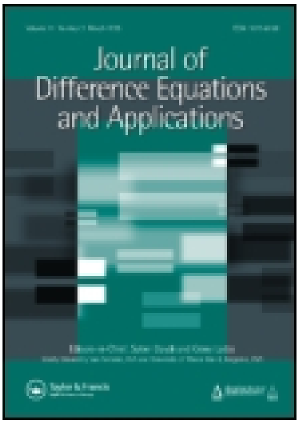

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Global attractivity in a class of non-autonomous, nonlinear, higher order difference equations

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Global attractivity in a class of non-autonomous, nonlinear, higher order difference equations

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Non-autonomous, higher order difference equations of type

$$x_{n+1} = \sum_{i=0}^{k} a_i x_{n-i} + g_n\left(\sum_{i=0}^{k} b_i x_{n-i}\right)$$

with real variables and parameters have appeared frequently in the literature. These equations are well defined on Banach algebras, and existing convergence results can be generalized from real numbers to algebras. Through this generalization and by using a recently obtained semiconjugate factorization of the above equation, new sufficient conditions are obtained for the convergence to zero of all solutions of nonlinear difference equations of the above type. Where reduction of order is possible, these conditions extend the ranges of parameters for which the origin is a global attractor even in the case of real variables and parameters.

Keywords: reduction of order; global attractivity; non-autonomous; higher order; difference equation; Banach algebra

AMS Subject Classification: 39A10; 39A30

1. Introduction

Special cases of the following type of higher order difference equation have frequently appeared in the literature in different contexts, both pure and applied:

$$x_{n+1} = \sum_{i=0}^{k} a_i x_{n-i} + g_n\left(\sum_{i=0}^{k} b_i x_{n-i}\right), \quad n = 0, 1, 2, \ldots. \tag{1}$$

We assume here that k is a fixed positive integer and for each n, the function $g_n:\mathbb{X} \to \mathbb{X}$ is defined on a real or complex Banach algebra $\mathbb{X}$ with identity. The parameters a_i, b_i are fixed elements in $\mathbb{X}$ such that

$$a_k \neq 0 \quad \text{or} \quad b_k \neq 0.$$

Upon iteration, equation (1) generates a unique sequence of points $\{x_n\}$ in $\mathbb{X}$ (its *solution*) from any given set of $k+1$ *initial values* $x_0, x_{-1}, \ldots, x_{-k} \in \mathbb{X}$. The number $k+1$ is the order of (1).

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The choice of a Banach algebra $\mathbb{X}$ is motivated by the fact that (1) is well defined on any algebra, and further, the foundations that support such a level of generality in the presence of a norm are already in place. Given that every normed algebra has a completion as a normed space, taking $\mathbb{X}$ to be a Banach algebra is not a significant loss of generality. On the other hand, all occurrences of (1) in the literature so far and the existing results for it involve $\mathbb{X} = \mathbb{R}$, the set of all real numbers. Thus, readers who are not interested in algebras may simply consider all parameters and variables to be real numbers.

Special cases of equation (1) on the set of real numbers appeared in the classical economic models of the business cycle in the twentieth century in the works of Hicks [8], Puu [17], Samuelson [18] and others; see [21], Section 5.1 for some background and references. Other special cases of (1) occurred later in mathematical studies of biological models ranging from whale populations to neuron activity; see, e.g. Clark [2], Fisher and Goh [5], Hamaya [7] and Section 2.5 in Kocic and Ladas [11].

The dynamics of special cases of (1) with $\mathbb{X} = \mathbb{R}$ have been investigated by several authors. Hamaya uses Liapunov and semicycle methods in [7] to obtain sufficient conditions for the global attractivity of the origin for the following special case of (1)

$$x_{n+1} = \alpha x_n + a \tan h\left(x_n - \sum_{i=1}^{k} b_i x_{n-i}\right)$$

with $0 \leq \alpha < 1$, $a > 0$ and $b_i \geq 0$. These results can also be obtained using only the contraction method in [20]. The results in [20] are also used in [21], Section 4.3D, to prove the global asymptotic stability of the origin for an autonomous special case of (1) with $a_i b_i \geq 0$ for all i and $g_n = g$ for all n, where g is a continuous, non-negative function. The study of global attractivity and stability of fixed points for other special cases of (1) appear in [6] and [9]; also see [11], Section 6.9.

The second-order case ($k = 1$) has been studied in greater depth. Kent and Sedaghat obtain sufficient conditions in [10] for the boundedness and global asymptotic stability of

$$x_{n+1} = cx_n + g(x_n - x_{n-1}). \tag{2}$$

Also see [22]. In [4], El-Morshedy improves the convergence results of [10] for (2) and also gives necessary and sufficient conditions for the occurrence of oscillations. The boundedness of solutions of (2) is studied in [19] and periodic and monotone solutions of (2) are discussed in [23]. Li and Zhang study the bifurcations of solutions of (2) in [13]; their results include the Neimark–Sacker bifurcation (discrete analogue of Hopf).

A more general form of (2), i.e. the following equation

$$x_{n+1} = ax_n + bx_{n-1} + g_n(x_n - cx_{n-1}) \tag{3}$$

is studied in [24] where sufficient conditions for the occurrence of periodic solutions, limit cycles and chaotic behaviour are obtained using reduction of order and factorization of the above difference equation into a pair of equations of lower order. See [26] for some background on order reduction methods. These methods are used in [3] to determine sufficient conditions on parameters for occurrence of limit cycles and chaos in those rational difference equations of the following type

$$x_{n+1} = \frac{ax_n^2 + bx_{n-1}^2 + cx_nx_{n-1} + dx_n + ex_{n-1} + f}{\alpha x_n + \beta x_{n-1} + \gamma}$$

that can be reduced to special cases of (3).

In this paper, by generalizing recent results on reduction of order, together with generalizations of some convergence results from the literature, we obtain sufficient conditions for the global attractivity of the origin for (1) in the context of Banach algebras. These results also extend previously known parameter ranges, even in the case of real numbers, i.e. $\mathbb{X} = \mathbb{R}$ and show that convergence may occur in some cases where the functions g_n or the unfolding map of (1) are not contractions.

Unless otherwise stated, throughout the rest of this paper $\mathbb{X}$ will denote a real or complex Banach algebra with identity 1 (since there is very little likelihood of confusion, 1 also denotes the identity of the underlying field of real or complex numbers). For the basics of Banach algebras, see, e.g. [12] or [27]. For convenience, we list a few basic features of Banach algebras here.

Each Banach algebra is a Banach space together with a multiplication operation xy that is associative, distributes over addition and satisfies the norm inequality

$$|xy| \leq |x||y| \tag{4}$$

with $|1| = 1$. The multiplication by real or complex numbers (or more generally, elements of an underlying field of scalars) that are inherited from the vector space structure of $\mathbb{X}$ is made consistent with the main multiplication by assuming that the following equalities hold for all scalars α:

$$\alpha(xy) = (\alpha x)y = x(\alpha y).$$

Elements of type $\alpha 1$ for every scalar α are the constants in $\mathbb{X}$. The set $\mathbb{R}$ ($\mathbb{C}$) is a real (complex) commutative Banach algebra with identity over the field of real (complex) numbers with respect to the ordinary addition and multiplication of complex numbers and the absolute value as norm. Less trivially, the Banach space $C[0,1]$ of all continuous real-valued functions on the interval [0,1] with the sup (or max) norm forms a commutative, real Banach algebra relative to the ordinary multiplication of functions. The identity element is the constant function $x(r) = 1$ for all $r \in [0,1]$. The other constants in $C[0,1]$ are just the constant functions on [0,1].

An element x of a Banach algebra $\mathbb{X}$ is invertible, or a unit, if there is $x^{-1} \in \mathbb{X}$ (the inverse of x) such that $x^{-1}x = 1$. The collection of all invertible elements of $\mathbb{X}$ forms a group $\mathcal{G}$ (the group of units) that contains all constants (non-zero). For each $u \in \mathcal{G}$ if $x \in \mathbb{X}$ satisfies the inequality

$$|x - u| \leq \frac{1}{|u^{-1}|},$$

then it can be shown that $x \in \mathcal{G}$. It follows that $\mathcal{G}$ is open relative to the metric topology of $\mathbb{X}$ and contains an open ball of radius $1/|u^{-1}|$ centred about each $u \in \mathcal{G}$. Since the zero element is not invertible, $\mathcal{G} \neq \mathbb{X}$. If $\mathbb{X}$ is either $\mathbb{R}$ or $\mathbb{C}$ then $\mathcal{G} = \mathbb{X} \backslash \{0\}$. In the algebra $C[0,1]$ units are functions that do not assume the (scalar) value 0 (i.e. their graphs do not cross the 'x-axis').

2. General results on convergence

Consider the non-autonomous difference equation

$$x_{n+1} = f_n(x_n, x_{n-1}, \ldots, x_{n-k}) \tag{5}$$

with a given sequence of functions $f_n : \mathbb{X}^{k+1} \to \mathbb{X}$. We say that the origin is *globally exponentially stable* if all solutions $\{x_n\}$ of (5) in $\mathbb{X}$ satisfy the norm inequality

$$|x_n| \leq c^n \mu,$$

where $c \in (0,1)$ and $\mu > 0$ are real constants such that c is independent of the initial values $x_0, x_{-1}, \ldots, x_{-k} \in \mathbb{X}$.

The next result, which is true for all Banach spaces (not just algebras), generalizes Theorem 3 in [20].

Lemma 1. *Let $\mathbb{X}$ be a Banach space and assume that for some real $\alpha \in (0,1)$ the functions f_n satisfy the norm inequality*

$$|f_n(\xi_0, \xi_1, \ldots, \xi_k)| \leq \alpha \max\{|\xi_0|, \ldots, |\xi_k|\}, \tag{6}$$

for every n and all $(\xi_0, \ldots, \xi_k) \in \mathbb{X}^{k+1}$. Then every solution $\{x_n\}$ of (5) with given initial values $x_0, x_{-1}, \ldots, x_{-k} \in \mathbb{X}$ satisfies

$$|x_n| \leq \alpha^{n/(k+1)} \max\{|x_0|, |x_{-1}|, \ldots, |x_{-k}|\}.$$

Therefore, the origin is globally exponentially stable.

Proof. Let $\mu = \max\{|x_0|, |x_{-1}|, \ldots, |x_{-k}|\}$. If $\{x_n\}$ is the solution of (5) with the given initial values, then we first claim that $|x_n| \leq \alpha\mu$ for all $n \geq 1$. By (6)

$$|x_1| = |f_0(x_0, x_{-1}, \ldots, x_{-k})| \leq \alpha \max\{|x_0|, \ldots, |x_{-k}|\} = \alpha\mu$$

and if for any $j \geq 1$ it is true that $|x_n| \leq \alpha\mu$ for $n = 1,2, \ldots, j$ then

$$|x_{j+1}| = |f_j(x_j, x_{j-1}, \ldots, x_{j-k})| \leq \alpha \max\{|x_j|, |x_{j-1}|, \ldots, |x_{j-k}|\} \leq \alpha \max\{\mu, \alpha\mu\} = \alpha\mu.$$

Therefore, our claim is true by induction. In particular, since $0 < \alpha < 1$ we have shown that $|x_n| \leq \alpha^{n/(k+1)}\mu$ for $n = 1,2, \ldots, k+1$. Now suppose that $|x_n| \leq \alpha^{n/(k+1)}\mu$ is true for $n \leq m$ where $m \geq k+1$. Then

$$\begin{aligned}
|x_{m+1}| = |f_m(x_m, x_{m-1}, \ldots, x_{m-k})| &\leq \alpha \max\{|x_m|, |x_{m-1}|, \ldots, |x_{m-k}|\} \\
&\leq \alpha\mu \max\left\{\alpha^{m/(k+1)}, \alpha^{(m-1)/(k+1)}, \ldots, \alpha^{(m-k)/(k+1)}\right\} \\
&= \alpha\mu\alpha^{(m-k)/(k+1)} \\
&= \alpha^{(m+1)/(k+1)}\mu
\end{aligned}$$

and the proof is complete by induction. □

The above induction argument is used by Berezansky, Braverman and Liz in [1] with $\mathbb{X} = \mathbb{R}$ and by Xiao and Yang in [28] in the autonomous case ($f_n = f$ is independent of n) for general Banach spaces. As we see above, this induction argument generalizes to non-autonomous equations in Banach spaces. Other approaches that yield convergence results such as Lemma 1 for $\mathbb{X} = \mathbb{R}$ are discussed by Liz in [14].

For $\mathbb{X} = \mathbb{R}$ Lemma 1 is also implied by Theorem 2 in [15] where Memarbashi uses a contraction argument adapted from Theorem 3 in [20] (exponential stability, autonomous case in $\mathbb{R}$). Contraction arguments with their geometric flavour are intuitively appealing and they also work for non-exponential asymptotic stability; see [20] for the autonomous case and [16] which extends the result in [20] to certain non-autonomous equations.

For a general Banach space, the type of convergence is dictated by the given norm. For instance, in $C[0,1]$ with the sup, or max, norm convergence to the zero function in Lemma 1 is uniform.

Next, define the following sequence of functions on a Banach algebra $\mathbb{X}$

$$f_n(\xi_0, \xi_1, \ldots, \xi_k) = \sum_{i=0}^{k} a_i \xi_i + g_n\left(\sum_{i=0}^{k} b_i \xi_i\right). \tag{7}$$

The following corollary of Lemma 1 generalizes previous convergence theorems proved for the autonomous case with $\mathbb{X} = \mathbb{R}$, e.g. the results in [7] or Theorem 4.3.9(b) in [21].

LEMMA 2. *Let $g_n:\mathbb{X} \to \mathbb{X}$ be a sequence of functions on a real or complex Banach algebra $\mathbb{X}$. Assume that there is a real number $\sigma > 0$ such that*

$$|g_n(\xi)| \leq \sigma|\xi|, \quad \xi \in \mathbb{X}, \tag{8}$$

for all n and further, for coefficients a_i, b_i (real or complex) we assume that the inequality

$$\sum_{i=0}^{k} (|a_i| + \sigma|b_i|) < 1 \tag{9}$$

holds. Then every solution $\{x_n\}$ of (1) with initial values $x_0, x_{-1}, \ldots, x_{-k} \in \mathbb{X}$ satisfies

$$|x_n| \leq \alpha^{n/(k+1)} \max\{|x_0|, |x_{-1}|, \ldots, |x_{-k}|\}, \quad \alpha = \sum_{i=0}^{k} (|a_i| + \sigma|b_i|).$$

Proof. If $(\xi_0, \xi_1, \ldots, \xi_k) \in \mathbb{X}^{k+1}$, then by the triangle inequality, (4) and (8)

$$\begin{aligned}\left|\sum_{i=0}^{k} a_i \xi_i + g_n\left(\sum_{i=0}^{k} b_i \xi_i\right)\right| &\leq \sum_{i=0}^{k} (|a_i| + \sigma|b_i|)|\xi_i| \\ &\leq \left[\sum_{i=0}^{k} (|a_i| + \sigma|b_i|)\right] \max\{|\xi_0|, \ldots, |\xi_k|\}.\end{aligned}$$

Therefore, given (9), by Lemma 1 the origin is globally asymptotically stable. □

Condition (8) implies that the origin is a fixed point of (1) since it implies that $g_n(0) = 0$ for all n. Except for this restriction, the functions g_n are completely arbitrary.

3. Reduction of order

Under certain conditions a special, *order-reducing* change of variables splits or factors equation (1) into a triangular system of two equations of lower order; see [26], Theorem 5.6. The next lemma extends that result from fields to algebras.

LEMMA 3. *Let $g_n:\mathbb{X}\to\mathbb{X}$ be a sequence of functions on an algebra $\mathbb{X}$ with identity (not necessarily normed) over a field $\mathcal{F}$. If for $a_i, b_i \in \mathbb{X}$ the polynomials*

$$P(\xi)=\xi^{k+1}-\sum_{i=0}^{k}a_i\xi^{k-i},\quad Q(\xi)=\sum_{i=0}^{k}b_i\xi^{k-i}$$

have a common root $\rho\in\mathcal{G}$, the group of units of $\mathbb{X}$, then each solution $\{x_n\}$ of (1) in $\mathbb{X}$ satisfies

$$x_{n+1}=\rho x_n+t_{n+1}, \tag{10}$$

where the sequence $\{t_n\}$ is the unique solution of the equation:

$$t_{n+1}=-\sum_{i=0}^{k-1}p_it_{n-i}+g_n\left(\sum_{i=0}^{k-1}q_it_{n-i}\right) \tag{11}$$

in $\mathbb{X}$ with initial values $t_{-i}=x_{-i}-\rho x_{-i-1}\in\mathbb{X}$ for $i = 0,\ldots,k-1$ *and coefficients*

$$p_i=\rho^{i+1}-a_0\rho^i-\cdots-a_i\quad\text{and}\quad q_i=b_0\rho^i+b_1\rho^{i-1}+\cdots+b_i$$

in $\mathbb{X}$. Conversely, if $\{t_n\}$ is a solution of (11) *with initial values $t_{-i}\in\mathbb{X}$ then the sequence $\{x_n\}$ that it generates in $\mathbb{X}$ via* (10) *is a solution of* (1).

Proof. Define the functions f_n as in (7) and for every $\xi_0, v_1, \ldots, v_k$ in $\mathbb{X}$, define $\zeta_0=\xi_0$ and for $j=1,\ldots,k$ and fixed $\gamma\in\mathcal{G}$ define

$$\zeta_j=(\gamma^{-1})^j\xi_0+\sum_{i=1}^{j}(\gamma^{-1})^{j-i+1}v_i.$$

Now the change of variables

$$t_n=x_n-\gamma x_{n-1} \tag{12}$$

in equation (1) reduces its order by 1 if and only if the quantity

$$f_n(\xi_0,\zeta_1,\ldots,\zeta_k)-\gamma\xi_0$$

is independent of ξ_0 ([26], Theorem 5.1). In this case, the above quantity defines a sequence of functions $\phi_n(v_1,\ldots,v_k)$ of k variables that yields a difference equation of order k ([26], Section 5.5). Now, by straightforward calculation

$$
\begin{aligned}
f_n(\xi_0, \zeta_1, \ldots, \zeta_k) - \gamma\xi_0 &= (a_0 - \gamma)\xi_0 + \sum_{j=1}^{k} a_j\left((\gamma^{-1})^j\xi_0 - \sum_{i=1}^{j}(\gamma^{-1})^{j-i+1}v_i\right) \\
&\quad + g_n\left(b_0\xi_0 + \sum_{j=1}^{k} b_j\left[(\gamma^{-1})^j\xi_0 - \sum_{i=1}^{j}(\gamma^{-1})^{j-i+1}v_i\right]\right) \\
&= \left[a_0 - \gamma + \sum_{j=1}^{k} a_j(\gamma^{-1})^j\right]\xi_0 - \sum_{j=1}^{k} a_j \sum_{i=1}^{j}(\gamma^{-1})^{j-i+1}v_i \\
&\quad + g_n\left(\left[b_0 + \sum_{j=1}^{k} b_j(\gamma^{-1})^j\right]\xi_0 - \sum_{j=1}^{k} b_j \sum_{i=1}^{j}(\gamma^{-1})^{j-i+1}v_i\right).
\end{aligned}
$$

The last expression above is independent of ξ_0 (for arbitrary ξ_0) if and only if γ can be chosen such that

$$
a_0 - \gamma + \sum_{j=1}^{k} a_j(\gamma^{-1})^j = 0 \quad \text{and} \quad b_0 + \sum_{j=1}^{k} b_j(\gamma^{-1})^j = 0.
$$

Multiplying the two equalities above on the right by γ^k yields

$$
\begin{aligned}
0 &= a_0\gamma^k - \gamma^{k+1} + \sum_{j=1}^{k} a_j(\gamma^{-1})^j\gamma^k = P(\gamma), \\
0 &= b_0\gamma^k + \sum_{j=1}^{k} b_j(\gamma^{-1})^j\gamma^k = Q(\gamma),
\end{aligned}
$$

so that γ must be a common root of the polynomials P and Q.

Now, let $\gamma = \rho$ be a common root of P and Q in $\mathcal{G}$ and define the aforementioned functions ϕ_n as

$$
\begin{aligned}
\phi_n(v_1, \ldots, v_k) &= f_n(\xi_0, \zeta_1, \ldots, \zeta_k) - \rho\xi_0 \\
&= -\sum_{j=1}^{k} a_j \sum_{i=1}^{j}(\rho^{-1})^{j-i+1}v_i + g_n\left(-\sum_{j=1}^{k} b_j \sum_{i=1}^{j}(\rho^{-1})^{j-i+1}v_i\right) \\
&= -\sum_{i=1}^{k}\sum_{j=i}^{k} a_j(\rho^{-1})^{j-i+1}v_i + g_n\left(-\sum_{i=1}^{k}\sum_{j=i}^{k} b_j(\rho^{-1})^{j-i+1}v_i\right).
\end{aligned}
$$

For each $i = 1, \ldots k$, since ρ is a root of the polynomial P it follows that

$$
\begin{aligned}
\sum_{j=i}^{k} a_j(\rho^{-1})^{j-i+1} &= \left(a_i\rho^{k-i} + a_{i+1}\rho^{k-i-1} + \cdots + a_{k-1}\rho + a_k\right)(\rho^{-1})^{k-i+1} \\
&= \left(\rho^{k+1} - a_0\rho^k - \cdots - a_{i-1}\rho^{k-i+1}\right)(\rho^{-1})^{k-i+1} \\
&= \rho^i - a_0\rho^{i-1} - \cdots - a_{i-1}.
\end{aligned}
$$

Similarly, since ρ is also a root of the polynomial Q it follows that

$$\begin{aligned}\sum_{j=i}^{k} b_j(\rho^{-1})^{j-i+1} &= \left(b_i\rho^{k-i} + b_{i+1}\rho^{k-i-1} + \cdots + b_{k-1}\rho + b_k\right)(\rho^{-1})^{k-i+1} \\ &= \left(-b_0\rho^k - b_1\rho^{k-1} - \cdots - b_{i-1}\rho^{k-i+1}\right)(\rho^{-1})^{k-i+1} \\ &= -b_0\rho^{i-1} - b_1\rho^{i-2} - \cdots - b_{i-1}.\end{aligned}$$

Now, if quantities p_i and q_i are defined as in the statement of this lemma, then the preceding calculations show that

$$\sum_{j=i}^{k} a_j(\rho^{-1})^{j-i+1} = p_{i-1} \quad \text{and} \quad \sum_{j=i}^{k} b_j(\rho^{-1})^{j-i+1} = -q_{i-1}.$$

Using these quantities the functions ϕ_n are determined as follows:

$$\phi_n(v_1, \ldots, v_k) = -\sum_{i=1}^{k} p_{i-1}v_i + g_n\left(\sum_{i=1}^{k} q_{i-1}v_i\right).$$

Identifying v_i with the new variables t_{n-i+1} yields a difference equation of order k as follows:

$$\begin{aligned}t_{n+1} &= \phi_n(t_n, \ldots, t_{n-k+1}) \\ &= -\sum_{i=1}^{k} p_{i-1}t_{n-i+1} + g_n\left(\sum_{i=1}^{k} q_{i-1}t_{n-i+1}\right) \\ &= -\sum_{i=0}^{k-1} p_i t_{n-i} + g_n\left(\sum_{i=0}^{k-1} q_i t_{n-i}\right),\end{aligned}$$

which is equation (11). From (12), we obtain (10) and the proof is complete. □

Remarks.

1. The preceding result shows that equation (1) splits into the equivalent pair of equations (10) and (11) via the change of variables (12) provided that the polynomials P and Q have a common non-zero root ρ in the group of units of $\mathbb{X}$. Equation (11) which is of the same type as (1) but with order reduced by 1 is the *factor equation* of (1). Equation (10) which bridges the order (or dimension) gap between (1) and (11) is the *cofactor equation*. The pair of equations (10) and (11) constitutes a *semiconjugate factorization* of (1). These concepts are from the general theory discussed in [26]; a limited exposure to some of this theory may be found in [25].
2. In the special case where $b_i = 0$ for all i (1) reduces to the linear non-homogeneous difference equation

$$x_{n+1} = \sum_{i=0}^{k} a_i x_{n-i} + g_n(0) \tag{13}$$

with constant coefficients. Since in this case Q is just the zero polynomial, Lemma 3 is applicable with ρ being any root of P in $\mathcal{G}$. So, as might be expected, when

$\mathbb{X} = \mathbb{R}$ the reduction of order is possible if the homogeneous part of (13) has non-zero eigenvalues. The reduction of order of general linear equations (non-homogeneous, non-autonomous) on arbitrary fields is discussed in [26], Chapter 7.

3. Solution of polynomials by factorization is problematic in a general Banach algebra $\mathbb{X}$ due to limitations of the cancellation law. Requiring all non-zero elements of $\mathbb{X}$ to be units reduces $\mathbb{X}$ to either $\mathbb{R}$ or $\mathbb{C}$ in commutative cases and to quaternions in non-commutative cases; see [27]. For general Banach algebras it is possible to determine, for special cases of (1), whether P and Q have a common root in $\mathcal{G}$; see the two corollaries in the next section.
4. Equation (11) preserves another aspect of (1): if $a_i, b_i \in \{0\} \cup \mathcal{G}$ and P and Q have a common root ρ in $\mathcal{G}$, then the numbers p_i, q_i in Lemma 3 are also units (or zero).

4. Extending the ranges of parameters

The solution of equation (10) in terms of t_n is

$$x_n = \rho^n x_0 + \sum_{j=1}^{n} \rho^{n-j} t_j. \tag{14}$$

This formula may be used to translate various properties of a solution $\{t_n\}$ of (11) into corresponding properties of the solution $\{x_n\}$ of (1). This is done for equation (3) in [24] for $\mathbb{X} = \mathbb{R}$. Doing the same for (1) more generally yields the following natural consequence of combining Lemmas 2 and 3.

THEOREM 4. *Let $g_n:\mathbb{X} \to \mathbb{X}$ be a sequence of functions that satisfy* (8) *for each* n. *Then every solution $\{x_n\}$ of* (1) *converges to zero if either* (*a*) *or* (*b*) *below is true:*

(a) *Inequality* (9) *holds;*
(b) *The polynomials P, Q in Lemma* 3 *have a common root $\rho \in \mathcal{G}$ such that $|\rho| < 1$ and*

$$\sum_{i=0}^{k-1} (|p_i| + \sigma|q_i|) < 1 \tag{15}$$

with the coefficients p_i, q_i in the factor equation (11), $i = 0,1,\ldots,k-1$.

Proof.

(a) Convergence in this case is an immediate consequence of Lemma 2.
(b) By an application of Lemma 3 we obtain (11). Then, given (15), an application of Lemma 2 to (11) implies that

$$|t_n| \leq \alpha^{n/(k+1)} \mu,$$

where $\mu = \max\{|t_0|, |t_{-1}|, \ldots, |t_{-k+1}|\}$ with $t_{-i} = x_{-i} - \rho x_{-i-1}$ for $i = 0,1,\ldots, k-1$ and

$$\alpha = \sum_{i=0}^{k-1} (|p_i| + \sigma|q_i|),$$

with p_i, q_i as given in Lemma 3. Since $|\rho^j| \le |\rho|^j$ for each j, taking norms in (14) yields

$$|x_n| \le |\rho|^n |x_0| + \sum_{j=1}^{n} |\rho|^{n-j} |t_j| \le |\rho|^n |x_0| + \mu |\rho|^n \sum_{j=0}^{n-1} \left(\frac{\alpha^{1/(k+1)}}{|\rho|} \right)^j. \tag{16}$$

If $\alpha^{1/(k+1)} \ne |\rho|$ then

$$\begin{aligned} |x_n| &\le |\rho|^n |x_0| + \mu \alpha^{1/(k+1)} |\rho|^{n-1} \frac{[\alpha^{1/(k+1)}/|\rho|]^n - 1}{[\alpha^{1/(k+1)}/|\rho|] - 1} \\ &= |\rho|^n |x_0| + \mu \alpha^{1/(k+1)} \frac{\alpha^{n/(k+1)} - |\rho|^n}{\alpha^{1/(k+1)} - |\rho|}. \end{aligned}$$

Since α, $|\rho| < 1$ it follows that $\{x_n\}$ converges to zero. If $\alpha^{1/(k+1)} = |\rho|$, then (16) reduces to

$$|x_n| \le |\rho|^n |x_0| + \mu |\rho|^n n$$

and by L'Hospital's rule $\{x_n\}$ again converges to zero. □

Corollary 5. *Let g_n be functions on $\mathbb{X}$ satisfying* (8) *for all $n \ge 0$. Every solution of the difference equation*

$$x_{n+1} = a x_n + g_n(b_0 x_n + b_1 x_{n-1} + \cdots + b_k x_{n-k}), \tag{17}$$

$$a \in \mathcal{G}, \quad b_i \in \mathbb{X}, \quad b_k \ne 0$$

converges to zero if $|a| < 1$ *and the following conditions hold:*

$$b_0 a^k + b_1 a^{k-1} + b_2 a^{k-2} + \cdots + b_k = 0, \tag{18}$$

$$\sum_{i=0}^{k-1} |b_0 a^i + b_1 a^{i-1} + \cdots + b_i| < \frac{1}{\sigma}. \tag{19}$$

Proof. For equation (17) the polynomials P, Q are

$$P(\xi) = \xi^{k+1} - a\xi^k, \quad Q(\xi) = b_0 \xi^k + b_1 \xi^{k-1} + \cdots + b_k.$$

Thus $\rho = a$ is their common root in $\mathcal{G}$ if (18) holds. The numbers p_i, q_i that define the factor equation (11) in this case are

$$p_i = \rho^{i+1} - a\rho^i = 0, \quad q_i = b_0 a^i + b_1 a^{i-1} + \cdots + b_i.$$

Thus, if $|a| < 1$ then by Theorem 4 every solution of (17) converges to zero. □

Remarks.

1. The parameter range determined by (15) is generally distinct from that given by (9); hence, Theorem 4 or Corollary 5 may imply convergence to 0 when Lemma 2 does

not apply and the unfolding map is not a contraction. To illustrate, consider the following equation on the set of real numbers:

$$x_{n+1} = ax_n + \alpha_n \tan h(x_n - bx_{n-k}), \tag{20}$$

which is a non-autonomous version of a type of equation discussed in [7]. Suppose that the sequence $\{\alpha_n\}$ of real numbers is bounded by $\sigma > 0$ and it is otherwise arbitrary. Then

$$|\alpha_n \tan ht| = |\alpha_n||\tan ht| \le |\alpha_n||t| \le \sigma|t|.$$

for all n and (8) holds. If $0 < |a| < 1$ and $b = a^k$, then by Corollary 5 every solution of (20) converges to zero if

$$\frac{1}{\sigma} > \sum_{i=0}^{k-1} |a|^i = \frac{1 - |a|^k}{1 - |a|}$$

i.e. if

$$\sigma < \frac{1 - |a|}{1 - |a|^k}. \tag{21}$$

On the other hand, applying Lemma 2 to (20) with $b = a^k$ produces the range

$$|a| + \sigma(1 + |a|^k) < 1 \Rightarrow \sigma < \frac{1 - |a|}{1 + |a|^k},$$

which is clearly more restricted than the one given by (21). Note that a and σ may satisfy (21) but with

$$|a| + \sigma(1 + |a|^k) > 1.$$

2. Equality (18) in Corollary 5 imposes a reduction in the dimension of the parameter space by permitting one variable to be determined in terms of all the others, e.g.

$$b_k = -b_0 a^k - b_1 a^{k-1} - \cdots - b_{k-1} a.$$

This restriction is in fact due to the use of the specific substitution (12) for reduction of order; this substitution, called the *linear form symmetry* in [26], need not be the only one that leads to a semiconjugate factorization. Indeed, form symmetries are generally not unique and for other types of difference equations, their known form symmetries do not impose any restrictions on the dimension of their natural parameter space; see [26] for examples and further details.

Corollary 6. *Let g_n be functions on $\mathbb{X}$ satisfying* (8) *for all $n \ge 0$. For the difference equation*

$$x_{n+1} = a_0 x_n + a_1 x_{n-1} + \cdots + a_k x_{n-k} + g_n(x_n - bx_{n-1}), \tag{22}$$

$$a_i \in \mathbb{X}, \quad b \in \mathcal{G}, \quad a_k \neq 0$$

assume that $|b| < 1$ *and the following conditions hold:*

$$a_0 b^k + a_1 b^{k-1} + \cdots + a_k = b^{k+1}, \tag{23}$$

$$\sum_{i=0}^{k-1} |b^{i+1} - a_0 b^i - \cdots - a_i| < 1 - \sigma. \tag{24}$$

Then every solution of (22) *converges to zero.*

Proof. The polynomials P, Q in this case are

$$P(\xi) = \xi^{k+1} - a_0\xi^k - \cdots - a_k, \quad Q(\xi) = \xi^k - b\xi^{k-1}.$$

Clearly, $Q(b) = 0$ and if equality (23) holds then $P(b) = 0$ too, so Theorem 4 applies. We calculate the coefficients of the factor equation (11) as $q_0 = 1$, $q_i = 0$ if $i \neq 0$ and

$$p_i = b^{i+1} - a_0 b^i - \cdots - a_i.$$

Now, inequality (15) yields (24) via a straightforward calculation:

$$\begin{aligned} 1 &> \sum_{i=0}^{k-1} |b^{i+1} - a_0 b^i - \cdots - a_i| + \sigma \\ &= \sum_{i=0}^{k-1} |b^{i+1} - a_0 b^i - \cdots - a_i| + \sigma. \end{aligned}$$

Thus, if $|b| < 1$ then by Theorem 4 every solution of (22) converges to zero. □

As an application of the preceding corollary, consider the case $k = 1$, i.e. the second-order equation

$$x_{n+1} = a_0 x_n + a_1 x_{n-1} + g_n(x_n - b x_{n-1}), \tag{25}$$

which is essentially equation (3) on a Banach algebra $\mathbb{X}$. By Corollary 6, every solution of (25) converges to zero if the functions g_n satisfy (8) and

$$b \in \mathcal{G}, \quad |b| < 1, \quad a_0 b + a_1 = b^2, \quad |a_0 - b| + \sigma < 1. \tag{26}$$

On the other hand, according to Lemma 2, every solution of (25) converges to zero if the functions g_n satisfy (8) and

$$|a_0| + |a_1| + \sigma(1 + |b|) < 1. \tag{27}$$

Parameter values that do not satisfy (27) may satisfy (26). For comparison, if $a_1 = b^2 - a_0 b$ then (27) may be solved for σ to obtain

$$\sigma < \frac{1 - |a_0| - |b||a_0 - b|}{1 + |b|}.$$

This is a stronger constraint on σ than $\sigma < 1 - |a_0 - b|$ from (26), especially if b is not near 0.

To illustrate the various aspects of the preceding results in a broader context, consider the following difference equation of order 2 on the real Banach algebra $C[0,1]$ with given initial functions $x_{-1}(r), x_0(r)$ in $C[0,1]$:

$$x_{n+1} = \frac{\alpha r}{r+1} x_n + \frac{\beta(\beta - \alpha r)}{(r+1)^2} x_{n-1} + \int_0^r \phi_n\left(x_n(s) - \frac{\beta}{s+1} x_{n-1}(s)\right) \mathrm{d}s. \tag{28}$$

The functions $\phi_n : \mathbb{R} \to \mathbb{R}$ are integrable and for each n they satisfy the absolute value inequality

$$|\phi_n(r)| \leq \sigma |r|, \quad r \in \mathbb{R},$$

for some $\sigma > 0$. Define the coefficient functions

$$a_0(r) = \frac{\alpha r}{r+1}, \quad a_1(r) = \frac{\beta(\beta - \alpha r)}{(r+1)^2}, \quad b(r) = \frac{\beta}{r+1}$$

and assume that the following inequalities hold:

$$0 < \beta < 1, \quad 3\beta \leq \alpha < 2 + \beta, \quad \sigma < \frac{2 + \beta - \alpha}{2}.$$

Then $b \in \mathcal{G}$, $b^2 - a_0 b = \beta(\beta - \alpha r)/(r+1)^2 = a_1$ and the norms of the coefficient functions satisfy:

$$|a_0| = \sup_{0 \leq r \leq 1} \frac{\alpha r}{r+1} = \frac{\alpha}{2}, \quad |b| = \sup_{0 \leq r \leq 1} \frac{\beta}{r+1} = \beta < 1$$

and

$$|a_0 - b| = \sup_{0 \leq r \leq 1} \left| \frac{\alpha r - \beta}{r+1} \right| = \max\left\{ \frac{\alpha - \beta}{2}, \beta \right\} = \frac{\alpha - \beta}{2} < 1 - \sigma.$$

It follows that the conditions in (26) are met.

Next, the functions $g_n : C[0,1] \to C[0,1]$ in (25) may be defined as

$$g_n(x)(r) = \int_0^r (\phi_n \circ x)(s) \mathrm{d}s,$$

for $r \in [0,1]$ and all n. Their norms satisfy

$$|g_n(x)| \leq \sup_{0 \leq r \leq 1} \int_0^r |\phi_n(x(s))| \mathrm{d}s \leq \sup_{0 \leq r \leq 1} \int_0^r \sigma |x(s)| \mathrm{d}s \leq \sigma |x| \sup_{0 \leq r \leq 1} r = \sigma |x|.$$

Therefore, Corollary 6 may be applied to conclude that for every pair of initial functions $x_0, x_{-1} \in C[0,1]$, the sequence of functions $x_n = x_n(r)$ that satisfy (28) in $C[0,1]$ converges exponentially (and uniformly) to the zero function.

It is finally worth mentioning that in the above example $a_0 \notin \mathcal{G}$ and $|a| \geq 1$ if $\alpha \geq 2$, in which case (27) does not hold.

5. Conclusion and future directions

The preceding corollaries and Theorem 4 are broad applications of the semiconjugate factorization method to very general equations. In particular, they show that different patterns of delays may be translated into algebraic problems about the polynomials P and Q and their root structures.

In some cases a more efficient application of Lemma 3 yields information about the behaviour of solutions beyond convergence. The next result offers a deeper use of order reduction in that sense and sets the stage for possible future investigations.

THEOREM 7. *In equation* (25) *assume that* $b \in \mathcal{G}$ *with* $|b| < 1$ *and* $a_0, a_1 \in \mathbb{X}$ *such that* $a_0 b + a_1 = b^2$. *If* x_0, x_{-1} *are given initial values for* (25) *for which the solution of the first-order equation*

$$t_{n+1} = (a_0 - b)t_n + g_n(t_n) \tag{29}$$

converges to zero with the initial value $t_0 = x_0 - bx_{-1}$, *then the corresponding solution of* (25) *converges to zero. In particular, if the origin is a global attractor of the solutions of the first-order equation* (29), *then it is also a global attractor of the solutions of* (25).

Proof. In this case $Q(\xi) = \xi - b$ so there is only one root b. Now Lemma 3 gives equation (29) if $P(b) = 0$, i.e. if $a_0 b + a_1 = b^2$. Finally, we complete the proof by arguing similarly to the proof of Theorem 4(b), using (14). □

As an application of Theorem 7, consider the following autonomous equation on the real numbers

$$x_{n+1} = ax_n + b(b - a)x_{n-1} + \sigma \tan h(x_n - bx_{n-1}), \tag{30}$$

where $\sigma > 0$, $0 < b < 1$ and $a < b$. Equation (29) in this case is

$$t_{n+1} = h(t_n), \quad h(\xi) = (a - b)\xi + \sigma \tan h\xi. \tag{31}$$

The function h has a fixed point at the origin since $h(0) = 0$. Furthermore, the origin is the unique fixed point of h if $|h(\xi)| < |\xi|$ for $\xi \neq 0$. Since h is an odd function, it is enough to consider $\xi > 0$. In this case, $h(\xi) < \xi$ if and only if $\sigma \tan h\xi < (1 - a + b)\xi$. Since $\tan h\xi < \xi$ for $\xi > 0$ it follows that $h(\xi) < \xi$ if

$$\sigma < 1 - a + b. \tag{32}$$

Given that $a < b$, it is possible to choose $1 \leq \sigma < 1 - a + b$ and extend the range of σ beyond what is possible with (26) or (27), which require that $\sigma < 1$. In particular, the function $\sigma \tan h\xi$ is not a contraction near the origin in this discussion.

Routine analysis of the properties of h leads to the following bifurcation scenario:

1. Suppose that $b - 1 \leq a < b < 1$ and (32) holds. Then $b - a \leq 1$ and all solutions of (31), hence, also all solutions of (30) converge to zero.
2. Now we fix b, σ and reduce the value of a so that $a < b - 1 < 0$. Then the function $h \circ h$ crosses the diagonal at two points $\tau > 0$ and $-\tau$ and a two-cycle $\{-\tau, \tau\}$ emerges for equation (31). Note that (32) still holds when a is reduced, but the origin is no longer globally attracting. The cycle $\{-\tau, \tau\}$ is repelling and generates a repelling two-cycle for (30); see [24] or [26], Section 5.5. The emergence of this

cycle implies that $\{t_n\}$ is unbounded if $|t_0| > \tau$ and it converges to 0 if $|t_0| > \tau$. Therefore, the corresponding solution $\{x_n\}$ of (30) also converges to 0 if

$$|x_0 - bx_{-1}| = |t_0| < \tau,$$

i.e. if the initial point (x_{-1}, x_0) is between the two parallel lines $y = b\xi + \tau$ and $y = b\xi - \tau$ in the (ξ, y) plane.

3. Suppose that a continues to decrease. Then the value of τ also decreases and reaches zero when

$$a = b - \sigma - 1,$$

i.e. when the slope of h at the origin is -1. Now, the cycle $\{-\tau, \tau\}$ collapses into the origin and turns it into a repelling fixed point. In this case, *all* non-zero solutions of (31) and (30) are unbounded.

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