The Global Stability of Equilibrium in A Nonlinear Second-Order Difference Equation

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Abstract. Consider the difference equation $x_{n+1} = cx_n + f(x_n - x_{n-1})$ where $0 \leq c < 1$ and f is continuous on $\mathbb{R}$ and has a global minimum (not necessarily unique) at the origin. Sufficient conditions are given on c and f for the unique fixed point $\bar{x} = f(0)/(1-c)$ to be globally asymptotically stable. Also, conditions under which solutions converge to $\bar{x}$ eventually monotonically are given, and we discuss cases in which the ratios $\{x_n/x_{n-1}\}$ are chaotic.

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1. Introduction

Consider the second order difference equation

$$x_{n+1} = cx_n + f(x_n - x_{n-1}), \qquad x_0, x_{-1} \in \mathbb{R}. \tag{1}$$

where f is continuous on $\mathbb{R}$ and that $0 \leq c < 1$. Special cases of this difference equation have appeared in the classical theories of the business cycle since 1939; see e.g., Hicks [1], Puu [6] and Samuelson [7]. Depending on the choice of the function f, this equation exhibits a remarkable variety of dynamical behaviors as discussed in Kent and Sedaghat [2] and Sedaghat [9]. In [2] it is generally assumed that f satisfies the condition $tf(t) \geq 0$ (e.g., f may be any *odd* function that is confined to the first and third quadrants).

In this paper we consider a complementary case where f is minimized at the origin (e.g., f is an *even* function with a global minimum at the origin). We establish conditions that are sufficient for the global asymptotic stability of the equilibrium, including conditions implying that convergence is monotonic. Although there are some similarities between the two cases, in general they result in very different types of dynamics. For example, with $tf(t) \geq 0$ oscillatory behavior is similar to that seen in linear equations (even when linearization is not possible). However, for functions f in this paper this is not the case; instead, oscillations occur off-equilibrium with chaotic relative rates so that there is no linear analog for these essentially nonlinear oscillations.

Note that (1) has a unique fixed point at $\bar{x} = f(0)/(1-c)$, so if $f(0) = 0$ then the only fixed point of (1) is at the origin which is also a point at which f attains its minimum value. The general background material for this paper is found in standard texts such as Kocic and Ladas [3] or Sedaghat [9].

2. Global Asymptotic Stability

We begin this section with a result from Sedaghat [8] which we quote here as a lemma.

Lemma 1. *Let $g : \mathbb{R}^m \to \mathbb{R}$ be continuous and let $\bar{x}$ be an isolated fixed point of*

$$x_{n+1} = g(x_n, x_{n-1}, \ldots, x_{n-m}).$$

Let $V_g(u_1, \ldots, u_m) = (g(u_1, \ldots, u_m), u_1, \ldots, u_{m-1})$ and for $\alpha \in (0, 1)$ define the set

$$A_\alpha = \{(u_1, \ldots, u_m) : |g(u_1, \ldots, u_m) - \bar{x}| \leq \alpha \max\{|u_1 - \bar{x}|, \ldots, |u_m - \bar{x}|\}$$

If S is a subset of A_α such that $V_g(S) \subset S$ and $(\bar{x}, \ldots, \bar{x}) \in S$, then $(\bar{x}, \ldots, \bar{x})$ is asymptotically (in fact, exponentially) stable relative to S.

The function g in Lemma 1 is said to be a *weak contraction* on the set A_α; see Sedaghat [9] for a general theory of weak contractions and weak expansions. The next result concerns monotonic positive solutions of (1).

Lemma 2. *If $0 \leq f(t) \leq a|t|$ for all t where $a < c$ then every positive solution of* (1) *is eventually decreasing.*

Proof. Let $\{x_n\}$ be a positive solution of (1). Then the ratios

$$r_n = \frac{x_n}{x_{n-1}}, \quad n \geq 0$$

are well defined and satisfy

$$r_{n+1} = c + \frac{f(x_n - x_{n-1})}{x_n} \leq c + \frac{a|x_n - x_{n-1}|}{x_n} = c + a\left|1 - \frac{1}{r_n}\right|.$$

Since it is also true that $r_{n+1} = c + f(x_n - x_{n-1})/x_n \geq c$ we have

$$c \leq r_{n+1} \leq c + a\left|1 - \frac{1}{r_n}\right|, \quad n \geq 0.$$

If $r_1 \leq 1$ then since $r_1 \geq c$, we have

$$c \leq r_2 \leq c - a + \frac{a}{r_1} \leq c - a + \frac{a}{c} < 1$$

where the last inequality holds because $a < c < 1$. Inductively, if for $k \geq 2$,

$$c \leq r_n < 1, \quad n < k$$

then

$$c \leq r_k \leq c - a + \frac{a}{c} < 1$$

so that

$$r_1 \leq 1 \Rightarrow r_n < 1 \quad \text{for all} \quad n > 1. \tag{2}$$

Now suppose that $r_1 > 1$. Then

$$c \leq r_2 \leq c + a - \frac{a}{r_1} < c + a.$$

If $c+a \leq 1$, then $r_2 < 1$ and (2) holds. Assume that $a+c > 1$ and $r_2 > 1$. Then

$$r_3 \leq c + a - \frac{a}{r_2} < r_2.$$

The last inequality holds because for every $r > 1$, $c + a - a/r < r$ if and only if

$$r^2 - (c+a)r + a > 0. \tag{3}$$

Inequality (3) is true because the quadratic on its left side can have zeros only for $r \leq 1$. Now, if $r_3 < 1$, then (2) holds for $n > 2$. Otherwise, using (3) we can show inductively that

$$r_1 > r_2 > r_3 > \cdots$$

so there is $k \geq 1$ such that $r_k \leq 1$ and (2) applies with $n > k$. Hence, we have shown that for any choice of r_0, the sequence r_n is eventually less than 1; i.e., $x_n < x_{n-1}$ for all n sufficiently large and the proof is complete.

Before stating Theorem 1, a main result of this paper, one more lemma is needed concerning non-positive solutions of (1). The simple proof is omitted.

Lemma 3. *If $f(t) \geq 0$ for all t and $f(0) = 0$ then every non-positive solution of* (1) *is nondecreasing and converges to zero.*

Theorem 1. *Let $f(0) \leq f(t) \leq a|t| + f(0)$ for all t so in particular f is minimized at zero. Then $\bar{x}$ is globally asymptotically stable in* (1) *if either (i) or (ii) below holds*:

(i) $0 < a < \max\{c, 1-c\}$;

(ii) $1 - b < a, c < b$, *where* $b = 2/(\sqrt{5}+1)$.

Proof. Because replacing $f(t)$ by $f(t) - f(0)$ does not change the asymptotic behaviors of solutions of (1), we may assume that $f(0) = 0$ and hence $\bar{x} = 0$. Now let $a < 1 - c$. Define $g(x, y) = cx + f(x - y)$ and for $x, y \geq 0$ notice that

$$\begin{aligned} g(x,y) &\leq cx + a|x-y| \\ &\leq cx + a\max\{x,\, y\} \\ &\leq (c+a)\max\{x\,, y\}. \end{aligned}$$

Since $c + a < 1$ by assumption, it follows that g is a weak contraction on the non-negative quadrant, i.e.,

$$[0, \infty)^2 \subset A_{a+c}.$$

Since $[0, \infty)^2$ is invariant under g, Lemma 1 implies that the origin is asymptotically (in fact, exponentially) stable relative to $[0, \infty)^2$. Thus the origin is stable and every solution $\{x_n\}$ of (1) for which $x_k \geq 0$ for some $k \geq 0$ converges to zero. If $x_0 < 0$ then solutions that remain negative for all n increase to zero by Lemma 3 for $n > -1$. It follows that the origin is globally asymptotically stable.

Next, if $a < c$ then by Lemmas 2 and 3 every solution of (1) is eventually monotonic and approaches zero. To show that the origin is stable, we need only consider the case where $0 \leq x_{-1} < x_0$. Let r_0 be as in the proof of

Lemma 2 and $r_0 > 1$. Then the sequence $\{r_n\}$ is decreasing until $r_k \leq 1$ for some $k \geq 1$. We showed that

$$r_j \leq a + c - \frac{a}{r_{j-1}}, \quad j = 1, \ldots, k$$

Define the mapping $\mu(r) = a + c - a/r$. Notice that μ is increasing and $r_j \leq \mu(r_{j-1})$ for each j. It follows that

$$r_k \leq \mu(r_{k-1}) \leq \mu(\mu(r_{k-2})) = \mu^2(r_{k-2}) \leq \cdots \leq \mu^{k-1}(r_1) \leq \mu^{k-1}(a+c).$$

In particular, k is no larger than the least integer κ that satisfies $\mu^{\kappa-1}(a+c) \leq 1$; i.e., $k \leq \kappa$ for all choices of the initial values x_0, x_{-1}. Since the peak value of $\{x_n\}$ occurs at x_{k-1} and since $r_j < a + c$ for $1 \leq j \leq k$, we have

$$x_{k-1} \leq r_{k-1}x_{k-2} \leq (a+c)x_{k-2} \leq (a+c)^2 x_{k-3} \leq \cdots \leq (a+c)^{k-1}x_0.$$

Therefore, as x_0 (and thus also $x_{-1} < x_0$) approach zero, so does the peak x_{k-1}. It follows that the origin is stable in this case.

Now assume that (ii) holds and note that because of Lemma 3, without loss of generality we may take positive initial values, i.e., $x_{-1}, x_0 > 0$. Define

$$\beta = \max\{a, c, 1-a, 1-c\}$$

and note that $\beta \in (1-b, b)$. For the term x_1 we have

$$\begin{aligned} x_1 &= cx_0 + f(x_0 - x_{-1}) \leq \beta x_0 + \mu, \\ |x_1 - x_0| &= |\,f(x_0 - x_{-1}) - (1-c)x_0| \leq \mu \end{aligned} \tag{4}$$

where

$$\mu = \max\{\,f(x_0 - x_{-1}), (1-c)x_0\}.$$

Further,

$$\begin{aligned} x_2 &\leq cx_1 + a|x_1 - x_0| \leq \beta x_1 + \beta\mu, \\ |x_2 - x_1| &\leq \max\{\,f(x_1 - x_0), (1-c)x_1\} \\ &\leq \max\{\,a|x_1 - x_0|, \beta x_1\} \\ &\leq \beta \max\{\mu, \beta x_0 + \mu\} \\ &= \beta(\beta x_0 + \mu). \end{aligned} \tag{5}$$

To highlight the main pattern inherent in the above calculations and those that follow, let us define

$$y_{-1} = \frac{\mu}{\beta^2}, \quad y_0 = x_0, \quad y_1 = \beta x_0 + \mu \tag{6}$$

Then inequalities (4) and (5) can be written as

$$\begin{aligned} x_1 &\leq \beta y_0 + \beta^2 y_{-1} = y_1 \\ x_2 &\leq \beta x_1 + \beta\mu \leq \beta y_1 + \beta^2 y_0 \\ |x_1 - x_0| &\leq \beta y_0, \quad |x_2 - x_1| \leq \beta y_1 \end{aligned} \tag{7}$$

where we have used the facts that $x_1 \leq y_1$ and $\mu \leq (1-c)x_0 \leq \beta y_0$. Now, based on the more transparent pattern in inequalities (7), let $\{y_n\}$ be the solution of the linear difference equation

$$y_{n+1} = \beta y_n + \beta^2 y_{n-1} \tag{8}$$

with initial values y_{-1}, y_0 as given in (6). Since the eigenvalues of (8) are

$$\left(\frac{1 \pm \sqrt{5}}{2}\right)\beta = \frac{\beta}{b}, -\beta b$$

we obtain

$$y_n = \alpha_1 \left(\frac{\beta}{b}\right)^n + \alpha_2(-\beta b)^n$$

for a suitable choice of constants α_1, α_2. Note that $y_n \to 0$ as $n \to \infty$ because $\beta < b$. To complete the proof of the theorem we show that $x_n \leq y_n$ for all n. Proceeding by way of induction, we note that $x_1 \leq y_1$ by (7). Also based on (7) assume for $k \geq 1$ that $x_n \leq y_n$ and $|x_n - x_{n-1}| \leq \beta y_{n-1}$ for $n < k$. Then

$$\begin{aligned} |x_k - x_{k-1}| &= |\, f(x_{k-1} - x_{k-2}) - (1-c)x_{k-1}| \\ &\leq \max\{a|x_{k-1} - x_{k-2}|, (1-c)x_{k-1}\} \\ &\leq \max\{a\beta y_{k-2}, (1-c)y_{k-1}\} \\ &\leq \beta \max\{\beta y_{k-2}, y_{k-1}\} \\ &\leq \beta y_{k-1} \end{aligned}$$

where the last inequality holds because $y_{k-1} = \beta y_{k-2} + \beta^2 y_{k-3} \geq \beta y_{k-2}$. Further

$$x_k \leq c x_{k-1} + a|x_{k-1} - x_{k-2}| \leq \beta x_{k-1} + a\beta y_{k-2} \leq y_k.$$

We have thus established for all $n \geq 1$ that

$$|x_n - x_{n-1}| \leq \beta y_{n-1}, \quad x_n \leq y_n.$$

Since $x_n \geq 0$ for all n, it follows that $x_n \to 0$ as $n \to \infty$, so the origin is globally attracting. The stability of the origin follows from the observation that the origin is stable for the linear equation (8) if $\beta < b$.

Remarks and a Conjecture.

1. (Global attractivity without stability) Conditions in Theorem 1 imply that f is not too steep near the origin. It turns out that this is important especially for *stability*; for if f is steep near zero, then the origin can be *globally attracting without being stable.* In fact, it is not hard to see that for certain non-decreasing mappings that are minimized at the origin, such as the piecewise linear

$$f(t) = \begin{cases} 0 & t \leq 0 \\ \gamma t & 0 \leq t \leq 1 \\ \gamma & t \geq 1 \end{cases}$$

with $\gamma \geq (1+\sqrt{1-c})^2$, the origin is unstable even though all solutions of (1) eventually converge to zero; see Chapter 5 in Sedaghat [9] for more details.

2. Theorem 1 is not true if $f(0)$ is not a minimum value for f. For example, if $f(t) = -at$, where

$$\frac{1+c}{2} < a < 1-c$$

then equation (1) is linear and its negative eigenvalue has magnitude less than -1 so the solutions are typically unbounded.

If $f(0)$ is not a minimum value for f, then the condition

$$|f(t)| \leq a|t| \tag{9}$$

with the additional restriction $a < (1-c)/2$ ensures that $g(x,y) = cx + f(x-y)$ is a weak contraction on the entire plane and thus the origin is globally exponentially stable. The origin may be still globally asymptotically stable when f is neither a weak contraction nor minimized at zero but instead satisfies the condition $tf(t) \geq 0$ for all t in addition to (9); see Kent and Sedaghat [3].

3. Theorem 1 covers most but not all of the possible c and a values in the unit square (non-closed) in the (c,a)-parameter space. The next conjecture

claims that the unique equilibrium is globally asymptotically stable for all points (c, a) in this unit square:

Conjecture. *Let $f(0) \leq f(t) \leq a|t| + f(0)$ for all t with $0 \leq a < 1$ and $0 \leq c < 1$. Then $\bar{x}$ is globally asymptotically stable in* (1).

This conjecture is not true if $a = 1$ or $c = 1$. For instance, if $f(t) = |t|$ with $c = 0$ then any solution of (1) with initial values $x_0 = x_{-1}$ yields the period-3 solution $\{x_0, x_0, 0, x_0, x_0, 0, \ldots\}$. Also, if $c = 1$, then with $f(t) = f(0)$ a contstant function, we find that solutions of the form $x_n = x_0 + f(0)n$ do not generally converge.

4. The occurrence of the (reciprocal) of the "golden mean," i.e., b in Theorem 1(ii) is of course due to the linear equation (8). This is a "damped" version of the Fibonacci equation, where the damping factor β ensures that the solutions of (8) converge to zero.

3. Non-monotonic Convergence and Chaotic Ratios

By Lemma 2, when $a < c$ the solutions of (1) in Theorem 1 converge to $\bar{x}$ monotonically. This is not true for $c < a < 1 - c$, in which case the solutions of (1) may converge while oscillating off-equilibrium. Theorem 2 below gives a detailed picture of this situation for the absolute value function $f(t) = a|t|$. In this case, we define the mapping

$$\phi(r) = c + a\left|1 - \frac{1}{r}\right|, \quad r > 0.$$

and note that ϕ has a unique positive fixed point

$$\bar{r} = \frac{1}{2}\left[\sqrt{(a-c)^2 + 4a} - (a - c)\right].$$

Before stating Theorem 2, for convenience we quote a fundamental result on chaos from Marotto [5] as Lemma 4. This result refers to the following concept: For a continuous map F of $\mathbb{R}^m$, an isolated fixed point $\bar{x}$ is a *snap-back repeller* (in the weak or non-smooth sense) if there is a sequence $\{B_k\}_{k=-\infty}^{l}$ of compact sets in $\mathbb{R}^m$ satisfying the following conditions:

(1) B_k converges to $\bar{x}$ as $k \to -\infty$;
(2) F is one-to-one on each B_k and $F(B_k) = B_{k+1}$ for every k;
(3) $\bar{x} \in \text{int}(B_l)$ and $B_l \cap B_k$ is empty for $1 \leq k < l$.

Snap-back repellers are more commonly defined in the differentiable setting where a more intuitive description is possible. However, the mapping ϕ to which Theorem 2 applies is not smooth so we need to use the more general defintion of snap-back repellers that was quoted above. For a proof of the following see [5] or [9].

Lemma 4. *If F has a snap-back repeller, then F is chaotic in the sense that*:

(I) *There is a positive integer N such that for each integer $p \geq N$, F has a point of period p (not necessarily stable)*;

(II) *F has a scrambled set, i.e., an uncountable set S containing no periodic points of F such that*:

(i) $F(S) \subset S$;

(ii) *For every $x \in S$ every y where either $y \in S$ and $x \neq y$, or y is a periodic point of F,*

$$\limsup_{k\to\infty} \left\| F^k(x) - F^k(y) \right\| > 0,$$

(III) *There is an uncountable set $S_0 \subset S$ such that for every $x, y \in S_0$*

$$\liminf_{k\to\infty} \left\| F^k(x) - F^k(y) \right\| > 0.$$

In particular, for one dimensional maps, i.e., $m = 1$, the existence of a snap-back repeller is a less stringent condition than the more familiar conditions of Li and Yorke in [4]. For instance, chaotic behavior in the sense of Lemma 4 above can occur without the existence of a period 3 point. See [9] for additional observations and for some applications of Lemma 4 to mathematical models in dimensions 1 and higher.

Theorem 2. *Let $f(t) = a|t|$ and let $\{x_n\}$ be any solution of* (1) *with initial values x_{-1}, x_0.*

(a) *If $c < a < 1 - c$ and $x_0/x_{-1} \notin \cup_{k=1}^{\infty} \phi^{-k}(\bar{r})$ then $\{x_n\}$ converges to zero in a non-monotonic fashion, oscillating above the unique equilibrium at the origin.*

(b) *If $a \in (1/2, 1)$ then there is $c_0 \in (0, a)$ such that for $c < \min\{c_0, 1-a\}$ the mapping ϕ has a scrambled set S; hence, the sequence $\{x_n/x_{n-1}\}$ of consecutive ratios is chaotic for $x_0/x_{-1} \in S$.*

Proof. (a) Define $r_{n+1} = \phi(r_n)$ for all $n \geq 0$ with $r_0 = x_0/x_{-1}$. If $r_0 \in \cup_{k=1}^{\infty} \phi^{-k}(\bar{r})$ then there is $k \geq 1$ such that $r_n = \bar{r}$ for $n \geq k$ so that $x_{n+1} =$

$\bar{r}x_n < x_n$ for $n \geq k$. Thus, eventually we obtain a monotonically decreasing solution. So in the rest of the proof we assume that $r_0 \notin \cup_{k=1}^{\infty} \phi^{-k}(\bar{r})$.

Note that the fixed point $\bar{r}$ of ϕ is unstable because $\bar{r} < \sqrt{a}$ and therefore, $|\phi'(\bar{r})| = a/\bar{r}^2 > 1$. Suppose that $r_0 < 1$, i.e., $x_0 < x_{-1}$ but $r_0 \neq \bar{r}$. Then $r_1 = \phi(r_0) = \phi_1(r_0)$ where ϕ_1 is the decreasing function

$$\phi_1(r) = c - a + \frac{a}{r}.$$

Since $\phi_1(r) > 1$ for $r \in (0, r^*)$ where $r^* = a/(1 + a - c)$, it follows that either $r_1 > 1$, or some iterate $r_k = \phi_1^k(r_0) > 1$. This means that $x_k > x_{k-1}$ while $x_1 > x_2 > \cdots > x_{k-1}$. Next, $r_{k+1} = \phi_2(r_k)$ where ϕ_2 is the increasing function

$$\phi_2(r) = a + c - \frac{a}{r}.$$

Since $\phi_2(r) < a + c < 1$ for all r we see that $r_{k+1} < 1$ and so the preceding process repeats itself ensuring that there are infinitely many terms x_{k_j}, $j = 1, 2, \ldots$ where the inequality $x_{k_j+1} > x_{k_j}$ holds.

The magnitude of the up-jump depends on the parameters; since c is the absolute minimum value of ϕ for $r > 0$, we see that

$$r_n \leq \phi(c) = \phi_1(c) = \frac{a}{c} - (a - c)$$

for all $n \geq 1$. Thus

$$x_{k_j} < x_{k_j+1} < \left[\frac{a}{c} - (a - c)\right] x_{k_j}, \quad j = 1, 2, \ldots$$

The differences $k_{j+1} - k_j$ are not necessarily constants, as seen in the next part.

(b) We show that $\bar{r}$ is a snap-back repeller for ϕ. Define $I_l = [\bar{r} - \delta, \bar{r} + \delta]$ for $\delta > 0$ small enough that $I_l \subset (c, 1)$. Then $\bar{r} \in \mathrm{int}(I_l)$ as required by condition (3) in the definition of snap-back repeller. To complete the proof, a little set up is necessary. Since $\bar{r} < a + c$,

$$\bar{r}_{-1} = \phi_2^{-1}(\bar{r}) = \frac{a}{a + c - \bar{r}}$$

is well defined and $\bar{r}_{-1} > 1$. So we may define $\bar{r}_{-2} = \phi_1^{-1}(\bar{r}_{-1})$ where

$$\phi_1^{-1}(r) = \frac{a}{r + a - c}, \quad r \geq c.$$

Now, since $\bar{r} < 1 < \bar{r}_{-1}$ and ϕ_1 is decreasing, it follows that

$$\bar{r} = \phi_1^{-1}(\bar{r}) > \phi_1^{-1}(\bar{r}_{-1}) = \bar{r}_{-2}. \tag{10}$$

Next, we require that

$$\bar{r}_{-2} > c. \tag{11}$$

Note that (11) is equivalent to $\bar{r}_{-1} < \phi_1(c)$ or equivalently,

$$\bar{r} < \phi_2(\phi_1(c)) = a + c - \frac{ac}{a - c(a-c)}. \tag{12}$$

Let us denote the right hand side of (12) by $g(c)$ and notice that $g(0) = a$ and g is continuous at $c = 0$ even though $\phi_1(c)$ is not. Using the definition of $\bar{r}$ we can write (12) in the equivalent form

$$a < (a-c)g(c) + [g(c)]^2. \tag{13}$$

Define the function

$$h(c) = a - (a-c)g(c) - [g(c)]^2, \quad 0 \leq c \leq a$$

and note that $h(0) = a - 2a^2 < 0$ since $a > 1/2$. Further, since $a < 1$,

$$h(a) = a - [g(a)]^2 = a - a^2 > 0.$$

It follows that there is c_0 in the interval $(0, a)$ such that $h(c) < 0$ for $c \in (0, c_0)$. For these values of c, (13) holds and thus, (11) holds also.

Now we are ready to complete the proof that $\bar{r}$ is a snap-back repeller. Define $\alpha_l = \bar{r} - \delta$, $\beta_l = \bar{r} + \delta$ and $I_{l-1} = \phi_2^{-1}(I_l) = [\alpha_{l-1}, \beta_{l-1}]$ where

$$\alpha_{l-1} = \phi_2^{-1}(\alpha_l) > 1, \qquad \beta_{l-1} = \phi_2^{-1}(\beta_l) > 1.$$

Then $I_{l-1} \subset (1, \infty)$ and $I_{l-1} \cap I_l$ is empty. Further,

$$\phi_1^{-1}(I_{l-1}) = [\phi_1^{-1}(\beta_{l-1}), \phi_1^{-1}(\alpha_{l-1})].$$

Let $\beta_{l-2} = \phi_1^{-1}(\alpha_{l-1})$ and $\alpha_{l-2} = \max\{c, \phi_1^{-1}(\beta_{l-1})\}$ and define $I_{l-2} = [\alpha_{l-2}, \beta_{l-2}]$. By this construction, $\bar{r}_{-2} \in I_{l-2}$ so $\beta_{l-2} > r_{-2} > c$ and I_{l-2} is not empty. Further,

$$\alpha_{l-1} > 1 > \bar{r} \Rightarrow \beta_{l-2} = \phi_1^{-1}(\alpha_{l-1}) < \phi_1^{-1}(\bar{r}) = \bar{r}$$

so that

$$I_{l-2} \subset [c, \bar{r}). \tag{14}$$

Next, we define

$$I_{l-3} = \phi_1^{-1}(I_{l-2}) = [\phi_1^{-1}(\beta_{l-2}), \phi_1^{-1}(\alpha_{l-2})] = [\alpha_{l-3}, \beta_{l-3}]$$

and notice that $\alpha_{l-3} > \phi_1^{-1}(\bar{r}) = \bar{r}$ and $\beta_{l-3} \leq \phi_1^{-1}(c) = 1$. Hence,

$$I_{l-3} \subset (\bar{r}, 1]. \tag{15}$$

Now, if for $j \geq 2$ we define the following sequence

$$I_{l-j} = \phi_1^{-1}(I_{l-j+1}) = [\alpha_{l-j}, \beta_{l-j}]$$

where

$$\begin{aligned} \alpha_{l-j} &= \phi_1^{-1}(\beta_{l-j+1}) = \frac{a}{\beta_{l-j+1} + a - c}, \\ \beta_{l-j} &= \phi_1^{-1}(\alpha_{l-j+1}) = \frac{a}{\alpha_{l-j+1} + a - c} \end{aligned}$$

then from (14) and (15) it follows that $c \leq \alpha_{l-j}, \beta_{l-j} \leq 1$ for $j \geq 2$ and thus, the intervals I_{l-j} are well-defined. In fact, if $\phi_1^{-2}(r) = \phi_1^{-1}(\phi_1^{-1}(r))$, then

$$\begin{aligned} \alpha_{l-2j} &= \phi_1^{-2}(\alpha_{l-2j+2}) \geq c, \\ \beta_{l-2j} &= \phi_1^{-2}(\beta_{l-2j+2}) < \bar{r}. \end{aligned}$$

We claim that

$$\alpha_{l-2j}, \beta_{l-2j} \to \bar{r} \quad \text{as} \quad j \to \infty. \tag{16}$$

If this is true, then

$$\alpha_{l-2j-1} = \phi_1^{-1}(\beta_{l-2j}) \to \bar{r}, \quad \beta_{l-2j-1} = \phi_1^{-1}(\alpha_{l-2j}) \to \bar{r}$$

and it follows that the compact intervals I_{l-j} converge to $\bar{r}$. From this and the fact that ϕ_1 is strictly decreasing on $(0, 1]$ it will necessarily follow that $\bar{r}$ is a snap-back repeller (in the definition of snap-back repeller we may take $k \geq 2$ to be the least integer j for which $I_{l-j} \cap I_l$ is non-empty).

To prove the claim (16), it suffices to show that if $\rho \in [c, \bar{r})$ then

$$\lim_{n\to\infty} \phi_1^{-2n}(\rho) = \bar{r}. \tag{17}$$

To see this, first note that

$$r < \phi_1^{-2}(r) < \bar{r} \quad \text{for} \quad r \in (0, \bar{r}). \tag{18}$$

To see this, we compute

$$\phi_1^{-2}(r) = \frac{a(r + a - c)}{(a - c)(r + a - c) + a}$$

and observe that if $r < \bar{r}$ then

$$\phi_1^{-2}(r) > r\frac{a[1 + (a - c)/\bar{r}]}{(a - c)(\bar{r} + a - c) + a} = r \tag{19}$$

since $\bar{r} + a - c = a/\bar{r}$. Further, the derivative $d\phi_1^{-2}/dr$ is positive so ϕ_1^{-2} is an increasing function on $(0, \bar{r})$ with $\phi_1^{-2}(\bar{r}) = \bar{r}$. This and (19) prove (18), which in particular shows that $\{\phi_1^{-2n}(\rho)\}$ is an increasing sequence in $[c, \bar{r})$. It follows that (17), and thus (16) is true and the proof that $\bar{r}$ is a snap-back repeller is complete. The proof of (b) is completed upon applying Lemma 4.

In closing, it may be relevant to point out that the mapping ϕ above that governs the dynamics of ratios $r_n = x_n/x_{n-1}$ is a *semiconjugate factor* (see Sedaghat [9]) of the mapping

$$F(x, y) = [cx + a|x - y|, x]$$

which governs the dynamics of (1) with $f(t) = a|t|$. The scalar map $H(x, y) = x/y$ links ϕ to F via the equation

$$H \circ F = \phi \circ H.$$

For more general choices of f, semiconjugate factors and links for the corresponding unfolding F are not generally known. However, in the special case $c = 1$, any f is easily seen to be a semiconjugate factor of its unfolding F via the link $H(x, y) = x - y$. This semiconjugacy arises in the business cycle model in Puu [6]; see Sedaghat [9] for details.

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