The Impossibility of Unstable, Globally Attracting Fixed Points for Continuous Mappings of the Line

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It is possible for a fixed point of a dynamical system to locally repel some trajectories, yet globally attract all trajectories. For example, consider the mapping

$$f_a(x) = \begin{cases} -2x & \text{if } x < a \\ 0 & \text{if } x \geq a \end{cases}$$

where a is any fixed positive real number. Then the first order difference equation

$$x_{n+1} = f_a(x_n) \qquad n = 0, 1, 2, 3, \ldots \tag{1}$$

has a solution

$$x_n = f_a^n(x_0) = \begin{cases} (-2)^n x_0 & \text{if } (-2)^{n-1} x_0 < a \\ 0 & \text{if } (-2)^{n-1} x_0 \geq a \end{cases}$$

for every choice of $x_0 \in \mathbf{R}$ (f_a^n represents the n-th iterate of f_a under function composition). Clearly, once $x_k \geq a$ for any k, then $x_n = 0$ for all $n \geq k$. In particular, every solution of (1) converges to zero, regardless of the choice of x_0. In this sense, the origin, which is the unique fixed point of f_a, is *globally attracting*. However, if $x_0 \neq 0$, then no matter how close x_0 is chosen to the origin, x_n must first exceed a before ultimately reaching the origin. Hence, the origin is *unstable* (in fact, locally repelling).

The preceding example shows that globally attracting fixed points that are *not* stable can easily occur in one-dimensional dynamical systems such as (1). Since f_a is discontinuous at $x = a$, it is natural to ask whether a *continuous* example of an unstable global point attractor can be constructed in one dimension. As the title of this note suggests, this is not possible. To see why continuous maps are nice in this sense, we need a local or asymptotic stability result from [7, p. 47]. Complete definitions of all concepts and terminology used here can be found in [2] and [5].

Criterion for asymptotic stability of fixed points: *A fixed point $\bar{x}$ of a continuous map f is asymptotically stable if and only if there is an open interval (a, b) containing $\bar{x}$ such that $f^2(x) > x$ for $a < x < \bar{x}$ and $f^2(x) < x$ for $\bar{x} < x < b$.*

The preceding criterion is remarkable for not requiring any differentiability conditions on f. Now we are ready to demonstrate our main result:

A continuous mapping of the real line cannot have an unstable fixed point that is globally attracting.

Suppose, on the contrary, that a continuous mapping f of the real line has an unstable fixed point $\bar{x}$ that is also globally attracting. Since there can be no periodic solutions, the iterate f^2 crosses the identity line only at $\bar{x}$. Hence, only one of the following two cases is possible:

(I) $f^2(x) > x$ for $x < \bar{x}$ and $f^2(x) < x$ for $x > \bar{x}$;

(II) $f^2(x) < x$ for $x < \bar{x}$, or $f^2(x) > x$ for $x > \bar{x}$.

By the preceding Criterion, Case (I) implies stability and must therefore be ruled out; this leaves Case (II). Assume that $f^2(x) > x$ for $x > \bar{x}$, and let $x_0 > \bar{x}$. Then $f^2(x_0) > x_0$; as this implies $f^2(x_0) > \bar{x}$, repeated applications of f^2 to x_0 generate the increasing sequence

$$\bar{x} < x_0 < f^2(x_0) < f^4(x_0) < \cdots$$

By continuity, $f^{2n}(x_0) \to \infty$ as $n \to \infty$, implying that $\{f^n(x_0)\}$ does not converge to $\bar{x}$. The case $f^2(x) < x$ for $x < \bar{x}$ reaches a similar contradiction, so we conclude that our original assumption on $\bar{x}$ was false.

A natural question with regard to the preceding impossibility result is whether *one dimensionality* is necessary (in addition to continuity) in order to rule out the existence of unstable global point attractors. The answer is indeed affirmative, and examples of continuous (in fact, differentiable) planar maps having unstable, globally attracting fixed points exist in the literature; see, e.g., [**4**, p. 90], or the discretization of the continuous time example in [**1**, p. 59]. Unstable fixed points that are globally attracting can also arise in a continuous second order difference equation, which is a very special type of a two dimensional system. Generally, a second order difference equation has the form

$$y_{n+1} = F(y_n, y_{n-1}) \qquad n = 0, 1, 2, 3, \dots, \tag{2}$$

where $F\colon \mathbf{R}^2 \to \mathbf{R}$ and real numbers y_0, y_{-1} are specified as initial conditions. A *fixed point* of (2) is a solution of $F(y, y) = y$. The particular F that we discuss here is not an artificial construct of purely theoretical interest; rather, it comes from the classical Hicks model of the trade cycle, an early mathematical model that aimed to explain well-documented fluctuations in economic output or GNP that cause recessions periodically; see [**3**]. The simplified, static Hicks model with a single-period lag is given by equation (2) in which F is the continuous, piecewise linear mapping

$$F(u, v) = \min\{K, a + bu + c\max\{u - v, d\}\} \tag{3}$$

with constants $a, c > 0$, $d < 0$, $0 < b < 1$, and $K > a/(1-b)$. The ratio $a/(1-b)$ gives the unique fixed point (or *equilibrium*) $\bar{y}$ of the Hicks equation; for an explanation of the general Hicks model and the details of all derivations, see [**6**]. In particular, the negative number d is what Hicks calls the "floor level of induced investment." It is shown in [**6**] that, under these hypotheses, every non-equilibrium solution of (3) executes bounded, non-decaying oscillations about the unstable (and non-attracting) fixed point, as is expected of the "business cycle." But what happens when d approaches zero? It is in the limiting case $d = 0$ of the Hicks equation that the fixed point $\bar{y}$ turns into a global attractor, which is unstable if

$$c > \left(1 + \sqrt{1-b}\right)^2. \tag{4}$$

To see this, choose $y_{-1} = \bar{y}$ and $y_0 = \bar{y} + \varepsilon$, where $\varepsilon > 0$ is small enough so that $y_0 < K$. Then the trajectory $\{y_n\}$ develops according to the linear difference equation

$$y_{n+1} = a + (b + c)y_n - cy_{n-1},$$

which has exponentially divergent solutions, since condition (4) implies the existence of eigenvalues with magnitude greater than 1. Hence, $\bar{y}$ is unstable. Upon reaching K, however, the trajectory bounces down and obeys the first order equation

$$y_{n+1} = a + by_n,$$

whose solution clearly converges to $\bar{y}$. Generalizing this argument to arbitrary pairs of initial conditions is not hard, and establishes that $\bar{y}$ is globally attracting.

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[The try-works are] a place also for profound mathematical meditation. It was in the left-hand try-pot of the Pequod, with the soapstone diligently circling round me, that I was first indirectly struck by the remarkable fact, that in geometry all bodies gliding along the cycloid, my soapstone for example, will descend from any point in precisely the same time.

Herman Melville, *Moby Dick*, Chapter XCVI, The Try-Works
Contributed by Karl David, Wells College