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INVERSE MAP CHARACTERIZATION OF ASYMPTOTIC STABILITY ON THE LINE

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ABSTRACT. By comparing a continuous function f with its inverse f^{-1} in a neighborhood of an isolated fixed point $\bar{x}$, a necessary and sufficient condition for the asymptotic stability of $\bar{x}$ is obtained.

1. Introduction. The notion of asymptotic stability of a fixed point $\bar{x}$ of a continuous mapping of the real line is indeed very familiar. However, until recently, conditions that are both necessary and sufficient for the asymptotic stability of $\bar{x}$ were not known; such conditions make it possible to resolve issues about dynamic systems on the line that cannot be settled using less complete characterizations, see, e.g., [**3**] and Section 3 below.

In this paper we prove a necessary and sufficient condition for asymptotic stability in which a continuous mapping f is compared to its inverse f^{-1} in a cut-and-paste sort of way. More specifically, we prove in Theorem 1 below that $\bar{x}$ is asymptotically stable if and only if the inverse image of the part of f to the right of $\bar{x}$ lies in the region of the plane that is above both the identity line and the part of f to the left of $\bar{x}$. The resulting geometric picture that emerges is appealing both for its generality and its simplicity. Also, if the aforementioned relationship between f and its inverse holds *globally*, then our stability results will also be global; see Example 1 below. In Theorem 2 several conditions, including the recent characterization in [**4**] based on f^2, are shown to be equivalent to that in Theorem 1. In the last section of the paper, we apply these characterization theorems to some specific mappings that defy analysis by the more familiar means.

2. Characterization theorems.

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General definitions and hypotheses. In this paper I denotes a nontrivial interval of real numbers, i.e., containing more than one point though not necessarily bounded or closed, and $f : I \to I$ is a continuous mapping of I. The mapping f defines a dynamical system on I in the sense that the successive iterates $f^2 \doteq f \circ f$, $f^3 \doteq f \circ f^2$, and so on, are all defined on I and when applied to a point $x_0 \in I$, they generate the orbit $\{f^n(x_0)\}$ which represents the (discrete) time evolution of the trajectory of x_0 as if it were moving in I. For each positive integer p, a solution $\bar{x}$ of the equation $f^p(x) = x$ is a period-p point of f and the finite orbit $\{\bar{x}, f(\bar{x}), \dots, f^{p-1}(\bar{x})\}$ is called a p-cycle. If $p = 1$, then $\bar{x}$ is a fixed point of f; such a point is said to be asymptotically stable if there is some neighborhood J of $\bar{x}$ in I such that $fJ \subset J$ and the decreasing sequence of sets

$$J \supset fJ \supset f^2J \supset \cdots$$

has limit $\cap_{n=1}^{\infty} f^n J = \{\bar{x}\}$ (here J may be assumed to be a compact interval).

It is clear from the preceding definitions that an asymptotically stable fixed point $\bar{x}$ is not a cluster point of a sequence of periodic points; in particular, $\bar{x}$ is isolated, i.e., it is not a cluster point of a sequence of other fixed points. Before stating the main stability theorem, we present a few preliminary results, beginning with the following useful fact quoted from [**4**, p. 48].

Lemma 1. *If K is a nontrivial compact interval such that $fK = K$, then K contains either at least two fixed points or a fixed point and a period-2 point.*

Definitions 1. (1) Let $\bar{x}$ be a fixed point of f and, for each subset $A \subset I$, define the right and left parts of A as

$$A_r \doteq A \cap [\bar{x}, \infty), \quad A_l \doteq A \cap (-\infty, \bar{x}].$$

(2) We denote by f_r and f_l the restrictions of f to I_r and I_l, respectively. Since $f_r I_r \subset I$, the inverse map f_r^{-1} may be generally defined on I if we allow the empty set as a possible value of f_r^{-1}. With this convention, we conclude that $f_r^{-1}(x) \subset I_r$ for all $x \in I$, with a similar conclusion holding for f_l and its inverse.

Lemma 2. *Let $a_i \in I_l$, $b_i \in f_r^{-1}(a_i)$, $i = 1, 2$. If $f(a_1) \geq b_1$ and $f(a_2) \leq b_2$, then there is a c between b_1 and b_2 such that $f^2(c) = c$, i.e., the graphs of f_l and f_r^{-1} intersect at c and $\{c, f_r(c)\}$ is a 2-cycle.*

Proof. Note that $f_l \circ f_r(b_1) = f_l(a_1) \geq b_1$ while $f_l \circ f_r(b_2) = f_l(a_2) \leq b_2$. Since $f_l \circ f_r$ is continuous, there is a c between b_1 and b_2 such that $f^2(c) = f_l \circ f_r(c) = c$. Note further that $b_1, b_2, c \in I_r$, while $f_r(c) \in I_l$. So $(f_r(c), c) \in f_r^{-1} \cap f_l$. □

Definition 2. If $\bar{x}$ is an isolated fixed point, then a bounded interval $U \subset I$ is a proper I-neighborhood of $\bar{x}$ if

(i) U is open in I and contains $\bar{x}$;

(ii) $\bar{x}$ is the only fixed point of f that is contained in the closure $\bar{U}$;

(iii) If a is an endpoint of I, then $a \in \bar{U}$ if and only if $a = \bar{x}$.

Note in particular that both U_r and U_l contain $\bar{x}$ and are nonempty; also, every interval neighborhood of $\bar{x}$ contains a proper I-neighborhood.

Definition 3. Let $\bar{x}$ be an isolated fixed point of f, and let U be a proper I-neighborhood of $\bar{x}$.

1) For each $x \in U$ define the lower envelope function of f_r^{-1} on U as

$$\phi(x) \doteq \inf f_r^{-1}(x) = \inf\{u \in U_r : f_r(u) = x\}.$$

Note that $\phi(x) \geq \bar{x} = \inf U_r$ for all $x \in U$ with equality holding if and only if $x = \bar{x}$. By usual convention, $\phi(x) = \infty$ if $f_r^{-1}(x)$ is empty.

2) For each $x \in U_l$ define the upper envelope function of f_l as

$$\mu(x) \doteq \sup_{x \leq u \leq \bar{x}} f_l(u).$$

Note that μ is bounded on U_l (because U is proper) and $\mu(x) \geq f(\bar{x}) = \bar{x}$ for all $x \in U_l$.

Lemma 3. *Let U be a proper I-neighborhood of an isolated fixed point $\bar{x}$ of f.*

(a) *μ is a continuous and nonincreasing function on U_l with $\mu(x) \geq f(x)$ for all $x \in U_l$.*

(b) *If* ϕ *is real valued on* U*, then* ϕ *is a decreasing function on* U_l *and an increasing function on* U_r*.*

(c) $\phi(x) > f(x)$ *for all* $x \in U_l$*,* $x \neq \bar{x}$*, if and only if* $\phi(x) > \mu(x)$ *for all* $x \in U_l$*,* $x \neq \bar{x}$*.*

Proof. (a) Assume, for nontriviality, that U_l contains points other than $\bar{x}$. It is clear from the definition that μ is nonincreasing and dominates f on U_l. To prove μ is continuous, let $a \in U_l$, $a \neq \bar{x}$ and consider two cases.

Case 1. $\mu(a) > f(a)$, so there is a least $b \in (a, \bar{x}]$ such that $\mu(a) = f(b)$. Choose $\delta > 0$ such that $a + \delta < b$, $V = (a - \delta, a + \delta) \subset U_l$ and $f(x) < \mu(a)$ for all $x \in V$. Now let $x \in V$ and note that if $x > a$ then

$$f(b) \leq \mu(x) \leq \mu(a) = f(b)$$

while if $x < a$ then

$$\mu(x) = \sup_{a \leq u \leq \bar{x}} f(u) = \mu(a).$$

Therefore, μ is constant, hence continuous, on V.

Case 2. $\mu(a) = f(a)$. If μ is not continuous at a, let $x_n \to a$ as $n \to \infty$ and first assume, by taking a subsequence if necessary, that there is an $\varepsilon > 0$ such that $\mu(x_n) - \mu(a) \leq -\varepsilon$ for all n; but then

$$f(x_n) - f(a) \leq \mu(x_n) - \mu(a) \leq -\varepsilon$$

for every n, contradicting the continuity of f. So assume, by taking a subsequence if necessary, that $\mu(x_n) \geq \mu(a) + \varepsilon$ for all n. Since μ is nonincreasing, it follows that $x_n < a$ for all n. For each n define $y_n \in [x_n, \bar{x}]$ by the equality $f(y_n) = \mu(x_n)$, and note that

$$f(y_n) > \mu(a) > f(x)$$

for $x \in [a, \bar{x}]$ and all n. Therefore, $x_n \leq y_n \leq a$ for all n, implying that $y_n \to a$ as $n \to \infty$; however, by the definition of y_n, $f(y_n)$ is not converging to $\mu(a) = f(a)$ which once again contradicts the continuity of f. This completes the proof of assertion (a).

To prove (b), note that since the sets $f_r^{-1}(x)$ are closed, $\phi(x) \in f_r^{-1}(x)$ for all $x \in U$. Therefore, for each $x \in U$, $\phi(x)$ is the smallest number in I_r with the property that $f_r(\phi(x)) = x$. Since f_r is continuous and $f_r(\bar{x}) = \bar{x}$, the minimality of $\phi(x)$ implies that, for $x \in U_l$,

$$f_r(y) \geq x \quad \text{for } y \in [\bar{x}, \phi(x)] \tag{1}$$

with the inequality reversed for $x \in U_r$. Now, if (b) is false and there are $u, v \in U_l$, $u < v$, such that $\phi(u) \leq \phi(v)$, then $\phi(u) \in [\bar{x}, \phi(v)]$ with $f_r(\phi(u)) = u < v$ which contradicts (1). The argument for $u, v \in U_r$ is similar.

With regard to (c), necessity being clear from the definition of μ, we proceed to prove the sufficiency; i.e., if there is $u \in U_l$ such that $\phi(u) \leq \mu(u)$, then for some $v \in U_l$, $\phi(v) \leq f(v)$. Choose $v \in [u, \bar{x}]$ so that $\mu(u) = f(v)$. Then by Part (b) and our assumption on u,

$$\phi(v) \leq \phi(u) \leq \mu(u) = f(v)$$

which is the desired inequality for v. □

Theorem 1. *A fixed point $\bar{x}$ of f is asymptotically stable if and only if there is a proper I-neighborhood U of $\bar{x}$ such that*

$$\begin{cases} \phi(x) > f(x) > x & \text{if } x \in U_l,\ x \neq \bar{x} \\ f(x) < x & \text{if } x \in U_r,\ x \neq \bar{x}. \end{cases} \tag{2}$$

Proof. Sufficiency. For convenience, we denote $U_l - \{\bar{x}\}$ by U_l^0, and similarly for U_r. First assume that $f_r^{-1}(x)$ is empty for all $x \in U_l^0$, or that U_l^0 is empty, so $f_r(x) \geq \bar{x}$ for all $x \in U_r$. Now if $x_0 \in U_r^0$ then, by (2), $\bar{x} \leq f_r(x_0) < x_0$ so we conclude by induction that $f^n(x_0) = f_r^n(x_0)$ decreases to $\bar{x}$ from the right. If $x_0 \in U_l^0$ then either $f^k(x_0) \geq \bar{x}$ for some $k \geq 1$ or $f^n(x_0) < \bar{x}$ for all $n \geq 1$. In the former case, assuming without loss of generality that $f^k(x_0) \in U_r^0$, the sequence $\{f^{k+n}(x_0)\}$ decreases as before to $\bar{x}$. In the second case, condition (2) shows that

$$x_0 < f^{n-1}(x_0) < f^n(x_0) < \bar{x}$$

for all n so that the terms $f^n(x_0)$ increase to $\bar{x}$ from the left.

Next assume that $f_r^{-1}(u)$ is nonempty for some $u \in U_l^0$, in which case $f_r^{-1}(x)$ is nonempty for all $x \in [u, \bar{x}]$ by Lemma 3(b). So we may choose $a \in U_l^0$ sufficiently close to $\bar{x}$ such that $f_r^{-1}(a)$ is nonempty, $\mu(a) \in U_r$ and thus $J = [a, \mu(a)] \subset U$. We now show that $fJ \subset J$. If $x \in [a, \bar{x}]$, then by (2)

$$a \leq x < f(x) \leq \mu(x) \leq \mu(a)$$

so that

$$f[a, \bar{x}] \subset (a, \mu(a)] \subset J. \tag{3}$$

Next suppose that $x \in [\bar{x}, \mu(a)]$. If $f(x) \geq \bar{x}$, then by (2) $f(x) \in [\bar{x}, x) \subset [\bar{x}, \mu(a)]$, while if $f(x) < \bar{x}$, then (2) and Lemma 3 (c) imply that

$$\mu(f(x)) < \phi(f(x)) = \inf f_r^{-1}(f_r(x)) \leq x \leq \mu(a)$$

which because of the nonincreasing nature of μ implies that $f(x) > a$. Thus

$$f[\bar{x}, \mu(a)] \subset (a, \mu(a)] \subset J. \tag{4}$$

Inequalities (3) and (4) imply that $fJ \subset J$. Now successive applications of f to J yield a decreasing sequence $J \supset fJ \supset f^2J \supset \cdots$ whose limit $K = \cap_{n=0}^{\infty} f^n J$ contains $\bar{x}$ and is thus nonempty. Since $f^n J$ is a compact interval for every n, it follows that K is a compact interval and $fK = K$. Given that $\bar{x}$ is the only fixed point of f in $K \subset J \subset U$, Lemma 1 implies that $K = \{\bar{x}\}$. Hence, $\bar{x}$ is asymptotically stable.

Necessity. Suppose that every proper I-neighborhood U of $\bar{x}$ contains a point x_U such that (2) fails at x_U. Thus either (i) $x_U \in U_r^0$ and $f_r(x_U) \geq x_U$ or (ii) $x_U \in U_l^0$ and $\phi(x_U) \leq f_l(x_U)$ or $f_l(x_U) \leq x_U$ (here ϕ defined with respect to some U works for all smaller neighborhoods contained in U). In case (i), the uniqueness of $\bar{x}$ in U implies that $f_r(x) > x$ for all $x \in (\bar{x}, x_U)$. But then for every $x_0 \in (\bar{x}, x_U)$, no matter how close to $\bar{x}$, the increasing sequence

$$x_0 < f(x_0) < \cdots < f^{n+1}(x_0), \quad \text{if } f^k(x_0) < x_U \quad \text{for } 1 \leq k \leq n$$

eventually exceeds x_U; it follows that $\bar{x}$ is not stable. In case (ii) the inequality $f_l(x_U) \leq x_U$ implies that $\bar{x}$ is not asymptotically stable in a manner similar to that just described for (i). It remains to show that the other inequality in (ii) also implies a lack of asymptotic stability. The first inequality in (ii) applied over a sequence $\{U_n\}$ of neighborhoods of $\bar{x}$ whose diameters approach zero, implies that there is a sequence $u_n \to \bar{x}$ from the left such that $\phi(u_n) \le f_l(u_n)$. Since, for each n, $\phi(u_n) \ge \bar{x}$ and also $f_l(u_n) \to \bar{x}$ as $n \to \infty$, we conclude that $\phi(u_n) \to \bar{x}$. Since, for each $x \in I_l^0$, $f_r^{-1}(x) \cap I_l^0$ is not empty, two possible cases arise.

Case 1. There is a $\delta > 0$ sufficiently small that $\sup[f_r^{-1}(x) \cap (\bar{x}, \bar{x} + \delta)] < f_l(x)$ for all $x \in (\bar{x} - \delta, \bar{x})$; i.e., the graph of f_r^{-1} near and to the left of $\bar{x}$ lies below the graph of f_l. Let $x_0 \in (\bar{x}, \bar{x} + \delta)$ and note that $f_r(x_0) < \bar{x}$. If $f_r(x_0) > \bar{x} - \delta$, then, since $x_0 \in f_r^{-1}(f_r(x_0))$, we see that $f^2(x_0) = f_l(f_r(x_0)) > x_0$. If $f^2(x_0) < \bar{x} + \delta$, then the preceding argument may be repeated; inductively, the sequence

$$x_0 < f^2(x_0) < f^4(x_0) < \cdots$$

is obtained which moves away from $\bar{x}$ until it exceeds $\bar{x} + \delta$, no matter how close x_0 is to $\bar{x}$. Therefore, $\bar{x}$ is not stable. Now let $x_0 \in (\bar{x} - \delta, \bar{x})$ and note that $f_l(x_0) > \bar{x}$. Repeating the above argument, the sequence $\{f^{2k+1}(x_0)\}$ is seen to increase away from $\bar{x}$, and once again $\bar{x}$ cannot be asymptotically stable.

Case 2. There is a sequence $v_n \to \bar{x}$, $v_n < \bar{x}$, such that

$$\sup[f_r^{-1}(v_n) \cap [\bar{x}, \bar{x} + 1/(n+1)]] \ge f_l(v_n), \quad n = 1, 2, 3, \dots,$$

i.e., for each n, there is $w_n \in f_r^{-1}(v_n) \cap [\bar{x}, \bar{x} + 1/(n+1)]$ such that $w_n \ge f_l(v_n)$ and $w_n \to \bar{x}$ as $n \to \infty$. The conditions of Lemma 2 are met with $a_1 = v_n$, $b_1 = w_n$, $a_2 = u_n$ and $b_2 = \phi(u_n)$. It follows that there is a c_n between w_n and $\phi(u_n)$ such that $f^2(c_n) = c_n$; i.e., there is a sequence of period-2 points $c_n \to \bar{x}$ as $n \to \infty$ and, therefore, $\bar{x}$ is again not asymptotically stable. This concludes the proof. □

Theorem 2. *Let $\bar{x}$ be a fixed point of f. The following statements are equivalent.*

(a) *$\bar{x}$ is asymptotically stable*;

(b) *there is a proper* I*-neighborhood* U *of* $\bar{x}$ *on which the following inequality holds*:

$$[f^2(x) - x](x - \bar{x}) < 0, \quad x \neq \bar{x}, \quad x \in U \subset I; \tag{5}$$

(c) *there is a proper* I*-neighborhood* U *of* $\bar{x}$ *on which* (2) *holds*;

(d) *there is a proper* I*-neighborhood* U *of* $\bar{x}$ *such that*

$$[f(x) - x](x - \bar{x}) < 0, \quad x \neq \bar{x}, \quad x \in U \subset I \tag{6}$$

and over $U_l - \{\bar{x}\}$, *the graph of* f_r^{-1} *lies above the graph of* μ;

(e) *inequality* (6) *holds on* U, *and if* $(x_l(t), y_l(t))$ *and* $(y_r(s), x_r(s))$ *are parametrizations of* f_l *and* f_r^{-1}, *respectively, then*

$$x_l(t) = y_r(s) \Longrightarrow x_r(s) > y_l(t).$$

(f) *There is a proper* I*-neighborhood* U *of* $\bar{x}$ *such that*

$$(\phi(x) - f(x))(\phi(x) - x)(f(x) - x)(x - \bar{x}) < 0, \quad x \in U - \{\bar{x}\}. \tag{7}$$

Proof. We show that (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (a) and (c) $\Rightarrow$ (d) $\Rightarrow$ (e) $\Rightarrow$ (f) $\Rightarrow$ (c). First, if (a) is true, then there is a sufficiently small I-neighborhood U of $\bar{x}$ on which the equality $f^2(x) = x$ holds only at $\bar{x}$ and every point in U is attracted to $\bar{x}$. Hence, the continuous function $f^2(x) - x$ either does not change its sign over U or, if it does, then the sign change can occur only at $\bar{x}$. If $f^2(x) > x$ for all $x \in U_r^0$ and $x_0 \in U_r^0$, then $f^2(x_0) > x_0$; if $f^2(x_0) \in U_r$ also, then another application of f^2 leads further away from $\bar{x}$ and the process continues until the trajectory $\{f^{2n}(x_0)\}$ exits U_r, no matter how close x_0 is to $\bar{x}$. Thus $\bar{x}$ cannot be stable, contradicting (a). Similarly, (a) is contradicted if $f^2(x) < x$ for all $x \in U_l^0$. Now (5) follows and (b) is established. Next, suppose that (b) is true. Then (6) must hold, since otherwise there is either a fixed point other than $\bar{x}$ in U at which (b) would be false or else $f(x) < x$, respectively, $f(x) > x$, for all $x < \bar{x}$, respectively, $x > \bar{x}$, in U, in which case choosing x_0 sufficiently close to $\bar{x}$ so that $f(x_0) \in U$ implies that $f^2(x_0) < x_0$, respectively, $f^2(x_0) > x_0$, also, again contradicting (b). To establish (c), it remains to show that $\phi(x) > f_l(x)$ for $x \in U_l$. This is clear if $\phi(x) \geq a > 0$ for all $x \in U_l^0$; otherwise, arguing as in the last two cases in the proof of Theorem 1, we conclude that there is either a sequence of period-2 points converging to $\bar{x}$ from the left or else there is $x' \in U_l^0$ close to $\bar{x}$ such that $x_0 = f(x') \in U_r$ and $f^2(x_0) \in U_r$ with $f^2(x_0) > x_0$. Since in either case (b) is contradicted, we must assume that (c) holds. Finally, in Theorem 1 it was established that (c) implies (a).

Next, note that (d) follows easily from (c) because conditions (2) imply (6) and, by Lemma 3 (c), ϕ, hence also the graph of f_r^{-1}, dominates f on U_l^0 if and only if ϕ dominates μ. In light of Lemma 3 (c), (e) is just a rephrasing of (d), hence equivalent to it. Statement (f) is an immediate consequence of (e), or equivalently (d), which implies that $\phi(x) > f(x)$ for all $x \in U - \{\bar{x}\}$ (for $x > \bar{x}$, the graph of f_r^{-1} lies above the identity line if and only if f_r lies below that line). Finally, assume (f) holds. For $x < \bar{x}$, $\phi(x) > \bar{x} > x$ so if $\phi(u) < f(u)$ for some $u < \bar{x}$, then $f(u) > \bar{x} > u$ and (7) fails. Hence, $\phi(x) > f(x)$ for all x, and so by (7) $f(x) > x$. For $x > \bar{x}$, the product $(\phi(x) - f(x))(\phi(x) - x)$ is always positive, since both f and the identity line always lie on the same side of f_r^{-1}. Therefore, by (7), $f(x) - x < 0$ and condition (2) is established. □

Remarks. (1) A general sufficient condition for $\bar{x}$ to be asymptotically stable is that

$$|f(x) - \bar{x}| < |x - \bar{x}|, \quad x \neq \bar{x}, \quad x \in U \subset I \tag{8}$$

which generalizes the well-known linearization condition $|f'(\bar{x})| < 1$. From Theorem 2 (d) we see explicitly how to extend (8) to obtain a necessary and sufficient condition for asymptotic stability. Geometrically, condition (8) requires that the graph of f be bounded by the lines $y = x$ and $y = 2\bar{x} - x$; therefore, for $x < \bar{x}$, the graph of f_r^{-1} lies above the graph of the line $y = 2\bar{x} - x$ while the graph of f_l lies below the same line. Certainly, this implies statement (d) in Theorem 2, in which the continuous nonincreasing function μ replaces the ray $y = 2\bar{x} - x$, $x < \bar{x}$.

(2) A different proof of the equivalence of (a) and (b) in Theorem 2 is given in [**4**, p. 47], which this author discovered after already proving Theorem 1. Not every condition in Theorem 2 is applicable with equal

ease to a given problem; for instance, in Example 3 below, Theorem 1 itself, as stated, is most easily applicable. Also, if we think of condition (2) as a "right" condition because of f_r^{-1}, then there is also a "left" analog of (2) which compares f_l^{-1} with f_r and can be useful when the left part f_l is simpler than the right part f_r for the purpose of inversion. These left versions are obtained from the right versions here by making a few minor modifications and will not be discussed here.

3. Further results and examples.

Definition 4. Let $\bar{x}$ be an isolated fixed point of f. If the left limit $\lim_{x\to\bar{x}^-}\phi(x) > \bar{x}$, then f is *ϕ-trivial* at $\bar{x}$. Here ∞ is a permissible value for ϕ.

Note that f is ϕ-trivial if and only if $f(x) > \bar{x}$ for all x near and to the right of $\bar{x}$, and so f is not ϕ-trivial if $f(x) < \bar{x}$ for all x near and to the right of $\bar{x}$. Thus, for a differentiable map f, the condition $f'(\bar{x}) > 0$ implies that f is ϕ-trivial while $f'(\bar{x}) < 0$ implies that f is not ϕ-trivial. If $f'(\bar{x}) = 0$, then f is ϕ-trivial if $\bar{x}$ is a local minimum and f is not ϕ-trivial if $\bar{x}$ is a local maximum. This line of reasoning gives obvious sufficient conditions for ϕ-triviality or nontriviality in terms of the higher derivatives, if the latter are defined at $\bar{x}$. The next result gives a complete description of trajectory behavior near a fixed point at which f is ϕ-trivial.

Corollary 1. *Let f be ϕ-trivial at an isolated fixed point $\bar{x}$. Then $\bar{x}$ is asymptotically stable if and only if* (6) *holds in some proper I-neighborhood of $\bar{x}$. If* (6) *does not hold near $\bar{x}$, then $\bar{x}$ is either strongly unstable or semi-stable* (*attracting from one side, repelling from the other*). *Also, every trajectory converges to $\bar{x}$ or diverges from it, as the case may be, monotonically after possibly a finite number of terms.*

Proof. Since f is ϕ-trivial, we may assume without loss of generality that $f_r(x) > \bar{x}$ for all x in some sufficiently small interval to the right of $\bar{x}$. So if (6) holds, then the trajectories of all points near and to the right of $\bar{x}$ decrease to $\bar{x}$. For points to the left of $\bar{x}$, either $f_l(x) < \bar{x}$ in which case trajectories near and to the left increase to $\bar{x}$, or else there is a least k such that $f^k(x_0) > \bar{x}$ for some $x_0 < \bar{x}$. In the latter case,

$f^n(x_0)$ decreases to $\bar{x}$ for $n > k$. The proof of the rest of the corollary is routine. □

Corollary 2. *Assume that f is not ϕ-trivial at an isolated fixed point $\bar{x}$. Then precisely one of the following is true.*

(i) *$\bar{x}$ is asymptotically stable*;

(ii) *$\bar{x}$ is unstable*;

(iii) *there is a sequence of period-2 points converging to $\bar{x}$.*

Proof. If the graph of f_r^{-1} is not entirely above or entirely below the graph of f_l near and to the left of $\bar{x}$, then f_r^{-1} must intersect f_l in every neighborhood of $\bar{x}$. □

The next result was proved in [**3**] using (5), a proof of which was already in print [**4**]; we now give a proof based on the results of this paper. The result is false for continuous mappings of the Euclidean plane, see [**3**]. A fixed point $\bar{x}$ is defined to be *globally attracting*, relative to I, if $f^n(x_0) \to \bar{x}$ as $n \to \infty$ for all $x_0 \in I$.

Corollary 3. *If $\bar{x}$ is globally attracting, then $\bar{x}$ is stable.*

Proof. Since $\bar{x}$ is globally attracting, (6) holds on I. If $\bar{x}$ is also unstable, then f is not ϕ-trivial and conditions (i) and (iii) in Corollary 2 cannot be satisfied. Indeed, $f_r^{-1} \cap f_l = \{\bar{x}\}$ on I due to global attractivity, so the strong instability requirement in Corollary 2 forces the graph of f_r^{-1} to stay below that of f_l on I, which is not possible. □

In conclusion, we discuss fixed point stability for some specific functions in order to illustrate the applicability and the computational feasibility of the results of the previous section.

Example 1. In applied models, linearization is the tool often used in establishing the (local) asymptotic stability of a fixed point. Yet, in many applications, one encounters parameter ranges that include the possibility $f'(\bar{x}) = \pm 1$ or even that $f'(\bar{x})$ does not exist. Further, even when linearization is applicable, it may be insufficient because it is often desirable to obtain information on the extent of attractivity of the fixed point, namely, the size of the "basin of attraction," about which no information is supplied by linearization. The results of this paper can be fruitfully applied in such cases to analyze the problem at hand. As an example of such a problem, consider the function

$$f(x) = xe^{4(1-x)/(1+x)}, \quad x \geq 0, \tag{9}$$

which comes from the analysis of a genotype selection model proposed by Robert May, see [**1**, p. 81]. Let us show, using the theorems of the preceding section, that the unique positive fixed point $\bar{x} = 1$ is asymptotically stable *globally* with respect to the domain $[0, \infty)$, i.e., every point of $[0, \infty)$ is attracted to 1, even though $f'(1) = -1$.

To see this, we reparametrize f by setting $t = -(1-x)/(1+x)$ so that in (9) $t = 0$ gives $x = 1$, while $t \in [-1, 0)$ corresponds to $x \in [0, 1)$ and $t \in (0, 1)$ corresponds to $x \in (0, \infty)$. The following representations are obtained:

$$f_l : x_l(t) = \frac{1+t}{1-t}, \qquad y_l(t) = \frac{1+t}{1-t} e^{-4t}, \quad -1 < t < 0$$

and

$$f_r^{-1} : y_r(s) = \frac{1+s}{1-s} e^{-4s}, \qquad x_r(s) = \frac{1+s}{1-s}, \quad 0 < s < 1.$$

Direct calculation now shows that condition (e) of Theorem 2 is satisfied for all $t \in [-1, 0)$, $s \in (0, 1)$. Therefore, $\bar{x} = 1$ is asymptotically stable, globally with respect to the domain $[0, \infty)$. We note that Theorem 2 (b) is applicable in this example with a comparable amount of effort.

Remark. When only *local* stability is of concern, because f is sufficiently smooth in Example 1, the specialized derivative condition involving $(f^2)'''$ in [**2**] may alternately be applied. According to this condition, if

$$(f^2)'''(\bar{x}) = -2f'''(\bar{x}) - 3[f''(\bar{x})]^2 < 0, \tag{10}$$

then $\bar{x}$ is locally asymptotically stable, while the reverse inequality implies instability. For Example 1, direct calculation shows that $(f^2)'''(1) = -2$. If, however, $(f^2)'''(\bar{x}) = 0$ then condition (10) is inconclusive with regard to stability or instability. This can happen for relatively commonplace functions, as seen in Example 2 below. It may be noted in passing that $(f^2)'''(\bar{x}) = 2Sf(\bar{x})$ whenever $f'(\bar{x}) = -1$, so in this case the negativity of the particular value $Sf(\bar{x})$ of the Schwarzian is equivalent to $(f^2)'''(\bar{x}) < 0$. For a discussion of the Schwarzian derivative Sf and its role in stability theory, see the original paper [**5**] or standard texts on dynamical systems.

Example 2. Functions of the type

$$f(x) = ax^{m/3} - x, \quad m \text{ a fixed positive integer,} \\ a \in (-\infty, \infty), \quad a \neq 0 \tag{11}$$

have a fixed point at the origin where derivatives of order 1 or higher may fail to exist. The only values of m at which condition (10) applies are $m = 6, 9$, since for all other values of m, $(f^2)'''(0)$ is either zero or undefined, e.g., for $m = 12$, we obtain quartic polynomials in (11) for which $(f^2)'''(0) = 0$. However, it is not difficult to show using Theorem 2 (b) or 2 (e) that the origin is asymptotically stable for all $m \geq 4$ and $a > 0$; for $a < 0$, the origin is asymptotically stable for all *even* $m \geq 4$ and it is unstable for *odd* m. The natural parametrizations

$$f_l : \begin{cases} x_l(t) = t^3 \\ y_l(t) = at^m - t^3, \end{cases} \qquad t < 0,$$
$$f_r^{-1} : \begin{cases} y_r(s) = as^m - s^3 \\ x_r(s) = s^3, \end{cases} \qquad s > 0,$$

and some straightforward calculations show that, for s, t satisfying $t^3 = as^m - s^3$ (such pairs exist for all a if $m \geq 4$), we obtain $x_r(s) > x_l(t)$ or the reverse of this inequality depending on the above mentioned values of a and m.

Example 3. The purpose of this example (and the next) is to illustrate the applicability of our results to certain functions that are rather ill-behaved at the fixed point. Consider the function

$$f(x) = \begin{cases} -x[1 + a\sin(\pi/x)] & x < 0, \\ (1-a)(x^2+x) & x \geq 0, \end{cases} \quad 1 < a < \sqrt{2} \tag{12}$$

which is continuous everywhere and has a unique fixed point at the origin, although f is not differentiable at the origin. To determine the stability character of the origin as a ranges over the values listed in (12), note that

$$f_r^{-1}(x) = \phi(x) = -\frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{4x}{a-1}}.$$

Since, for $x < 0$, we have $-x[1 + a\sin(\pi/x)] \leq -x(1+a)$, Theorem 1 implies that, for all a satisfying the inequality

$$-\frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{4x}{a-1}} > -x(1+a), \tag{13}$$

the origin is asymptotically stable. Clearly, if the derivative of $f_r^{-1} = \phi$ is greater in magnitude than $-(1+a)$, then (13) holds for all $x < 0$ of small enough magnitude. Since $\phi'(0) = -1/(a-1)$, we require $-1/(a-1) > -(1+a)$ for the asymptotic stability of 0. Solving the last inequality gives $1 < a < \sqrt{2}$.

Example 4. In this final example, given constants $a > 0$ and $b > 1$, a continuous piecewise linear mapping f is defined as follows.

f_l consists of line segments joining the point P_1 to Q_1 to P_3 to Q_3, etc., where

$$P_{2k-1} = \left(\frac{-a}{(4b^2)^{k-1}}, \frac{ab}{(4b^2)^{k-1}}\right), \qquad Q_{2k-1} = \left(\frac{-a}{2(4b^2)^{k-1}}, 0\right).$$

Note that, for each k, P_{2k-1} lies on the line $y = -bx$. Next, f_r^{-1} consists of the line segments connecting the points P_2 to Q_2 to P_4 to Q_4, etc., where

$$P_{2k} = \left(\frac{-a}{2(4b^2)^{k-1}}, \frac{a}{2b(4b^2)^{k-1}}\right), \qquad Q_{2k} = \left(0, \frac{ab}{(4b^2)^k}\right).$$

Note that P_{2k} is on the line $y = -x/b$ and f_r can be easily obtained from f_r^{-1} so as to build f. Also, both ϕ and μ are easy to construct in this example.

By its construction, f has a unique fixed point at the origin and the graph of f_r^{-1} lies above the graph of f_l so the origin is asymptotically stable by Theorem 2 (d). This is true no matter how large the value of b is and illustrates the point made in Remark (1) following Theorem 2. It seems difficult to prove the stability in this case using any other method.

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