of each cardiac unit in cycle (or beat) n. Equation (18) holds when an action potential pulse reenters the ring and causes abnormal fast beating of the heart (tachyarrhythmia) that may lead to cardiac arrest. For additional background information on the ring model and equation (18) see [5,11,20].

Equation (18) which provided the initial motivation for Theorem 1, was shown in [20] to have an asymptotically stable fixed point $x*$ under certain conditions on A and C. This result may now also be stated as a corollary of Theorem 1. We define

$$f(u_1, \ldots, u_m) = \sum_{i=1}^{m} C(u_i) - A(u_m)$$

and

$$g(u) = f(u, \ldots, u) = mC(u) - A(u).$$

The following is an immediate consequence of Theorem 1.

Corollary 6

(a) *Assume that there is* $r > 0$ *such that* $mC(r) > A(r) + r$ (i.e. $g(r) > r$). *Then equation* (18) *has a unique fixed point* $x* \in (r, g(r))$.
(b) *If the hypothesis in* (a) *is satisfied and there is* $s \in [r, x*)$ *such that* $g^2(u) > u$ *for all* $u \in (s, x*)$ *then* $x*$ *is a stable fixed point of equation* (18) *that attracts every solution starting in the inverval* $(s, g(s))$.

The physical interpretation of Corollary 6 is that the period of the circulating pulse approaches the fixed value $x*$ that in cardiac physiology is usually measured in milliseconds. Often exponential type functions are used to represent A and C in the cardiac ring literature, e.g.

$$A(t) = a - be^{-\sigma t}, \quad C(t) = c + de^{-\omega t}$$

where the parameters $a, b, c, d, \sigma, \omega > 0$. Over certain ranges of these parameters the hypotheses of Corollary 6 are satisfied and thus every solution of

$$x_n = d\sum_{i=1}^{m} e^{-\omega x_{n-i}} + be^{-\sigma x_{n-m}} + mc - a$$

that starts in a suitable invariant interval converges to a unique fixed point $x*$. See [20] for examples and more details.

5. Conclusion

The preceding corollaries indicate a range of different problems that can be studied with the aid of Theorems 1 and 2. However, it is clear that we cannot give a full indication of the range of applicability of Theorems 1 and 2 and their mirror image version Theorem A in a single paper. Other applications have appeared in the literature as sited above and additional uses of these theorems will undoubtedly be observed again. These facts also highlight the potential importance of the more recent Theorem B in future applications because of the high degree of versatility of that result.

Certainly what may be known about the one dimensional map g in Theorem A or Theorem 1 often tells us just a part of the story as far as the full behavior of the difference equation (1) is concerned. This is especially apparent in the case of equation (12) and similar equations that involve mappings that are not monotonic over their entire domains, though the non-necessity exists also in the completely monotonic cases as well. On the other hand, Theorems A, B and Theorem 1 can provide, in many cases where other tools are lacking, a solid starting point from which a deeper study of a particular equation of higher order may commence.

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