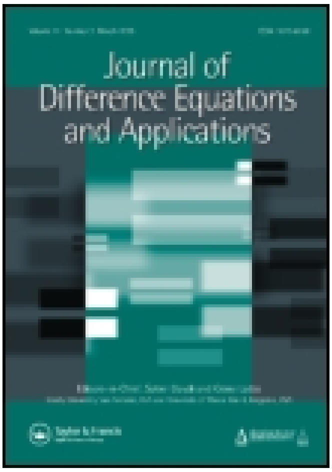

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Global behaviour of the Riccati difference equation of order two

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Global behaviour of the Riccati difference equation of order two

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The second order rational difference equation

$$x_{n+1} = a + \frac{b}{x_n} + \frac{c}{x_n x_{n-1}}, \quad x_0, x_{-1} \in \mathbb{R}$$

is associated with a linear third order difference equation in the same way that the first order Riccati equation ($c = 0$) is associated with a linear second order equation. This association and other features are used to study the global behaviour of solutions. If $a, b \geq 0$ and $a + b, c > 0$ then the above equation has a unique positive fixed point that is stable and attracts all orbits with initial points outside a set M of Lebesgue measure zero in the plane. However, within M there is an invariant subset containing periodic orbits of all possible periods.

Keywords: rational; Riccati; global asymptotic stability; forbidden set; periodic solutions

2000 Mathematics Subject Classification: 39A10; 39A11

1. Introduction

Consider the rational difference equation

$$x_{n+1} = a + \frac{b}{x_n} + \frac{c}{x_n x_{n-1}}. \tag{1}$$

If $c = 0$ then (1) reduces to the first order Riccati rational equation which has been studied thoroughly; see Refs. [1–4]. In particular, these references show that the discrete Riccati equation of order one can be transformed into a linear difference equation of order two. The linear equation is then used to obtain detailed information about the solutions of the first order Riccati equation.

In this paper, we generally assume that

$$a, b \geq 0, \quad c > 0, \quad x_0, x_{-1} \in (-\infty, \infty). \tag{2}$$

Equation (1) is not an arbitrary generalization of the first order Riccati rational equation ($c = 0$). Similarly to the latter equation, equation (1) is associated with a linear difference equation as follows: Define a new variable $x_n = y_n / y_{n-1}$ where x_n is a solution

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of (1) and change variables in (1) to obtain the following linear homogeneous equation of order 3

$$y_{n+1} = ay_n + by_{n-1} + cy_{n-2}. \tag{3}$$

If we define the initial values for (3) as

$$\begin{aligned} y_0 = x_0 y_{-1}, \quad y_{-1} = x_{-1} y_{-2} \quad \text{and} \quad \text{set} \\ y_{-2} = 1(\text{or any fixed nonzero real number}), \end{aligned} \tag{4}$$

then we obtain a one to one correspondence between the solutions of (1) and those solutions of (3) that do not contain zero; i.e. each solution of (1) uniquely defines a solution of (3) that does not pass through the origin and vice versa. If $\{y_n\}$ is a solution of (3) with $y_k = 0$ for some least k then $x_{k+1} = y_{k+1}/y_k$ is not defined. Under conditions (4), the correspondence between solutions of (3) that pass through the origin and those of (1) that become undefined is also one to one.

In Section 2, we find the solutions of the linear equation (3) and use them to determine the forbidden set F of (1), i.e. the set of all initial values in $\mathbb{R}^2$ that lead to a zero in the denominator of (1) after a finite number of iterations. In Section 3, we prove that the unique positive fixed point of (1) under conditions (2) is globally asymptotically stable for initial points (x_0, x_{-1}) outside a set M of Lebesgue measure zero in the Euclidean plane that contains F. In Section 4, we show that equation (1) is more complicated than the first order Riccati equation ($c = 0$) because there are exceptional solutions of (1) that originate in M and which do not converge to the positive fixed point under conditions (2), i.e. with non-negative parameters. These solutions include periodic solutions of all possible periods as well as oscillatory nonperiodic solutions.

2. The forbidden set

The characteristic equation of (3) is

$$P(\lambda) = \lambda^3 - a\lambda^2 - b\lambda - c = 0. \tag{5}$$

Note that the real solutions of (5) also give the fixed points of equation (1). The cubic polynomial P has at least one real root. The next two results give more precise information about the roots of P.

LEMMA 1. *Assume that conditions* (2) *hold. Then the polynomial P has precisely one positive real root ρ that satisfies*

$$\rho \geq \max\left\{\sqrt[3]{c}, \frac{a + \sqrt{a^2 + 4b}}{2}\right\} \tag{6}$$

with equality holding if and only if $a = b = 0$.

Proof. By the Descartes rule of signs P has only one positive root ρ under conditions (2). Further

$$P(\lambda) = \lambda(\lambda^2 - a\lambda - b) - c$$

and the roots of $\lambda^2 - a\lambda - b$ are $(a \pm \sqrt{a^2+4b})/2$. If λ_0 is the non-negative root then since $P(\lambda_0) = -c < 0$ it follows that $\rho > \lambda_0$. Next, from $P(\rho) = 0$ we obtain

$$\rho^2 - a\rho - b = \frac{c}{\rho}, \tag{7}$$

which implies $c/\rho \leq \rho^2$, i.e. $\rho \geq \sqrt[3]{c}$. Finally, if equality holds in (6) then $\rho = \sqrt[3]{c} \neq \lambda_0$ since $P(\lambda_0) = -c \neq 0$. But then $P(\sqrt[3]{c}) = 0$ implies that $a\sqrt[3]{c} + b = 0$ which implies $a = b = 0$ because $a, b \geq 0$. Conversely if $a = b = 0$ then $\lambda^3 - c = 0$ so $\rho = \sqrt[3]{c}$ and equality holds in (6). □

It is possible to find a formula for ρ in terms of radicals (see Ref. [5]) but we omit it since that information is not particularly useful here.

Lemma 2. *Assume that conditions* (2) *hold and let ρ be the positive root of* (5).

(a) *Equation* (5) *has two other roots that can be calculated in terms of ρ as*

$$r^{\pm} = -\frac{\rho - a}{2} \pm \sqrt{\left(\frac{\rho + a}{2}\right)^2 - \rho^2 + b}. \tag{8}$$

(b) *If $(\rho + a)^2 \geq 4(\rho^2 - b)$ then the real roots $r^{\pm}$ are negative and*

$$-\rho < r^- \leq -\frac{\rho - a}{2} \leq r^+ < 0.$$

(c) *If $(\rho + a)^2 < 4(\rho^2 - b)$ (e.g. if $b = 0$) then the complex roots $r^{\pm}$ satisfy*

$$|r^{\pm}| = \sqrt{\rho^2 - a\rho - b} = \sqrt{\frac{c}{\rho}}. \tag{9}$$

Proof.

(a) Dividing $P(\lambda)$ by $\lambda - \rho$ gives the quadratic polynomial $Q(\lambda) = \lambda^2 + (\rho - a)\lambda + \rho^2 - a\rho - b$. The two roots $r^{\pm}$ of Q are given by (8).
(b) In this case,

$$r^- > -\rho \quad \text{iff} \quad \rho - \frac{\rho - a}{2} > \sqrt{\left(\frac{\rho + a}{2}\right)^2 - \rho^2 + b} \quad \text{iff} \quad \rho^2 > b.$$

The last inequality is true by (7) in the proof of Lemma 1. Similarly,

$$r^+ < 0 \quad \text{iff} \quad \sqrt{\left(\frac{\rho + a}{2}\right)^2 - \rho^2 + b} < \frac{\rho - a}{2} \quad \text{iff} \quad \rho^2 - a\rho - b > 0.$$

The last inequality is true again by (7).
(c) In this case, the moduli of $r^{\pm}$ are easily found to be given by (9). If $b = 0$ then since by (6) $\rho > a$ it follows that $(\rho + a)^2 < (2\rho)^2 = 4\rho^2$ and roots $r^{\pm}$ are complex.

□

Based on Lemma 2, the next result summarizes the standard facts about the solutions of the linear equation (3). Of particular interest to us is the fact that the coefficients of solutions all have the same general formula.

LEMMA 3. *Suppose that conditions* (2) *hold.*

(a) *If* $(\rho + a)^2 > 4(\rho^2 - b)$ *then for all* $n \geq 0$

$$y_n = C_1 \rho^n + C_2 (r^+)^n + C_3 (r^-)^n,$$

where the coefficients C_j, $j = 1, 2, 3$ *are given by*

$$C_j(x_0, x_{-1}) = \alpha_{1j} x_0 x_{-1} + \alpha_{2j} x_{-1} + \alpha_{3j} \tag{10}$$

for suitable constants α_{ij}, $i, j = 1, 2, 3$ *that do not depend on the initial values.*

(b) *If* $(\rho + a)^2 = 4(\rho^2 - b)$ *then for all* $n \geq 0$

$$y_n = C_1 \rho^n + (C_2 + C_3 n) r^n \quad \text{where } r = r^+ = r^- = -\frac{\rho - a}{2},$$

where the coefficients C_j *are given by* (10) *with constants* α_{ij}, $i, j = 1, 2, 3$ *appropriate to this case.*

(c) *If* $(\rho + a)^2 < 4(\rho^2 - b)$ *then for all* $n \geq 0$

$$y_n = C_1 \rho^n + (\rho^2 - a\rho - b)^{n/2} (C_2 \cos n\theta + C_3 \sin n\theta)$$

where $\theta \in (\pi/2, \pi)$ *is a constant and the coefficients* C_j *are given by* (10) *with constants* α_{ij}, $i, j = 1, 2, 3$ *appropriate to this case.*

Proof. The solutions $\{y_n\}$ in each case are obtained routinely from the basic linear theory so we only explain about (10) and the range of θ in (c).

(a) The coefficients C_j satisfy the system

$$C_1 + C_2 + C_3 = x_0 x_{-1}, \quad C_1/\rho + C_2/(r^+) + C_3/(r^-) = x_{-1}, \quad \text{and}$$
$$C_1/\rho^2 + C_2/(r^+)^2 + C_3/(r^-)^2 = 1.$$

This system which is linear in the C_j can be easily solved to obtain

$$C_1 = \frac{\rho^2 [x_0 x_{-1} - (r^+ + r^-) x_{-1} + r^+ r^-]}{(\rho - r^+)(\rho - r^-)} \tag{11}$$

from which we can read off the values of the constants α_{1j}. Further,

$$C_2 = \frac{-(r^+)^2 [x_0 x_{-1} - (r^+ + r^-) x_{-1} + r^+ r^-]}{(\rho - r^+)(r^+ - r^-)}$$

gives the constants α_{2j} and

$$C_3 = x_0 x_{-1} - C_1 - C_2 = (1 - \alpha_{11} - \alpha_{12}) x_0 x_{-1} - (\alpha_{21} + \alpha_{22}) x_{-1} - (\alpha_{31} + \alpha_{32})$$

which gives α_{3j}.

(b) In this case, the coefficients C_j satisfy

$$C_1 + C_2 = x_0x_{-1}, \quad C_1/\rho - 2(C_2 - C_3)/(\rho - a) = x_{-1}, \quad \text{and}$$
$$C_1/\rho^2 + 4(C_2 - 2C_3)/(\rho - a)^2 = 1.$$

From these, we obtain

$$C_1 = \frac{4\rho^2(\rho - a)}{(3\rho - a)^2}x_0x_{-1} + \frac{4\rho^2}{(\rho - a)(3\rho - a)}x_{-1} + \frac{\rho^2(\rho - a)^2}{(3\rho - a)^2} \tag{12}$$

from which we can read off the values of the constants α_{1j}. Further,

$$C_2 = x_0x_{-1} - C_1 = (1 - \alpha_{11})x_0x_{-1} - \alpha_{21}x_{-1} - \alpha_{31}$$

gives the constants α_{2j} for this case and

$$C_3 = \frac{\rho - a}{2}x_{-1} - \frac{\rho - a}{2\rho}C_1 + C_2$$

from which α_{3j} can be calculated.

(c) In this case, the coefficients C_j satisfy

$$\begin{gathered} C_1 + C_2 = x_0x_{-1}, \quad C_1/\rho + (C_2 \cos\theta - C_3 \sin\theta)/\sqrt{\rho^2 - a\rho - b} = x_{-1}, \\ \text{and} \quad C_1/\rho^2 + (C_2 \cos 2\theta - C_3 \sin 2\theta)/(\rho^2 - a\rho - b) = 1 \end{gathered} \tag{13}$$

where θ is defined by the equalities

$$\cos\theta = -\sqrt{\frac{\rho}{c}}\frac{\rho - a}{2}, \quad \sin\theta = \sqrt{\frac{\rho}{c}}\sqrt{\rho^2 - b - \left(\frac{\rho + a}{2}\right)^2} \tag{14}$$

which also show that $\theta \in (\pi/2, \pi)$. From (13), we obtain using $\rho^2 - a\rho - b = c/\rho$,

$$C_1 = \frac{\rho^2 c}{\rho^3 + c - 2\sqrt{\rho^3 c}\cos\theta}\left[\frac{\rho}{c}x_0x_{-1} - 2\sqrt{\frac{\rho}{c}}(\cos\theta)x_{-1} + 1\right] \tag{15}$$

from which we can read off the values of the constants α_{1j}. Further,

$$C_2 = x_0x_{-1} - C_1, \quad C_3 = \frac{c \sin 2\theta}{\rho^3}C_1 + \frac{\cos 2\theta}{\sin 2\theta}C_2 - \frac{c}{\rho}\sin 2\theta$$

from which α_{ij}, $i = 2, 3$ can be calculated. □

Since each of the coefficients C_j depends on the two initial values, each solution of the linear equation (3) is a function $y_n(u, v)$ of two variables, all other parameters being fixed. Thus the forbidden set F of equation (1) can be written as

$$F = \bigcup_{n=-1}^{\infty} \{(u, v) : y_n(u, v) = 0\}.$$

We note that $F \subset \mathbb{R}^2 \backslash (0,\infty)^2$ because under conditions (2) each solution $\{x_n\}$ of (1) with $(x_0, x_{-1}) \in (0,\infty)^2$ satisfies $x_n > 0$ for all $n \geq -1$ and thus there are no undefined values. Now the next result is an immediate consequence of Lemma 3.

THEOREM 4. *Suppose that conditions* (2) *hold. Then the forbidden set of equation* (1) *is the following sequence of hyperbolas*

$$F = \bigcup_{n=-1}^{\infty} \{(u,v) : \beta_{1n}uv + \beta_{2n}v + \beta_{3n} = 0\} \subset \mathbb{R}^2 \backslash (0,\infty)^2,$$

where the sequences β_{in} *are defined as follows:*

(a) *If* $(\rho + a)^2 > 4(\rho^2 - b)$ *then*

$$\beta_{in} = \alpha_{i1} + \alpha_{i2}(r^+/\rho)^n + \alpha_{i3}(r^-/\rho)^n,$$

where α_{ij} *are the constants in Lemma* 3(*a*).

(b) *If* $(\rho + a)^2 = 4(\rho^2 - b)$ *then*

$$\beta_{in} = \alpha_{i1} + (-1/2)^n(1 - a/\rho)^n(\alpha_{i2} + \alpha_{i3}n),$$

where α_{ij} *are the constants in Lemma* 3(*b*).

(c) *If* $(\rho + a)^2 < 4(\rho^2 - b)$ *then*

$$\beta_{in} = \alpha_{i1} + (c/\rho^3)^{n/2}(\alpha_{i2}\cos n\theta + \alpha_{i3}\sin n\theta)$$

where α_{ij} *are the constants in Lemma* 3(*c*).

3. Global asymptotic stability

In this section, we use the preceding results to show that under conditions (2) almost all solutions of equation (1) converge to the positive fixed point ρ if at least one of the parameters a or b is positive.

LEMMA 5. *Under conditions* (2), ρ *is the unique positive fixed point of* (1) *and if* $a + b > 0$ *then* ρ *is locally asymptotically stable.*

Proof. Define

$$f(u,v) = a + \frac{b}{u} + \frac{c}{uv}.$$

Since the fixed points of (1) correspond to the roots of the polynomial P in (5), the uniqueness of ρ follows from Lemma 1. Next, the characteristic equation of the linearization of (1) at the fixed point (ρ,ρ) is

$$\lambda^2 - f_u(\rho,\rho)\lambda - f_v(\rho,\rho) = 0, \tag{16}$$

where

$$f_u = \frac{-1}{u^2}\left(b + \frac{c}{v}\right), \quad f_v = \frac{-c}{uv^2}.$$

These and the fact that $b\rho + c = \rho^3 - a\rho^2$ determine equation (16) as

$$\lambda^2 + \frac{\rho - a}{\rho}\lambda + \frac{c}{\rho^3} = 0.$$

The zeros of this quadratic are

$$\lambda^{\pm} = \frac{\rho - a}{2\rho}\left[-1 \pm \sqrt{1 - \frac{4c}{\rho(\rho - a)^2}}\right].$$

If

$$\rho(\rho - a)^2 \geq 4c \tag{17}$$

then the numbers $\lambda^{\pm}$ are real and $\lambda^{-} \leq \lambda^{+} < 0$. Further, a little algebra shows that $\lambda^{-} > -1$ if and only if

$$\sqrt{1 - \frac{4c}{\rho(\rho - a)^2}} < \frac{\rho + a}{\rho - a},$$

which is obviously true since the left side is less than 1 and the right side greater than 1. Thus if (17) holds then ρ is a stable node for (1). Next suppose that (17) is false. Then $\lambda^{\pm}$ are complex with $|\lambda^{\pm}| = \sqrt{c/\rho^3} < 1$ where the inequality holds by Lemma 1 when $a + b > 0$. Thus if (17) is false then ρ is a stable focus for (1). These cases exhaust all possibilities so ρ is locally asymptotically stable. □

In considering the global behaviour of solutions of eqaution (1), the following set must be considered:

$$M = F \cup \{(u, v) : C_1(u, v) = 0\} = F \cup \{(u, v) : \alpha_{11}uv + \alpha_{21}v + \alpha_{31} = 0\}, \tag{18}$$

where F is the forbidden set of (1) as determined in Theorem 4 and α_{i1} are the constants defined in Lemma 3.

THEOREM 6. *Assume that conditions* (2) *hold with* $a + b > 0$. *Then the positive fixed point* ρ *is globally asymptotically stable relative to* $\mathbb{R}^2 \backslash M$ *where the set* $M \subset \mathbb{R}^2 \backslash (0, \infty)^2$ *defined by* (18) *has Lebesgue measure zero.*

Proof. By Lemma 5, ρ is stable so it only remains to prove global attractivity. If $\{x_n\}$ is a solution of equation (1) then we claim that $\lim_{n\to\infty} x_n = \rho$ if $(x_0, x_{-1}) \notin M$.

First, consider the case where $r^{\pm}$ are real and distinct. In this case, Lemma 3 implies that

$$x_n = \frac{y_n}{y_{n-1}} = \frac{C_1\rho^n + C_2(r^+)^n + C_3(r^-)^n}{C_1\rho^{n-1} + C_2(r^+)^{n-1} + C_3(r^-)^{n-1}}. \tag{19}$$

Since $(x_0, x_{-1}) \notin M$, we have $C_1 = C_1(x_0, x_{-1}) \neq 0$. Now dividing by $C_1\rho^{n-1}$ yields

$$x_n = \frac{\rho + (C_2\rho/C_1)(r^+/\rho)^n + (C_3\rho/C_1)(r^-/\rho)^n}{1 + (C_2/C_1)(r^+/\rho)^{n-1} + (C_3/C_1)(r^-/\rho)^{n-1}},$$

which implies, by Lemma 2(b), that $\lim_{n\to\infty} x_n = \rho$. Next, in the case of equal real roots a similar calculation gives

$$x_n = \frac{\rho + r(C_2/C_1 + C_3 n/C_1)(r/\rho)^{n-1}}{1 + [C_2/C_1 + C_3(n-1)/C_1](r/\rho)^{n-1}}.$$

Since by Lemma 2(b) $|r/\rho| < 1$ it follows that $\lim_{n\to\infty} x_n = \rho$. Next, in the case of complex roots

$$x_n = \frac{\rho + \sqrt{\rho^2 - a\rho - b}(1 - a/\rho - b/\rho^2)^{(n-1)/2}(C_2 \cos n\theta + C_3 \sin n\theta)/C_1}{1 + (1 - a/\rho - b/\rho^2)^{(n-1)/2}[C_2 \cos(n-1)\theta + C_3 \sin(n-1)\theta]/C_1},$$

so clearly $\lim_{n\to\infty} x_n = \rho$.

Finally, since M is a countable collection of hyperbolas it has Lebesgue measure zero in $\mathbb{R}^2$. To establish that $M \subset \mathbb{R}^2 \backslash (0,\infty)^2$, it remains to show that the set

$$\{(u,v) : C_1(u,v) = 0\} = \{(u,v) : \alpha_{11} uv + \alpha_{21} v + \alpha_{31} = 0\} \tag{20}$$

does not intersect the positive quadrant $(0,\infty)^2$. From expressions (11), (12) and (15) above we see that $\alpha_{i1} > 0$ for $i = 1, 2, 3$ in each of the three possible cases. Thus the set (20) cannot contain points (u, v) with $u, v > 0$. □

In the boundary case $a = b = 0$ in (2), Theorem 6 is false; as the next proposition shows the solutions of (1) exhibit a completely different behaviour in this case.

PROPOSITION 7. *If neither of the initial values x_0, x_{-1} is zero then the corresponding solution of*

$$x_{n+1} = \frac{c}{x_n x_{n-1}}, \quad c \neq 0 \tag{21}$$

is given as

$$\left\{x_{-1}, x_0, \frac{c}{x_0 x_{-1}}, x_{-1}, x_0, \frac{c}{x_0 x_{-1}}, \ldots\right\}.$$

In particular, every non-constant solution of (21), *i.e.* $(x_0, x_{-1}) \neq (\sqrt[3]{c}, \sqrt[3]{c})$ *has period* 3.

The next result applies Theorem 6 to an equation that is similar to (1).

COROLLARY 8. *Assume that conditions* (2) *hold for the following equation*

$$z_{n+1} = \frac{1}{a + b z_n + c z_n z_{n-1}}. \tag{22}$$

If $a > 0$ and $z_0, z_{-1} \geq 0$ or if $a + b > 0$ and $z_0 > 0$, $z_{-1} \geq 0$ then $\lim_{n\to\infty} z_n = 1/\rho$ *where ρ is defined in Lemma* 1.

Proof. Since the change of variables $x_n = 1/z_n$ transforms (22) into (1), if $z_0, z_{-1} > 0$ then Theorem 6 implies that $\lim_{n\to\infty} z_n = \lim_{n\to\infty}(1/x_n) = 1/\rho$. If $a > 0$ and either $z_0 = 0$ or $z_{-1} = 0$ then from (22) we find that $z_1, z_2 > 0$ so again Theorem 6 applies. The last case is argued similarly. □

4. Exceptional solutions

The proof of Theorem 6 contains information about solutions that do not converge to ρ. These are exceptional solutions of (1) since they can only originate in the set $M \backslash F$. They can be either convergent or oscillatory. In the first order case ($c = 0$), it is shown in Ref. [1] that non-convergent solutions do not occur when $a, b > 0$ (i.e. cases with $R \leq 1/4$). So the fact that they do occur in the second order case with positive parameters is an indication of the greater complexity of the higher order Riccati equation.

THEOREM 9. *Assume that conditions* (2) *hold with* $a + b > 0$.

(a) *The hyperbola* $H = \{(u, v) : uv - (r^+ + r^-)v + r^+r^- = 0\}$ *is an invariant subset of* M.
(b) *If* $(\rho + a)^2 \geq 4(\rho^2 - b)$ *and* $(x_0, x_{-1}) \in H \backslash \{(r^+, r^+)\}$ *then* $\lim_{n\to\infty} x_n = r^-$.
(c) *Let* $(\rho + a)^2 < 4(\rho^2 - b)$. *If* $\theta = \pi q/p$ *satisfies* (14) *for positive integers p, q that are relatively prime then for each* $(x_0, x_{-1}) \in H$ *the corresponding solution* $\{x_n\}$ *of* (1) *has period p. If* θ *is an irrational multiple of* π *then the corresponding solution of* (1) *is oscillatory but not periodic and the orbit* $\{(x_n, x_{n-1})\}$ *is dense in H.*

Proof. Notice from (11), (12) and (15) that the expression for C_1 is real even if $r^\pm$ are complex and that $C_1(x_0, x_{-1}) = 0$ if and only if

$$x_0x_{-1} - (r^+ + r^-)x_{-1} + r^+r^- = 0. \tag{23}$$

Indeed, from (15) we obtain $C_1 = 0$ if and only if

$$x_0x_{-1} - 2\sqrt{\frac{c}{\rho}}(\cos\theta)x_{-1} + \frac{c}{\rho} = 0,$$

which is identical to (23) if $r^\pm$ are complex. Thus $C_1 = 0$ in all cases if it is shown that $C_1(x_{n+1}, x_n) = 0$ for all $n \geq 0$ whenever x_0 and x_{-1} satisfy (23).

Now

$$\begin{aligned} C_1(x_1, x_0) &= x_1x_0 - (r^+ + r^-)x_0 + r^+r^- \\ &= x_0\left(a + \frac{b}{x_0} + \frac{c}{x_0x_{-1}}\right) - (r^+ + r^-)x_0 + r^+r^- \\ &= ax_0 + b + \frac{c}{x_{-1}} + (\rho - a)x_0 + \rho^2 - a\rho - b \\ &= \frac{c}{x_{-1}} + \rho x_0 + \rho^2 - a\rho. \end{aligned} \tag{24}$$

If (23) holds then

$$x_0 = -(\rho - a) - \frac{\rho^2 - a\rho - b}{x_{-1}} = -(\rho - a) - \frac{c}{\rho x_{-1}}$$

which if inserted into (24) yields $C_1(x_1, x_0) = 0$. The proof of (a) can now be easily completed by induction.

(b) In this case, the roots $r^{\pm}$ are real. First suppose that $(\rho+a)^2 > 4(\rho^2 - b)$. If $(x_0, x_{-1}) \in H$ then $C_1 = 0$ in (19) and thus

$$x_n = \frac{y_n}{y_{n-1}} = \frac{C_2(r^+)^n + C_3(r^-)^n}{C_2(r^+)^{n-1} + C_3(r^-)^{n-1}}.$$

If $C_3 = 0$ then $x_n = r^+$ for all n which can occur only if $x_1 = x_0 = r^+$. If $C_3 \neq 0$ then dividing by $C_3(r^-)^{n-1}$ and taking the limit gives

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{(C_2/C_3)(r^+/r^-)^n\, \bar{r} + r^-}{(C_2/C_3)(r^+/r^-)^{n-1} + 1} = r^-.$$

The argument for the case $(\rho+a)^2 = 4(\rho^2 - b)$ is similar but using Lemma 3(b); we omit the straightforward details.

(c) In this case, the roots $r^{\pm}$ are complex and if $C_1 = 0$ then from Lemma 3 we obtain

$$\begin{aligned} x_n &= \frac{(\rho^2 - a\rho - b)(C_2 \cos n\theta + C_3 \sin n\theta)}{C_2 \cos(n-1)\theta + C_3 \sin(n-1)\theta} \\ &= \frac{c}{\rho}\cos\theta + \frac{c}{\rho}\sin\theta \frac{C_3\cos(n-1)\theta - C_2 \sin(n-1)\theta}{C_2\cos(n-1)\theta + C_3\sin(n-1)\theta}. \end{aligned}$$

Define $\cos\phi = C_2/\sqrt{C_2^2 + C_3^2}$ and $\sin\phi = C_3/\sqrt{C_2^2 + C_3^2}$. Then

$$\begin{aligned} x_n &= \frac{c}{\rho}\cos\theta + \frac{c}{\rho}\sin\theta \frac{\sin\phi\cos(n-1)\theta - \cos\phi\sin(n-1)\theta}{\cos\phi\cos(n-1)\theta + \sin\phi\sin(n-1)\theta} \\ &= \frac{c}{\rho}\cos\theta - \frac{c}{\rho}\sin\theta \frac{\sin[(n-1)\theta - \phi]}{\cos[(n-1)\theta - \phi]} \\ &= \frac{c}{\rho}\cos\theta - \frac{c}{\rho}\sin\theta \tan[(n-1)\theta - \phi]. \end{aligned} \tag{25}$$

Now if $\theta = \pi q/p$ is a rational multiple of π then it follows from (25) that x_n is periodic (with period p if q/p is in reduced form). If θ is not a rational multiple of π then the angles $(n-1)\theta - \phi$ form a dense subset of the circle as $n \to \infty$. Given that $\tan x$ is a homeomorphism from $(-\pi/2, \pi/2)$ to $\mathbb{R}$ we conclude from (25) that the sequence $\{x_n\}$ is dense in $\mathbb{R}$. It follows that the orbit $\{(x_n, x_{n-1})\}$ is dense in H. □

5. Concluding remarks

In the preceding sections, we examined the global behaviour of equation (1) subject to conditions (2). The natural question to ponder is what kinds of behaviours are possible for the solutions of (1) if conditions (2) do not hold. Numerical simulations suggest that oscillatory solutions (periodic or not) occur non-exceptionally so a greater variety of behaviours are observable. We leave further exploration of equation (1) with $a, b, c \in \mathbb{R}$, $c \neq 0$ to future studies but note that for certain negative parameter values the cases discussed in the preceding sections are revisited in disguise as the next corollary shows.

Corollary 10. *Assume that* $a, c < 0 \leq b$ *in equation* (1). *If* $x_0, x_{-1} \notin -M$ *where* M *is the set defined in* (18) *then* $\lim_{n\to\infty} x_n = -\rho$.

Proof. Since setting $w_n = -x_n$ gives

$$w_{n+1} = -x_{n+1} = -a - \frac{b}{x_n} - \frac{c}{x_n x_{n-1}} = -a + \frac{b}{w_n} - \frac{c}{w_n w_{n-1}}$$

it follows that w_n satisfies (1) subject to conditions (2). The proof is completed by applying Theorem 6. □

Riccati difference equations of order greater than 2 can also be defined. Moreover, these equations can be defined on arbitrary algebraic fields including finite fields. For additional facts, conjectures and issues pertaining to higher order, discrete Riccati equations we refer to the web site: http://www.discretedynamics.net/Articles/articles.htm.

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