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A note: Every homogeneous difference equation of degree one admits a reduction in order

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Every difference equation $x_{n+1} = f_n(x_n, x_{n-1}, \ldots, x_{n-k})$ of order $k+1$ with each mapping f_n being homogeneous of degree 1 on a group G is shown to be equivalent to a system consisting of an equation of order k and a linear equation of order 1.

Keywords: order reduction; homogeneous; degree 1; non-autonomous

Let G be a nontrivial group and consider the non-autonomous difference equation of order $k+1$

$$x_{n+1} = f_n(x_n, x_{n-1}, \ldots, x_{n-k}), \quad n = 0, 1, 2, \ldots, \tag{1}$$

where $x_n \in G$ and $f_n : D \to G$ for every n. The domain $D \subset G^{k+1}$ is such that the projection of D into each of the $k+1$ coordinates equals G and that $F_n(D) \subset D$ i.e. D is invariant under the unfolding $F_n(u_0, \ldots, u_k) = (f_n(u_0, \ldots, u_k), u_0, \ldots, u_{k-1})$ of f_n for every n. Equation (1) recursively generates a *solution* or *orbit* $\{x_n\}_{n=-1}^{\infty}$ in G from a set of initial values $x_0, x_{-1}, \ldots, x_{-k}$ in G.

The group structure provides a suitable context for our main result, but in most applications G is a substructure of a more complex object such as a vector space or an algebra possessing a metric topology relative to which the mappings f_n are continuous. In such cases, each f_n may be defined on the ambient structure as long as the invariance condition $f_n(D) \subset G$ holds for all n.

We call a function $f : D \to G$ homogeneous of degree 1 (or HD1) on D relative to G if

$$f(tu_0, \ldots, tu_k) = tf(u_0, \ldots, u_k) \quad \text{for all } t, u_i \in G, i = 0, \ldots, k. \tag{2}$$

If each f_n in (1) is HD1 on D then we say that equation (1) is homogeneous of degree 1.

The case $k = 1$ for equation (1) is discussed in [3] where the second-order equation $x_{n+1} = f_n(x_n, x_{n-1})$ with HD1 maps f_n is seen to be equivalent to a system of two first-order equations

$$u_{n+1} = h_n(u_n), \quad v_{n+1} = v_n u_{n+1}.$$

The next theorem extends this result to an arbitrary integer $k \geq 1$. For further comments on homogeneous functions and their abundance on groups we refer to [3]; these comments extent to any number of variables.

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THEOREM. *Let G be a nontrivial group. If f_n is HD1 relative to G for $n \geq 1$ then equation* (1) *is equivalent to the following system of equations*

$$r_{n+1} = f_n(1, r_n^{-1}, (r_{n-1}r_n)^{-1}, \ldots, (r_{n-k+1}\cdots r_{n-1}r_n)^{-1})$$

$$s_{n+1} = s_n r_{n+1}.$$

Note that the first difference equation above has order k and the second is linear of order one in s_n.

Proof. For each solution $\{x_n\}_{n=-k}^{\infty}$ of (1) define $r_n = x_{n-1}^{-1}x_n$ for each $n = -k+1, -k+2, \ldots$ Then $x_{n+1} = x_n r_{n+1}$ and

$$\begin{aligned} r_{n+1} &= x_n^{-1}x_{n+1} = x_n^{-1}f_n(x_n, x_{n-1}, x_{n-2}, \ldots, x_{n-k}) = f_n\left(1, x_n^{-1}x_{n-1}, x_n^{-1}x_{n-2}, \ldots, x_n^{-1}x_{n-k}\right) \\ &= f_n\left(1, x_n^{-1}x_{n-1}, \left(x_n^{-1}x_{n-1}\right)\left(x_{n-1}^{-1}x_{n-2}\right), \ldots, \left(x_n^{-1}x_{n-1}\right)\left(x_{n-1}^{-1}x_{n-2}\right)\cdots\left(x_{n-k+1}^{-1}x_{n-k}\right)\right) \\ &= f_n\left(1, r_n^{-1}, (r_{n-1}r_n)^{-1}, \ldots, \left(r_{n-k+1}\cdots r_{n-1}r_n\right)^{-1}\right). \end{aligned}$$

It follows that $\{r_n\}_{n=-k+1}^{\infty}$ is a solution of the first equation so that $\{(r_n, s_n)\}_{n=-k+1}^{\infty}$ is a solution of the system with $s_n = x_n$ for $n = -k+1, -k+2, \ldots$

Conversely, let $\{(r_n, s_n)\}_{n=-k+1}^{\infty}$ be a solution of the system. Then, $\{r_n\}_{n=-k+1}^{\infty}$ is a solution of the first equation. Choose $x_{-k} \in G$ and set $x_n = s_n$ for $n = -k+1, -k+2, \ldots$ Then, $x_{n+1} = s_{n+1} = x_n r_{n+1}$ so that

$$\begin{aligned} x_{n+1} &= x_n f_n\left(1, r_n^{-1}, (r_{n-1}r_n)^{-1}, \ldots, (r_{n-k+1}\cdots r_{n-1}r_n)^{-1}\right) \\ &= f_n\left(x_n, x_n\left(x_{n-1}^{-1}x_n\right)^{-1}, x_n\left(x_{n-2}^{-1}x_n\right)^{-1} \ldots, x_n\left(x_{n-k}^{-1}x_n\right)^{-1}\right) \\ &= f_n(x_n, x_{n-1}, x_{n-2}, \ldots, x_{n-k}). \end{aligned}$$

It follows that the sequence $\{x_n\}_{n=-k}^{\infty}$ is a solution of (1).

Remarks 1. (a) The system in the above Theorem can be solved explicitly in terms of a solution $\{r_n\}_{n=-k+1}^{\infty}$ of the first equation as follows:

$$s_n = s_0 r_1 r_2 \cdots r_n \quad n = 1, 2, 3, \ldots \tag{3}$$

Thus for HD1 functions, the above theorem essentially reduces the study of equation (1) with order $k+1$ to that of the first equation of the system which has order k. In the additive case, (3) takes the form

$$s_n = s_0 + r_1 + r_2 + \cdots + r_n. \tag{4}$$

(b) We can quickly construct the first equation of the system in the above Theorem directly from (1) in the HD1 case by making the substitutions

$$1 \to x_n, \quad (r_{n-i+1}\cdots r_{n-1}r_n)^{-1} \to x_{n-i} \ \text{ for } \ i = 1, 2, \ldots, k. \tag{5}$$

Recall that 1 represents the group identity. In the additive case, (5) takes the form

$$0 \to x_n, \quad -r_n - r_{n-1}\cdots - r_{n-i+1} \to x_{n-i} \quad \text{for} \ \ i = 1, 2, \ldots, k. \tag{6}$$

The number of different types of HD1 equations is unlimited. Previous studies involving HD1 equations implicitly use the idea behind the theorem above to reduce second-order equations to first-order ones; see e.g. [1]. Here, we discuss a few equations with orders >2 to illustrate the theorem above and some associated concepts.

Example 1. Consider the autonomous rational difference equation

$$x_{n+1} = x_n\left(\frac{ax_{n-k+1}}{x_{n-k}} + b\right), \quad a, b > 0, \ \ a + b \neq 1. \tag{7}$$

With positive initial values, this equation is clearly HD1 relative to the multiplicative group G of positive real numbers $(0,\infty)$. The Theorem above states that equation (7) which has order $k+1$ can be reduced to an equation of order k using (5) as follows:

$$r_{n+1} = 1\left(\frac{a(r_{n-k+2}\cdots r_{n-1}r_n)^{-1}}{(r_{n-k+1}\cdots r_{n-1}r_n)^{-1}} + b\right) = ar_{n-k+1} + b. \tag{8}$$

Using the linear (non-homogeneous) equations (8) and (3) it can be shown easily that if $a+b<1$ then all solutions of equation (7) converge to zero, eventually monotonically and that if $a+b>1$ then all solutions of equation (7) converge to ∞, eventually monotonically.

Remarks 2. (a) Example 1 can be extended to the non-autonomous equation

$$x_{n+1} = x_n\left(\frac{a_n x_{n-k+1}}{x_{n-k}} + b_n\right),$$

whose order-reduced form is $r_{n+1} = a_n r_{n-k+1} + b_n$. In particular, if $a_n \to a$ and $b_n \to b$ with a, b as above, then the conclusions of example 1 are essentially unchanged.

(b) Note that an HD1 equation on a group G cannot have any isolated fixed points in G. But after reduction of order, the resulting equation is usually not HD1 and often has isolated fixed points. This is seen both in examples 1 and 2 below. Thus, the theorem above is a necessary 'starter' for analyzing HD1 equations, because conventional methods of analysis (e.g., linearization, semicycles, etc.) can often be applied only to the lower order, non-HD1 equation.

(c) System (2) is a special type of semiconjugate factorization; see [4].

Example 2. The difference equation

$$x_{n+1} = x_n + \frac{b}{a + x_{n-j} - x_{n-k}}, \quad a, b > 0, \ \ k \geq 1, \ \ 0 \leq j \leq k-1 \tag{9}$$

is HD1 relative to the additive group $\mathbb{R}$. Using (6) we order-reduce it to

$$r_{n+1} = \frac{b}{a + r_{n-j} + r_{n-j+1} + \cdots + r_{n-k+1}}, \quad r_n = x_n - x_{n-1}. \tag{10}$$

Initial values satisfying $x_0 > x_{-1} > \cdots > x_{-k}$ result in $r_0, \ldots, r_{-k+1} > 0$. This implies that $r_n > 0$ for $n \geq 1$, so the corresponding solution x_n of (9) is increasing and eventually positive since by (4) $x_n = x_0 + \sum_{j=1}^{n} r_j$. Equation (10) has known properties; substituting $t_n = b/r_n$ transforms (10) into the more familiar $t_{n+1} = a + b\sum_{i=j}^{k-1} 1/t_{n-j}$. It is shown in [2] that all positive solutions of this version of (10) converge to its unique positive fixed point $L = \left(a + \sqrt{a^2 + 4b(k-j)}\right)/2$. Thus a straightforward argument shows that $x_n/n \to b/L$ as $n \to \infty$; i.e. x_n converges to ∞ asymptotically as $(b/L)n$.

Example 3. This example illustrates a situation where (1) and its order-reduction are both HD1, although with respect to different groups. We examine the third order equation

$$x_{n+1} = x_n + \frac{a(x_n - x_{n-1})^2}{x_{n-1} - x_{n-2}}, \quad a > 0. \tag{11}$$

Relative to the additive group $\mathbb{R}$, this equation is HD1 and reducible to $r_{n+1} = ar_n^2/r_{n-1}$ with $r_n = x_n - x_{n-1}$. Note that $r_n \neq 0$ for $n \geq 1$ if initial values satisfy

$$x_0, x_{-2} \neq x_{-1}. \tag{12}$$

Relative to the multiplicative group of all nonzero real numbers, the second-order equation above is HD1 and reducible to the first-order linear equation $t_{n+1} = at_n$ with $t_n = r_n/r_{n-1}$. Now using (3) and (4) we obtain the following formula for solutions of (11) subject to (12):

$$x_n = x_0 + r_0 \sum_{k=1}^{n} t_0^k a^{k(k+1)/2}, \quad t_0 = \frac{r_0}{r_{-1}} = \frac{x_0 - x_{-1}}{x_{-1} - x_{-2}}.$$

This representation and standard analysis establish the following types of behavior for (11): given the increasing nature of x_n, if $a > 1$ then all positive solutions subject to (12) converge to ∞; if $a < 1$ then all positive solutions subject to (12) converge to a finite limit that depends on the initial values. If $a = 1$ then bounded and unbounded solutions coexist: if $x_0 + x_{-2} < 2x_{-1}$ then $\lim_{n\to\infty} x_n = x_0 + (x_0 - x_{-1})^2/(2x_{-1} - x_{-2} - x_0)$ but if $x_0 + x_{-2} \geq 2x_{-1}$ then $x_n \to \infty$ as $n \to \infty$.

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