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Bounded Oscillations in the Hicks Business Cycle Model and other Delay Equations

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Sufficient conditions for the persistent (non-decaying) oscillations of the bounded trajectories of the delay difference equation

$$x_n = F(x_{n-1}, \dots, x_{n-m})$$

are obtained. Applications include the Hicks equation for the trade cycle with arbitrary lag structure. The case $m = 2$ of second order equations is discussed in greater detail with applications to certain rational recursive sequences and to a generic extension of the Hicks equation with a one-period lag.

Keywords: persistent oscillations; boundedness; repelling fixed points; Hicks equation

Classification Categories: 39A11, 39A12, 90A16

1 INTRODUCTION

In a classic 1950 monograph [7], the Nobel Prize-winning economist J.R. Hicks proposed a model for the business or trade cycle that was based on a nonlinear delay difference equation. Widely recognized as a classic in macroeconomics, the Hicks model is perhaps the oldest application of nonlinear difference equations of order greater than 1 outside of mathematics. Hicks claimed (and so believed) that his hypotheses on the various economic parameters in his equation implied persistent oscillation of the output trajectories. However, Hicks never gave a proof of his claim that was independent of the delay pattern (see [18] and the Historical Remarks below). The autonomous Hicks equation for relative output is an instance of the m-th order equation

$$x_n = F(x_{n-1}, x_{n-2}, \dots, x_{n-m}), \qquad n = 1, 2, 3, \dots \tag{1}$$

with initial values $x_{1-m}, \dots, x_0 \in [0,\infty)$. We assume throughout this paper that F: $[0,\infty)^m \to [0,\infty)$ is continuous, and obtain sufficient conditions for the persistent and bounded oscillation of non-trivial solutions of (1) and some of its special cases. In particular, we prove the oscillatory character of Hicksian output trajectories regardless of the investment-consumption lag structure.

Another significant instance of (1) occurs in the mathematical literature in the form of the following general equation

$$x_n = x_{n-1} f(x_{n-1}, x_{n-2}, \ldots, x_{n-m}), \qquad n = 1, 2, 3, \ldots \tag{2}$$

whose permanence and global attractivity are analyzed in [11, pp. 35-45] under certain conditions on the function f (see Lemma 1 below with regard to permanence). The study of this equation is motivated in part by the growing effort in analyzing the dynamical behavior of rational recursive sequences; the latter sequences and the equations which generate them are seen in several biological models (see, e.g., [11, Chapters 3, 4]).

Concerning Eq.(2), we supplement the existing permanence and global attractivity results by furnishing conditions that imply oscillatory behavior. The oscillation results discussed here are not of the linearized type analyzed in, e.g., [6, Sec. 7.4] or [10]. For instance, the linearization of the Hicks equation at its unique fixed point yields monotonically divergent solutions for a notable range of legitimate parameter values.

2 GENERAL OSCILLATION RESULTS

Definitions and Notations. Define the *vector form* or the *standard representation* of (1) as a first order system in the usual way:

$$X_n = V_F(X_{n-1}), \qquad n = 1, 2, 3, \ldots \tag{3}$$

where $V_F : [0,\infty)^m \to [0,\infty)^m$ is defined as

$$V_F(x^1, x^2, \ldots, x^m) = [F(x^1, \ldots, x^m), x^1, \ldots, x^{m-1}]$$

We assume that $F \in C\,[[0,\infty)^m, [0,\infty)]$, so that $V_F \in C\,[[0,\infty)^m, [0,\infty)^m]$. Clearly a fixed point of (3) takes the form $\overline{X} = (\bar{x}, \ldots, \bar{x})$ where $\bar{x}$ is a real solution of

$$F(x, \ldots, x) = x$$

and a fixed point of (1). Note that V_F is continuously differentiable at $\overline{X}$ if and only if F is; in particular, this is the case if all partial derivatives of F are continuous at $\overline{X}$. Then the fixed point $\overline{X}$ is said to be linearly *repelling* (or a "source") if all the eigenvalues of the derivative (or Jacobian) $DV_F(\overline{X})$ of V_F have modulus greater than 1. Specifically, given the definition of V_F this means that all the roots of the (characteristic) polynomial equation

$$\lambda^m - \sum_{i=1}^{m} \frac{\partial F}{\partial x^i}(\overline{X})\lambda^{m-i} = 0$$

have modulus greater than 1 (see, e.g., [11, p.14]). Under these differentiable circumstances, the familiar Hartman-Grobman linearization theorem (see, e.g., [1, p. 68]) implies that if V_F is locally a C^1 diffeomorphism, then $\bar{X}$ is repelling for V_F if and only if the origin is repelling for the linearization $DV_F(\bar{X})$.

We say that a bounded solution $\{x_n\}$ of (1) *oscillates persistently* if the sequence $\{x_n\}$ has at least two distinct limit points. Notice that not every oscillating sequence oscillates persistently.

Finally, we call Eq.(1) *permanent* if there is a positive number M such that for each solution $\{x_n\}$ we have $x_n \in [0, M]$ for all n larger than some positive integer n_0. Further, if there is $L \in [0, M]$ such that $x_n \in [L, M]$ for all $n \geq n_0$, then we say that (1) is *positively permanent*. Note that the numbers L, M do not depend on the initial conditions. The intervals $[0, M]$ or $[L, M]$ may be called *absorbing* intervals for Eq.(1).

We now state the basic result of this paper on which are based several persistent oscillation results for permanent systems and other systems possessing non-trivial bounded solutions.

THEOREM 1. *Assume that the following conditions hold:*

a. *The equation* $F(x,\ldots, x) = x$ *has a finite number of solutions* $0 < \bar{x}_1 < \ldots < \bar{x}_k < \infty$;
b. *For* $i = 1, \ldots, m$, $\partial F/\partial x^i$ *exist continuously at* $\bar{X}_j = (\bar{x}_j, \ldots, \bar{x}_j)$, *and every root of the characteristic polynomial* $\lambda^m - \sum_{i=1}^{m} \partial F/\partial x^i(X_j)\lambda^{m-i}$ *has modulus greater than* 1 *for each* $j = 1, \ldots, k$;
c. *For every* $j = 1, \ldots, k$, $F(\bar{x}_j, \ldots, \bar{x}_j, x) \neq \bar{x}_j$ *if* $x \neq \bar{x}_j$.

Then all bounded solutions of (1) except the trivial solutions $\bar{x}_j$, $j = 1, \ldots, k$, oscillate persistently.

Proof. Let $\{x_n\}$ be a bounded solution of (1); then due to a compact range, $\{x_n\}$ must have limit points. Suppose that $\{x_n\}$ has a unique limit point $\tilde{x}$. Then

$$\tilde{x} = \lim_{n \to \infty} x_n = \lim_{n \to \infty} F(x_{n-1}, \ldots, x_{n-m}) = F(\tilde{x}, \ldots, \tilde{x})$$

so by (a) we have $\tilde{x} = \bar{x}_j$ for some j; i.e., if the solution $\{x_n\}$ converges, it must converge to some one of the fixed points $\bar{x}_j$, $1 \leq j \leq k$.

Next, suppose that $x_n = \bar{x}_j$ for all n larger than some positive integer r. Then

$$\bar{x}_j = x_{r+m} = F(x_{r+m-1}, \ldots, x_{r+1}, x_r) = F(\bar{x}_j, \ldots, \bar{x}_j, x_r)$$

so that by (c), $x_r = \bar{x}_j$. Repeating this argument inductively shows that $x_n = \bar{x}_j$ for all $n \geq 1 - m$. Therefore, the only solution of (1) which is constantly equal to $\bar{x}_j$ in a finite number of iterations is the trivial solution $\bar{x}_j$.

The preceding argument shows that if $\{x_n\}$ is not a trivial solution, then x_n is approaching some fixed point $\bar{x}_j$, in the sense that $|x_n - \bar{x}_j| \to 0$ as $n \to \infty$, but $|x_n-$

$\bar{x}_j| \neq 0$ for infinitly many n. Now (b) in particular implies that $\partial F/\partial x^m(\bar{X}) \neq 0$, so that by the implicit function theorem there is an open neighborhood U_j of $\bar{X}_j$ on which V_F is a C^1 diffeomorphism, and $\bar{X}_i \notin U_j$ if $i \neq j$. Defining $X_n = (x_n, \ldots, x_{n-m+1})$ for $n \geq 0$, we see that $\{X_n\}$ is a bounded solution of (3) that is eventually in U_j and $X_n \to \bar{X}_j$. However, this is prevented by (b) and the Hartman-Grobman theorem. To avoid this contradiction we conclude that every non-trivial solution of (1) must have more than one limit point; i.e., they must oscillate persistently. □

COROLLARY 1. *The autonomous piecewise linear Hicks Equation:*

$$y_n = \min\left\{Y_c, c_0 + \sum_{i=1}^{m} c_i y_{n-i} + \max\left\{d, \sum_{i=1}^{m-1} v_i(y_{n-i} - y_{n-i-1})\right\}\right\} \quad (m \geq 2) \quad (4)$$

where y_n is the output in period n and:

a. $d \leq 0$, $c_0 + d > 0$, $Y_c > 0$;
b. $v_i \geq 0$ *for* $i = 1, \ldots, m-1$;
c. $c_i \geq 0$ *for* $i = 1, \ldots, m$ *and* $\sum_{i=1}^{m} c_i < 1$;
d. $Y_c(1 - \sum_{i=1}^{m} c_i) > c_0$;

has a unique positive fixed point:

$$\bar{y} = \frac{c_0}{1 - \sum_{i=1}^{m} c_i}.$$

If in addition:

e. $d < 0$ *and* $c_m = 0$;
f. *Every root of the polynomial* $\lambda^m - (c_1 + v_1)\lambda^{m-1} - \sum_{i=2}^{m-1} (c_i + v_i - v_{i-1})\lambda^{m-i} + v_{m-1}$ *has modulus greater than* 1;

then every non-trivial solution of (4) *oscillates persistently within* $[c_0 + d, Y_c]$.

Proof. The statement about $\bar{y}$ is easy to verify, so that in particular, (a) in Theorem 1 holds. To prove the rest of the corollary, assume $d < 0$ and define the Hicks function $H \in C([0,\infty)^m, [c_0 + d, Y_c])$ as

$$H(y^1, \ldots, y^m) = \min\left\{Y_c, c_0 + \sum_{i=1}^{m} c_i y^i + \max\left\{d, \sum_{i=1}^{m-1} v_i(y^i - y^{i+1})\right\}\right\}$$

Then $H(\bar{Y}) = \bar{y}$,

$$\frac{\partial H}{\partial y^i}(\bar{Y}) = c_i + v_i - v_{i-1}, \quad i = 2, \ldots, m-1$$

and

$$\frac{\partial H}{\partial y^1}(\bar{Y}) = c_1 + v_1, \quad \frac{\partial H}{\partial y^m}(\bar{Y}) = -v_{m-1}.$$

Thus, (b) in Theorem 1 holds by (f) here. We show next that Theorem 1(c) is also true. We note that

$$H(\bar{y}, \ldots, \bar{y}, y) = \min\left\{Y_c, c_0 + \bar{y}\sum_{i=1}^{m-1} c_i + \max\{d, v_{m-1}(\bar{y} - y)\}\right\}$$

so if $H(\bar{y}, \ldots, \bar{y}, y) = \bar{y}$, then we have

$$\max\{v_{m-1}(\bar{y} - y), d\} = -c_0 + \left(1 - \sum_{i-1}^{m-1} c_i\right)\bar{y}.$$

From the definition of $\bar{y}$ it follows that the right hand side of preceding equation is zero, so that the only solution of the equation is $y = \bar{y}$. Hence (c) in Theorem 1 is also satisfied. Now, observing that for every solution of (4), $y_n \in [c_0 + d, Y_c])$ for all $n \geq 1$, Theorem 1 may be applied to conclude the proof.

Remarks

1. It may be noticed that with H having a compact range in the positive reals, the Hicks equation is trivially positively permanent. The problem of permanence is more difficult in generic extensions of the Hicks equation where the "ceiling" term Y_c does not appear. See Lemma 4 below.
2. Hicks did not explicitly require that $c_m = 0$. This condition is necessary for Theorem 1(c) and for local invertibility in (b); without them non-trivial, non-oscillating trajectories can arise. For instance, let $m = 2$ in (4) and suppose that $v_1 > 1$, $c_2 > 0$. If $(y_{k-1}, y_k) = (\bar{y}-d/c_2, \bar{y})$ for some k, then $y_n = \bar{y}$ for $n \geq k$; hence, the set of all initial pairs (y_{-1}, y_0) for which such a k exists, leads to eventually constant solutions for the second order Hicks equation. This example also shows that while condition (b) implies (c) locally in Theorem 1, we need (c) globally there.
3. If $m = 1$ in Theorem 1, then condition (c) of the theorem is to be interpreted as $F(x) \neq \bar{x}$ if $x \neq \bar{x}$. This condition requires that $k = 1$ as well; i.e., the fixed point must be unique in the first order case for Theorem 1 to apply. The existence of additional fixed points gives rise to eventually constant solutions which need not be rare in general.

4. It is not enough for the $\overline{X}_j$ to be unstable; if some eigenvalue of $DV_F(\overline{X}_j)$ has modulus less than or equal to 1 for any j, then the stable set of $\overline{X}_j$ complicates the situation as seen in Remark 2 above. Still, if the instability is of linear or hyperbolic type, Theorem 1 might hold under additional restrictions. If $\overline{X}_j$ is unstable in a nonlinear way for some j (i.e., $DV_F(\overline{X}_j)$ is either undefined or else has eigenvalues of unit modulus), then Theorem 1 may be false, as demonstrated by the following example.

Example. (*An unstable global point attractor in the Hicks equation*) Consider the special second order, boundary equation ($d = 0$) of Hicks

$$y_n = \min\{Y_c, c_0 + cy_{n-1} + v\max\{y_{n-1} - y_{n-2}, 0\}\} \tag{5}$$

obtained from (4) by setting $m = 2$, $d = 0$, $c_1 = c \in [0,1)$, $c_2 = 0$, and $v \geq (1 + \sqrt{1-c})^2$. Then the fixed point $\bar{y} = c_0/(1-c)$ is unique and every solution of the linear equation

$$y_n = c_0 + cy_{n-1} + v(y_{n-1} - y_{n-2})$$

is monotonically divergent. So if $y_{-1} = \bar{y}$ and $y_0 = \bar{y} + \varepsilon$ for any $\varepsilon > 0$, then y_n increases monotonically until it reaches Y_c; it follows that $\bar{y}$ is unstable. Note, however, that the Hicks function in this case, namely,

$$H(x^1, x^2) = \min\{Y_c, c_0 + cx^1 + v\max\{x^1 - x^2, 0\}\}$$

is not differentiable at $(\bar{y}, \bar{y})$, so that the instablility of $\bar{y}$ is of nonlinear type. We now show that every solution of (5) converges to $\bar{y}$ so that Theorem 1 fails. Let $\{y_n\}$ be any solution of (5), and note that because of Y_c and the instability of $\bar{y}$, there is a $k \geq 2$ such that $\Delta y_{k-1} = y_k - y_{k-1} \leq 0$. If also $\Delta y_{k-2} \leq 0$, then using (5) and subtracting, we have

$$y_{k+1} - y_k = \Delta y_k = c\Delta y_{k-1} \leq 0$$

which implies $\Delta y_{k+1} = c\Delta y_k$. Applying this argument inductively, we find that

$$\Delta y_n = c^{n-k}\Delta y_k \quad n \geq k. \tag{6}$$

Now suppose that $\Delta y_{k-2} > 0 \geq \Delta y_{k-1}$; then the trajectory has turned around because of Y_c. Thus either $y_{k-1} = Y_c$ or $y_k = Y_c$, but $y_{k-2} < Y_c$. In either case, $y_{k+1} \leq y_k$ so that $\Delta y_k \leq 0 \leq d$ and thus $\Delta y_{k+1} = c\Delta y_k$. Again, using induction we obtain (6). In particular, for all $n \geq k$, (6) implies that $\Delta y_n \leq 0$, so that (5) reduces to the first order equation

$$y_n = cy_{n-1} + c_0 \quad n \geq k$$

whose solution $y_n = \beta c^{n-k} + \bar{y}$ where β is a constant, converges to $\bar{y}$. This shows that the fixed point $\bar{y}$ is an unstable global attractor. Such fixed points have been noted before in the literature for general planar systems; see [4, p.109], [13, p.90] and for a continuous time example, see [19, p. 65] (although the last one has actually two fixed points). Our example here displays a globally attracting, unstable fixed point in a second order *equation*, which is, of course, a very special type of planar system. The sensitivity of solutions of (5) to slight external perturbations is discussed in [18].

Historical Remarks

The theory of the "business cycle," namely, the observed and well-documented fluctuations in employment, total output (or the GNP), investment, etc. which are responsible for recurrent recessions and changing levels of inflation and unemployment has been and continues to be an important source of significant mathematical problems in economic dynamics. Mathematical models analyzing the economic mechanisms capable of producing steady oscillations in the total output of an economy date back to the 1930's. One of the earliest discrete models is the second-order, linear model of P. Samuelson which appeared in a 1939 paper [15] and was influenced by the work of the economist A. Hansen. This model attracted some attention because with its aid, Samuelson was able to show the emergence of cycles simply and rigorously as a consequence of the interaction between the Aftalion-Clark "acceleration principle" and the famous Keynesian "multiplier effect."

It was soon noticed that the linear model had fundamental defects, namely, exempting the exceptional periodic case, the oscillations in output were either damped or explosive; even the periodic case was questioned since its amplitude depended on the initial conditions. These criticisms highlighted the need for the introduction of nonlinear stablizing effects that could give rise to bounded and persistent oscillations. With the exception of one additional, though largely ignored effort by Samuelson [16], all well-known nonlinear models before 1950 were proposed in continuous time (the most prominent of which is Kaldor's model; see [9] or the more rigorous mathematical exposition in [3]). Then, in 1950, Hicks formulated his model in terms of a *nonlinear delay difference equation*. At its core, Hicks's oscillator is essentially Samuelson's linear accelerator-multiplier mechanism, although Hicks adds on some of the nonlinear stablizers that one observes in the economy (i.e., a growth ceiling due to full utilization of existing labor and natural resources, as well as a minimum or "floor" level of net investment). The Hicks model has since appeared in various simplified (and occasionally erroneous) forms throughout the economic literature. Other discrete models of the business cycle were proposed after [7] debuted, some of which included economic parameters that were either ignored or not properly treated by Hicks while others considered non-accelerator type mechanisms; a sampling of some of these works (as well as several past and recent continuous time models such as Goodwin's classic accelerator-based nonlinear model [5]) may be found in [8].

The model proposed by Hicks is based on a non-autonomous, piecewise linear delay difference equation which generates oscillations about an exponentially increasing "equilibrium path" (which represents economic growth in the model). Hicks shows that the ratio of the amplitudes of these oscillations over the rising equilibrium value satisfies (in the uniform limit) an *autonomous* delay difference equation of the same order as the original equation; then he argues (relying on piecewise linearity) that all non-trivial solutions of the autonomous equation execute bounded, persistent oscillations when the stationary equilibrium is unstable.

From a strictly mathematical point of view, Hicks's arguments are rather difficult to follow as they are blended with economics and scattered all over his 200-page monograph, not just in the "Mathematical Appendix" (which is concerned primarily with a detailed though incomplete analysis of the various *linear* components of his equation). Equation (4) is a distillation of Hicks's ideas; the middle linear section which represents the lagged (or delay) version of Samuelson's accelerator-multiplier mechanism, is listed explicitly by Hicks as Eq.(19.1) in [7, p.185]. The terms Y_c and d (the nonlinearities introduced by Hicks) specify the output ceiling and the investment floor, respectively. Our Corollary 1 gives sufficient conditions for the oscillations of Hicksian trajectories with arbitrary lag structure.

We now turn to Eq.(2). Before applying Theorem 1 to this equation, we state a fundamental permanence theorem due to G. Ladas and V.L. Kocic [11, p.35] as a lemma. Permanence and other boundedness results are often needed in conjuction with Theorem 1 since the conditions in the latter do not necessarily imply the *existence* of non-trivial bounded solutions.

LEMMA 1. *Assume that the function f in (2) satisfies the following conditions for $m \geq 2$:*

a. $f \in C[(0,\infty) \times [0,\infty)^{m-1}, (0,\infty)]$ *and* $\lim_{x^1 \to 0^+} x^1 f(x^1, x^2, \ldots, x^m) > 0$ *exists;*

b. $f(x^1, x^2, \ldots, x^m)$ *is nonincreasing in* $x^2, \ldots, x^m$;

c. *The equation* $f(x, \ldots, x) = 1$ *has a unique positive solution* $\bar{x}$;

d. *For every* $x > 0$, *and* $u \geq 0$:

$$[f(x, u, \ldots, u) - f(\bar{x}, u, \ldots, u)](x - \bar{x}) \leq 0$$

with

$$[f(x, \bar{x}, \ldots, \bar{x}) - f(\bar{x}, \bar{x}, \ldots, \bar{x})](x - \bar{x}) < 0 \quad \text{for} \quad x \neq \bar{x}.$$

Then (2) *is positively permanent with an absorbing interval* $[L, M]$ *where:*

$$M = \max\{2\bar{x}, [f(\bar{x}, 0, \ldots, 0)]^m \sup_{0 \leq x \leq \bar{x}} xf(x, 0, \ldots, 0)\}$$

and:

$$L = \min\{\frac{x}{2}\ [f(\bar{x}, M, \ldots, M)]^m \inf_{0 \le x \le \bar{x}} xf(x, 0, \ldots, 0)\}.$$

PROPOSITION 1. *In addition to (a)-(d) in Lemma 1, assume that f satisfies the following condition:*

e. *For $i = 1, \ldots, m$, $\partial f/\partial x^i$ exist continuously at $\bar{X} = (\bar{x}, \ldots, \bar{x})$ and all roots of the polynomial $\lambda^m - \bar{x}[1 + \partial f/\partial x^1(\bar{X})]\lambda^{m-1} - \bar{x}\sum_{i=2}^{m} \partial f/\partial x^i(\bar{X})\lambda^{m-i}$ have modulus greater than 1.*

Then every non-trivial solution of (2) *eventually oscillates persistently in the absorbing interval* $[L, M]$ *of Lemma 1.*

Proof. We need only verify Condition (c) in THEOREM 1. Condition (e) in this proposition in particular implies that $\partial f/\partial x^m(\bar{X}) \neq 0$ which together with (b) in Lemma 1 yields $f(\bar{x}, \ldots, \bar{x}, x) \neq 1$ for $x \neq \bar{x}$. □

We close this section with the following special case of Eq.(1).

COROLLARY 2. *Let $h \in C[[0,\infty),[0,\infty)]$ and consider the equation:*

$$x_n = h(x_{n-m}) \quad (m \ge 1) \tag{7}$$

Assume that the following conditions hold:

a. *The equation $h(x) = x$ has a unique solution $\bar{x} > 0$;*
b. $h^{-1}(\bar{x}) = \{\bar{x}\}$;
c. *The derivative h' exists continuously at $\bar{x}$ with $|h'(\bar{x})| > 1$.*

Then except for the trivial solution $\bar{x}$, all solutions of (7) *eventually oscillate persistently in the absorbing interval* $[0, M]$ *where* $M = \sup_{0 \le x \le \bar{x}} h(x)$.

Proof. If we define $F \in C[[0,\infty)^m, [0,\infty)]$ by $F(x^1, \ldots, x^m) = h(x^m)$, then $\bar{X} = (\bar{x}, \ldots, \bar{x})$ are the fixed points of V_F and the conditions of Theorem 1 are all satisfied. To show that (7) is permanent, observe that since each solution of (7) is obtained by interlacing m solutions of the same equation in first order form, we need only consider the first order case $m = 1$.

Let M be as defined in the statement of the corollary. Due to (a)-(c) and the continuity of h, we have $h(0) > 0$ and

$$h(x) < x \quad \text{for} \quad x > \bar{x} \tag{8}$$

Also due to continuity of h' at $\bar{x}$, we have $h'(x) < -1$ for all x in some sufficiently small interval $(\bar{x} - \varepsilon, \bar{x} + \varepsilon)$. Now the mean-value theorem implies that

$$M - \bar{x} \ge h(x) - h(\bar{x}) > \bar{x} - x > 0$$

for each $x \in (\bar{x} - \varepsilon, \bar{x})$. Hence, $M > \bar{x}$ and therefore, $h(M) < M$. Furthermore,

$$x \le M \text{ implies } h(x) \le M. \tag{9}$$

The preceding statement is true because if $x \le \bar{x}$, then $h(x) \le M$ by the definition of M, while if $\bar{x} < x \le M$, then $h(x) < x \le M$.

Now let $\{x_n\}$ be an arbitrary solution of (7) with $m = 1$. From (8) it follows that there is an integer $r \ge 1$ such that $x_r \le M$. Thus by (9) $x_n \le M$ for all $n \ge r$. □

Remarks

The preceding corollary in particular gives a persistent oscillation result for first order equations. Its conditions can be considerably weakened if we allow for solutions that are constant in a finite number of iterations; however, additional conditions will then be required to ensure that the stable sets of fixed point(s) are not too large. Note that if $\inf_{x>0} h(x) > 0$ then (7) is positively permanent. On the other hand, if $h \in C[(0,\infty), (0,\infty)]$ then (7) need not be permanent; a counterexample would be $x_n = 1/x_{n-1}^2$ whose only bounded solution is the trivial one $\bar{x} = 1$. Requiring that $\lim_{x\to 0^+} h(x) > 0$ exist would, of course, be equivalent to defining $h(0)$ as the limit.

3 OSCILLATIONS OF SECOND-ORDER EQUATIONS

In this section, we consider some second-order special cases of the results of the preceding section. We begin with the next lemma that gives the necessary and sufficient restrictions on parameters for the fixed point to be repelling.

LEMMA 2. *The following statements are equivalent:*

a. *The origin is a repelling fixed point of the linear equation*:

$$x_n = px_{n-1} + qx_{n-2}$$

b. *Both roots of the polynomial*:

$$\lambda^2 - p\lambda - q = 0 \tag{10}$$

have modulus greater than 1.

c. $|q| > 1$ *and* $|q - 1| > |p|$.

Proof. We need only show that condition (c) is equivalent to both roots of (10) having modulus greater than 1. These roots are

$$\lambda_1 = \frac{1}{2}[p - \sqrt{p^2 + 4q}] \quad \text{and} \quad \lambda_2 = \frac{1}{2}[p + \sqrt{p^2 + 4q}].$$

CASE 1. Both roots are complex: $p^2 + 4q < 0$; in this case $|\lambda_i|^2 = -q$, $i = 1, 2$ so $|\lambda_i| > 1$ if and only if $q < -1$. Also note that since $q < -p^2/4$ we have $q < 1-|p|$ in this case.

CASE 2. $p^2 + 4q \geq 0$ so that both roots are real and

$$\lambda_1 \leq p/2 \leq \lambda_2 \tag{1}$$

Now $\lambda_1 > 1$ if and only if $p^2 + 4q < (p-2)^2$, i.e., $q < 1 - p$. Because of (11) we also have $p > 2$ and $\lambda_2 > 1$. Similarly, $\lambda_2 < -1$ if and only if $q < 1 + p$, with $p < -2$ and $\lambda_1 < -1$ due to (11). Together with Case 1, we have now shown that when $q < -1$, then both roots have modulus greater than 1 if and only if $q < 1 - |p|$.

Finally, it remaines to consider the possibility that $\lambda_2 > 1$ and $\lambda_1 < -1$. This is easily seen to be equivalent to the inequality $q > 1 + |p|$. Since this implies $q > 1$, we have completed the proof. □

The next result is the second-order version of Theorem 1.

PROPOSITION 2. *Consider Eq.*(1) *with* $m = 2$ *and* $F = F(x,y)$. *Assume that the following conditions hold:*

a. *The equation* $F(x, x) = x$ *has a finite number of solutions* $0 < \bar{x}_1 < \ldots < \bar{x}_k$;
b. *For every* $j = 1, \ldots, k$, $F(\bar{x}_j, y) \neq \bar{x}_j$; *if* $y \neq \bar{x}_j$;
c. $\partial F/\partial x$ *and* $\partial F/\partial y$ *both exist continuously at* $(\bar{x}_j, \bar{x}_j)$ *for all* $j = 1, \ldots, k$, *with:*

$$\left|\frac{\partial F}{\partial y}(\bar{x}_j, \bar{x}_j)\right| > 1, \qquad \left|\frac{\partial F}{\partial y}(\bar{x}_j, \bar{x}_j) - 1\right| > \left|\frac{\partial F}{\partial x}(\bar{x}_j, \bar{x}_j)\right|.$$

Then all non-trivial bounded solutions oscillate persistently.

PROPOSITION 3. *Consider Eq.*(2) *with* $m = 2$ *and* $f = f(x, y)$. *Assume that the following condition holds in addition to (a)-(d) in Lemma* 1*:*

e. $\partial f/\partial x$ *and* $\partial f/\partial y$ *both exist continuously at* $(\bar{x},\bar{x})$, $i = 1,2$, *with*:

$$\left|\frac{\partial f}{\partial y}(\bar{x}, \bar{x})\right| > \frac{1}{\bar{x}}, \qquad \left|\frac{\partial f}{\partial y}(\bar{x}, \bar{x}) - \frac{1}{\bar{x}}\right| > \left|\frac{\partial f}{\partial x}(\bar{x}, \bar{x}) + \frac{1}{\bar{x}}\right|.$$

Then all non-trivial solutions of (2) *with* $m = 2$ *eventually oscillate persistently in the absorbing interval* $[L, M]$ *defined in Lemma* 1.

Proposition 3 is easily applicable, as demonstrated by the following.

COROLLARY 3. *The equation:*

$$x_n = \frac{a + bx_{n-1}}{c + x_{n-2}^d} \qquad (a, b, c > 0, \quad d > 1) \tag{12}$$

has a unique positive fixed point $\bar{x}$. *If also*:

$$\alpha \geq \frac{d}{d-1}, \quad c \leq b \leq \alpha c, \quad a > (\alpha c - b)[(b-c)^{1/d} + a^{1/(d+1)}] \tag{13}$$

then all non-trivial solutions of (12) eventually oscillate persistently in the interval $[L, M]$ *where:*
$M = \max\{2\bar{x}, \bar{x}(1 + \bar{x}^d/c)^2\}$ $\quad L = \min\{\bar{x}/2, \bar{x}(1 + \bar{x}^d/c)[(c + \bar{x}^d)/(c + M^d)]^2\}$.

Proof. The fixed points of (12) are zeros of the function

$$\phi(x) \doteq x^{d+1} - (b - c)x - a.$$

Since $\phi(0) = -a < 0$ and ϕ is twice continuously differentiable, using elementary calculus it can be easily seen that $\phi(x)$ has precisely one positive zero $\bar{x}$ for $d > 1$ and all positive a, b, c; $\bar{x}$ is thus the unique positive fixed point of (12). If (13) also holds, then we have the following bounds on $\bar{x}$:

$$(b - c)^{1/d} < \bar{x} \le (b - c)^{1/d} + a^{1/(d+1)}. \tag{14}$$

The lower bound is easy to see as $\phi\,((b - c)^{1/d}) = -a < 0$; as for the upper bound, define $\beta \doteq (b - c)^{1/d}$ and $\gamma \doteq a^{1/(d+1)}$ and note that if $b = c$ then $\phi(\beta + \gamma) = 0$ while if $b > c$ then

$$\phi(\beta + \gamma) = a[(1 + \beta/\gamma)^d - 1] + (\beta + \gamma)(b - c)[(1 + \gamma/\beta)^{d-1} - 1] > 0$$

so that $\phi(\beta + \gamma) \ge 0$.

Next, define

$$f(x, y) = \frac{ax^{-1} + b}{c + y^d}$$

and note that conditions (a)-(d) of Proposition 3 are easily verified. It remains to verify (e). Direct computation gives

$$\frac{\partial f}{\partial y}(\bar{x}, \bar{x}) = \frac{-d\bar{x}^{-d-1}}{c + \bar{x}^{-d}}, \qquad \frac{\partial f}{\partial x}(\bar{x}, \bar{x}) = \frac{-a\bar{x}^{-2}}{c + \bar{x}^{-d}},$$

so that

$$\left|\frac{\partial f}{\partial y}(\bar{x}, \bar{x}) - \frac{1}{\bar{x}}\right| = \frac{(d + 1)\bar{x}^{-d} + c}{\bar{x}(c + \bar{x}^{-d})} \qquad \left|\frac{\partial f}{\partial x}(\bar{x}, \bar{x}) + \frac{1}{\bar{x}}\right| = \frac{b}{\bar{x}(c + \bar{x}^{-d})}$$

By (14) $\bar{x}^d > (b - c) \ge (b - c)/(d + 1)$, so we have the second inequality in (e). Further, from (13) and (14) we see that

$$a > (\alpha c - b)(\beta + \gamma) \ge (\alpha c - b)\bar{x}$$

which implies that

$$\alpha c < \frac{a}{\bar{x}} + b = \bar{x}^{-d} + c$$

i.e., $\bar{x}^{-d} > (\alpha - 1)c \ge c/(d - 1)$. This readily implies the other inequality in (e). The bounds L and M of the absorbing interval are easily computed using the formulas in Lemma 1. $\square$

Remark

The results in [11, Section 3.4] show that $\bar{x}$ in Corollary 3 may be globally asymptotically stable when $d = 1$, even if the inequalities concerning a, b, c hold for some $\alpha \geq 1$ in (13). In such a case, persistent oscillations obviously do not arise; however, decaying oscillations could exist, and their existence may be established using linearized oscillation results.

The next boundedness result, proved in [2], is similar to Lemma 1 although the conclusion is not as strong.

LEMMA 3. *Consider the equation:*

$$x_n = g(x_{n-1})f(x_{n-2}) \tag{15}$$

and assume that the following conditions hold:

a. $g \in C[[0,\infty),(0,\infty)]$ *and* $f \in [[0,\infty), [0,\infty)]$;

b. *g is increasing and f nonincreasing;*

c. *There exist l, p, $q \geq 0$ and A, $B > 0$ such that $g(x) \leq Ax^p$ and $f(x) \leq Bx^{-q}$ for all $x \geq l$;*

d. *Either $p = 0$ or $0 < p^2 < 4q$;*

e. *The equation $g(x)f(x) = x$ has precisely $k + 1$ solutions $0 \leq \bar{x}_0 < \bar{x}_1 < \ldots < \bar{x}_k < \infty$, $k \geq 0$.*

Then every solution of (15) *is bounded.*

PROPOSITION 4. *Assume that the following conditions hold in addition to* (a)-(d) *in Lemma* 3:

e. *The equation $g(x)f(x) = x$ has a finite number of solutions $0 < \bar{x}_1 < \ldots < \bar{x}_k < \infty$;*

f. *f and g are continuously differentiable at $\bar{x}_j$ for every $j = 1, \ldots, k$ and:*

$$\left|\frac{f'(\bar{x}_j)}{f(\bar{x}_j)}\right| > \frac{1}{\bar{x}_j} \qquad \left|\frac{f'(\bar{x}_j)}{f(\bar{x}_j)} - \frac{1}{\bar{x}_j}\right| > \left|\frac{g'(\bar{x}_j)}{g(\bar{x}_j)} + \frac{1}{\bar{x}_j}\right| \tag{6}$$

Then every non-trivial solution of (15) *is bounded and oscillating persistently.*

Proof. Inequalities (16) being just restatements of those in (c) of Proposition 2, we need only show that condition (b) in Proposition 2 holds, with $F(x,y)=g(x)f(y)$. Note for each j that if $g(\bar{x}_j)f(y) = \bar{x}_j$, then since f is nonincreasing and by condition (f) of this corollary, $f'(x) \neq 0$ for all x in some small neighborhood of $\bar{x}_j$, we must have $y = \bar{x}_j$. □

Proposition 4 may be used, in particular, to establish persistent oscillatory behavior where Proposition 3 does not apply. The next result is a case in point.

COROLLARY 4. *Consider the equation:*

$$x_n = \frac{a + bx_{n-1}^2}{c + x_{n-2}^2} \qquad (a, b, c > 0) \tag{7}$$

and assume that the following conditions hold:

$$b^2 \leq 3c, \quad a > \max\{bc, (2\sqrt{c} - b)c, \ (\sqrt{c} - b)^3\} \tag{8}$$

Then (17) *has a unique fixed point* $\bar{x} > 0$ *and all non-trivial solutions of* (17) *are bounded and persistently oscillating.*

Proof. The fixed points of (17) are the roots of the cubic polynomial

$$P(x) \doteq x^3 - bx^2 + cx - a$$

Under the restriction $b^2 - 3c \leq 0$ in (18), $P(x)$ has a non-negative derivative everywhere, and hence, only one real root. Since $P(0) = -a < 0$, this real root must be positive and it is thus $\bar{x}$. Also note that due to (18), $b + a^{1/3} > c^{1/2}$ and

$$P(b) = bc - a < 0, \ P(\sqrt{c}) = 2c\sqrt{c} - bc - a < 0, \ P(b + \sqrt[3]{a}) > 0.$$

Therefore,

$$\max\{b, \sqrt{c}\} < \bar{x} < b + \sqrt[3]{a}. \tag{9}$$

Now to apply Proposition 4, define $g(x) = a + bx^2$ and $f(x) = 1/(c+x^2)$. Clearly (a), (b) and (e) of Proposition 4 hold; also with $p = q = 2$, $B = 1$, $A = b + 1$ conditions (c) and (d) of Proposition 4 hold with $l = a^{1/2}$. We now verify (f). Note that

$$\left|\frac{f'(\bar{x})}{f(\bar{x})}\right| = \frac{2\bar{x}}{c + \bar{x}^2} > \frac{1}{\bar{x}}$$

since $\bar{x}^2 > c$ by (19). Further,

$$\left|\frac{f'(\bar{x})}{f(\bar{x})} - \frac{1}{\bar{x}}\right| = \frac{2\bar{x}}{c + \bar{x}^2} + \frac{1}{\bar{x}} \qquad \left|\frac{g'(\bar{x})}{g(\bar{x})} + \frac{1}{\bar{x}}\right| = \frac{2b\bar{x}}{a + b\bar{x}^2} + \frac{1}{\bar{x}}$$

so that the second inequality in (16) holds if

$$\frac{2\bar{x}}{c + \bar{x}^2} > \frac{2b\bar{x}}{a + b\bar{x}^2}$$

i.e., if

$$\bar{x} = \frac{a + b\bar{x}^2}{c + \bar{x}^2} > b$$

which is true by (19). Now the application of Proposition 4 completes the proof. $\square$

We close with a direct application of Proposition 2 in the form of an oscillation result for a generic extension of the second-order Hicks equation called the *Goodwin-Hicks equation*, namely,

$$x_n = a + cx_{n-1} + g(x_{n-1} - x_{n-2}). \tag{10}$$

The background on this equation is given in [18]. The next result, proved in [17], states sufficient conditions for the permanence of (20). We note that if $c \neq 1$ then

$$\bar{x} = \frac{a + g(0)}{1 - c}$$

is the unique fixed point of (20).

LEMMA 4. *Assume that a, d are real numbers with $d < a$ and let $0 \le c < 1$. If g: $(-\infty, \infty) \to [d, \infty)$ is nondecreasing and if there is $b \in (0,1)$ and $u_0 > 0$ such that $g(u) \le bu - a$ for all $u \ge u_0$, then for all large n every solution $\{x_n\}$ of (20) is eventually in the interval $[L,M]$, where:*

$$L = \frac{a+d}{1-c} - 1, \qquad M = \max\left\{\frac{u_0}{1-c}, \frac{-2L}{1-b}\right\} - L + 1..$$

COROLLARY 5. *Assume the following conditions:*

a. $g \in C[(-\infty, \infty), [d, \infty)]$, $d < 0$;
b. $a + d \ge 0$ *and* $0 \le c < 1$;
c. *g is nondecreasing everywhere and* $g(0) = 0$;
d. *g is continuously differentiable at the origin with* $g'(0) > 1$;
e. *There are constants $u_0 > 0$, $0 < b < 1$ such that $g(u) \le bu - a$ for all $u \ge u_0$.*

Then all non-trivial solutions of (20) with initial values $x_{-1}, x_0 \ge 0$ eventually oscillate persistently in the interval $[a + d, u_0/(1 - c) + 1]$.

Proof. Given (a) and (b) above, define $F \in C[[0,\infty)^2, [a + d, \infty)]$ as

$$F(x, y) = a + cx + g(x - y), \quad x, y \ge 0$$

and observe that $\bar{x} = a/(1 - c)$ is now the unique positive fixed point of (20). Note also that $0 = g(0) \le g(u_0)$ so

$$\bar{x} \le \frac{a + g(u_0)}{1 - c} \le \frac{bu_0}{1 - c} < \frac{u_0}{1 - c}$$

implying that $\bar{x}$ is in the interior of the absorbing interval in the statement of the corollary. Further, if $\bar{x} = F(\bar{x}, y) = \bar{x} + g(\bar{x} - y)$, then (c) and (d) above imply that $y = \bar{x}$. Hence conditions (a) and (b) of Proposition 2 are satisfied. Since

$$\frac{\partial F}{\partial y}(\bar{x}, \bar{x}) = -g'(0), \qquad \frac{\partial F}{\partial x}(\bar{x}, \bar{x}) = c + g'(0)$$

condition (c) of Proposition 2 is also easily seen to hold. The proof may now be completed upon applying Proposition 2 and Lemma 4. With regard to the absorbing interval, we note that if $\{x_n\}$ is any solution of (20), then the definition of F above shows that $x_n \geq a + d$ for all $n \geq 1$. Thus we may set $L = 0$ in Lemma 4 and obtain $M = u_0/(1-c) + 1$. □

4 CONCLUSION

We derived above sufficient conditions implying persistent oscillatory behavior for many types of difference equations. From the stand-point of the theory, if it is known that a particular difference equation possesses non-trivial bounded solutions and a strongly unstable fixed point, then the results of this paper can quickly establish the oscillatory behavior, when it exists. Therefore, extensions of Lemmas 1, 3 and 4 would be desirable. In applications (such as the Hicks equation) one often has grounds for assuming the existence of non-trivial bounded solutions, although as Lemma 4 shows, sometimes it is necessary to give a proof of such a fact.

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