JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **213,** 496–510 (1997)
ARTICLE NO. AY975553

General Permanence Conditions for Nonlinear Difference Equations of Higher Order

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Submitted by Gerry Ladas

Received December 2, 1996

1. INTRODUCTION

Consider the mth order nonlinear difference equation

$$x_n = f(x_{n-1}, x_{n-2}, \dots, x_{n-m}), \qquad n = 1, 2, 3, \dots, \tag{1}$$

where $f: [0, \infty)^m \to [0, \infty)$ and the initial values $x_0, \dots, x_{1-m} \in [0, \infty)$. In studying the global behavior of the solutions of Eq. (1), we often need to establish that (1) is permanent, i.e., it has the property that every one of its solutions is eventually confined within a fixed compact interval regardless of the initial values chosen. Permanence is needed directly or indirectly in establishing other properties such as persistent bounded oscillations or the global attractivity of fixed points. For example, in [6] permanence of the equation

$$x_n = x_{n-1} g(x_{n-1}, \dots, x_{n-m}) \tag{2}$$

under certain conditions on the function g is used directly in the derivation of global attractivity results. More recently, the oscillations results in [9] require the existence of nontrivial bounded solutions to (1). Other instances where permanence is applied are noted in the sequel.

In this paper we obtain general sufficient conditions that imply Eq. (1) is permanent. Our approach here utilizes the existence of linear bounds on minimally restricted functions f (i.e., when f is sublinear in the large).

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0022-247X/97 $25.00

Thus we will generally not require that f be continuous or monotonic in some of its arguments; nor is it necessary to assume that (1) possesses a simplified fixed point structure as is commonly done when dealing with global stability or with oscillations. One consequence of this generalization is that the results in this paper extend and unify the existing permanence and boundedness results in the literature. On the other hand, the restrictive hypotheses needed for stability or for oscillations but not assumed here, sometimes involve parameter ranges that go beyond what is allowed for f here. Hence, the results of this paper do not entirely subsume the existing permanence results.

In addition to the unifying aspect mentioned above, we also obtain completely new results concerning broad classes of difference equations of arbitrary order. Many of these results immediately yield new bounded oscillation theorems when coupled with the results in [9].

2. PRELIMINARIES

DEFINITION 1. Equation (1) is said to be *permanent* if there exist constants $L, M \in [0, \infty)$, such that for each solution $\{x_n\}$ of (1), there is a positive integer $n_0 = n_0(x_{1-m}, \dots, x_0)$ such that $x_n \in [L, M]$ for all $n \geq n_0$. Any compact interval having this property may be called an *absorbing interval* for (1).

Remarks.

(1) Note that each absorbing interval must contain all attracting points and limit sets, so permanence puts restrictions on trajectories that mere boundedness does not. Indeed, boundedness of all solutions of (1) does not imply permanence (see below).

(2) If the function f in (1) is bounded, then (1) is trivially permanent.

(3) In [6] it is also required that $L > 0$. We shall not assume this condition, but when it does hold we may refer to (1) as *positively* permanent. In any case, it is important to note that constants L, M do not depend on the initial conditions; further, in the absence of sharpness requirements or other considerations, we may let $L = 0$.

DEFINITION 2. Bold-faced letters denote vectors in the m-dimensional space $\mathbf{R}^m$ or the cone $[0, \infty)^m$, unless otherwise noted; we now list three of the most familiar norms on $\mathbf{R}^m$ for reference:

$$\|\mathbf{u}\|_\infty \doteq \max\{|u_1|,\dots,|u_m|\} \qquad \text{(the max-norm, sup-norm, or } l^\infty\text{-norm)}$$

$$\|\mathbf{u}\|_2 \doteq \left(u_1^2 + \cdots + u_m^2\right)^{1/2} \qquad \text{(the Euclidean or } l^2 \text{ norm)}$$

$$\|\mathbf{u}\|_1 \doteq |u_1| + \cdots + |u_m| \qquad \text{(the sum or } l^1 \text{ norm)}.$$

An elementary fact about finite dimensional spaces is that all norms on them are equivalent (see, e.g., [7, p. 75]). In particular, this means that for any norm $\|\cdot\|$, there are constants $C, C' > 0$ such that

$$C'\|\mathbf{u}\|_\infty \le \|\mathbf{u}\| \le C\|\mathbf{u}\|_\infty$$

for all $\mathbf{u} \in \mathbf{R}^m$. In the sequel we refer to C (and C') as the *upper* (*and lower*, *respectively*) *max-norm coefficients* of $\|\cdot\|$. Clearly $C = 1$ for the max-norm itself, and it is easily verified that $C = m$ for the sum norm and that $C = \sqrt{m}$ for the Euclidean norm. For both of the last two norms, $C' = 1$.

We use the notation $\mathbf{u} \cdot \mathbf{v}$ to define the familiar scalar product in $\mathbf{R}^m$:

$$\mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^{m} u_i v_i.$$

The following variation on a rather familiar result (see, e.g., [8, p. 20]) is needed in the sequel.

LEMMA 1. *Let* $f, g\colon [0,\infty)^m \to [0,\infty)$, *and assume that for all* $\mathbf{u} \in [0,\infty)^m$, $f(\mathbf{u}) \le g(\mathbf{u})$. *If* g *is nondecreasing in each of its* m *arguments and the equation*

$$z_n = g(z_{n-1},\dots,z_{n-m}) \tag{3}$$

is permanent, then Eq. (1) *is permanent. Also, if all solutions of* (3) *are bounded, then so are all solutions of* (1).

Proof. We may assume that $[0, M]$ is an absorbing interval for (3) where M is some positive real number. Suppose that $x_0,\dots,x_{1-m}$ is an arbitrary set of initial values for (1) and define $z_i = x_i$ for $i = 1-m,\dots,0$. Note that

$$x_1 = f(x_0,\dots,x_{1-m}) \le g(z_0,\dots,z_{1-m}) = z_1$$

and by induction

$$x_n = f(x_{n-1},\dots,x_{n-m}) \le g(z_{n-1},\dots,z_{n-m}) = z_n.$$

Since there is an integer $n_0 \geq 1$ such that $z_n \leq M$ for all $n \geq n_0$, it follows that $[0, M]$ is also an absorbing interval for (1). The final assertion about boundedness is similarly proved. ∎

3. THE MAIN RESULTS

LEMMA 2. *Let $a \in [0,1)$ and $b \in [0,\infty)$. Then $\bar{x} = b/(1-a)$ is the unique fixed point of*

$$x_n = a \max\{x_{n-1}, \dots, x_{n-m}\} + b \tag{4}$$

and $\bar{x}$ is globally asymptotically stable.

Proof. Since $\bar{x}$ is the only solution of the equation $x = a\max\{x, \dots, x\} + b$, it follows that $\bar{x}$ is the unique fixed point of (4) and also that $a\bar{x} + b = \bar{x}$. To show that $\bar{x}$ is globally attracting, we consider two cases:

Case 1. $x_k \leq \bar{x}$ for all $k = 1-m, \dots, 0$. Define

$$M \doteq \max\{x_0, \dots, x_{1-m}\}.$$

Note that $M \leq \bar{x}$, which implies that

$$M \leq aM + b \leq a\bar{x} + b = \bar{x}.$$

Since $x_1 = aM + b$, it follows that $M \leq x_1 \leq \bar{x}$. Therefore,

$$x_2 = a\max\{x_1, x_0, \dots, x_{2-m}\} + b = a(aM+b) + b = a^2M + b(1+a)$$

and

$$M \leq aM + b \leq a(aM+b) + b \leq a\bar{x} + b$$

so that

$$M \leq x_1 \leq x_2 \leq \bar{x}.$$

Proceeding in this manner, we find inductively that

$$x_n = a^n M + b\sum_{i=0}^{n-1} a^i$$

is a nondecreasing sequence which clearly approaches $\bar{x}$ as $n \to \infty$.

Case 2. $x_k > \bar{x}$ for some $k = 1-m, \dots, 0$. Defining M as above, we see that $M > \bar{x}$ so that

$$M > aM + b > \bar{x}.$$

Thus

$$\bar{x} < x_1 = aM + b < M$$

implying

$$ax_1 + b \le x_2 \le aM + b.$$

Therefore, $\bar{x} < x_2 \le x_1 < M$. Since x_0, x_1 are both present in $\max\{x_{n-1}, \dots, x_{n-m}\}$ for $n = 1, \dots, m$, it is evident that for these values of n,

$$\bar{x} < x_n \le aM + b < M \qquad (1 \le n \le m).$$

However,

$$x_{m+1} = a \max\{x_m, x_{m-1}, \dots, x_1\} + b \le a(aM + b) + b < aM + b$$

and x_m is present in $\max\{x_{n-1}, \dots, x_{n-m}\}$ for $n = m + 1, \dots, 2m$. It follows that

$$\bar{x} < x_n \le a^2 M + b(1 + a) < aM + b < M \qquad (m + 1 \le n \le 2m).$$

Continuing inductively, for $j = 1, 2, 3, \dots,$ we have

$$\bar{x} < x_n \le a^j M + b \sum_{i=0}^{j-1} a^i \qquad ((j - 1)m + 1 \le n \le jm).$$

Once again, it is clear that $x_n \to \bar{x}$ as $n \to \infty$.

Finally, the monotonic nature of the convergence of x_n to $\bar{x}$ in both of the above cases implies that $\bar{x}$ is in fact stable, thus completing the proof. ∎

Remarks.

(1) When $a = 1$, Eq. (4) is not permanent. In this case, when $b = 0$, (4) has the trivial solution $x_n = M$, $n = 1, 2, 3, \dots$ with M as defined in the proof of Lemma 2. Clearly all such solutions are bounded, although the dependence of M on the initial values that define it means that (4) is not permanent. When $a = 1$ and $b > 0$, then it is easily seen that solutions of (4) are all of the unbounded variety $x_n = M + bn$.

(2) It may be worth noting that in Lemma 2, $\bar{x}$ is not a hyperbolic or linearly stable fixed point when $m \ge 2$ and $a > 0$, since the partial derivatives of the max function

$$f(u_1, \dots, u_m) = a \max\{u_1, \dots, u_m\} + b$$

do not exist at any point $(x, \dots, x)$ for $x \in (0, \infty)$.

LEMMA 3. *Let* $f: [0, \infty)^m \to [0, \infty)$, *and assume that there exist* $a \in [0, 1)$ *and* $b \in [0, \infty)$ *such that*

$$f(\mathbf{u}) \le a\|\mathbf{u}\|_\infty + b$$

for all $\mathbf{u} \in [0, \infty)^m$. *Then Eq.* (1) *is permanent.*

Proof. If we define $g(\mathbf{u}) \doteq a\|\mathbf{u}\|_\infty + b$, then g is clearly nondecreasing in each coordinate of $\mathbf{u} = (u_1, \dots, u_m)$. Hence, an application of Lemmas 1 and 2 completes the proof. ∎

THEOREM 1. *Let* $f: [0, \infty)^m \to [0, \infty)$. *If either of the following two sublinearity conditions hold, then* (1) *is permanent.*

(I) f *is bounded on the compact subsets of* $[0, \infty)^m$ *and*

$$\limsup_{\|\mathbf{u}\| \to \infty} \frac{f(\mathbf{u})}{\|\mathbf{u}\|} < \frac{1}{C},$$

where C *is the upper max-norm coefficient.*

(II) *There is* $\mathbf{a} \in [0, \infty)^m$ *with* $\|\mathbf{a}\|_1 < 1$ *and* $b \in [0, \infty)$ *such that for all* $\mathbf{u} \in [0, \infty)^m$,

$$f(\mathbf{u}) \le b + \mathbf{a} \cdot \mathbf{u}. \tag{5}$$

Proof. First, we show that (I) implies Eq. (1) is permanent. The function

$$h(x) \doteq \sup\left\{ \frac{f(\mathbf{u})}{\|\mathbf{u}\|} : \|\mathbf{u}\| > x \right\}$$

is nonincreasing and $\lim_{x \to \infty} h(x) < 1/C$. Thus there is $a \in [0, 1)$ and $r > 0$ such that $h(r) = a/C$. It follows that

$$\frac{f(\mathbf{u})}{\|\mathbf{u}\|} \le \frac{a}{C} \qquad \text{for all } \|\mathbf{u}\| > r$$

which implies that

$$f(\mathbf{u}) \le a\|\mathbf{u}\|_\infty \qquad \text{for all } \|\mathbf{u}\|_\infty > \frac{r}{C'},$$

where C' is the lower max-norm coefficient. Now, $\|\mathbf{u}\|_\infty > r/C'$ if and only if $\max\{u_1, \dots, u_m\} > r/C'$ if and only if $\mathbf{u} \notin [0, r/C']^m$. Define

$$b = \sup\left\{ f(\mathbf{u}) : \mathbf{u} \in [0, r/C']^m \right\} < \infty$$

and note that for all $\mathbf{u} \in [0, \infty)^m$ it is true that $f(\mathbf{u}) \le a\|\mathbf{u}\|_\infty + b$. Lemma 3 now establishes the permanence of (1).

Now suppose that condition (II) holds. Then

$$f(\mathbf{u}) \le b + \mathbf{a}\cdot\mathbf{u} \le b + \sum_{i=1}^{m} a_i \max\{u_1,\dots,u_m\} = b + \|\mathbf{a}\|_1\|\mathbf{u}\|_\infty$$

so that once again Lemma 3 applies. ∎

Remarks.

(1) Theorem 1(I) is false if $1/C$ is replaced by 1 for norms other than the max-norm. Indeed, the function

$$f(u,v) = b + \frac{u+v}{2}$$

results in a non-permanent second order equation, as may easily be checked. Note that f has the property that

$$\frac{f(u,v)}{\|(u,v)\|_1} = \frac{b}{u+v} + \frac{1}{2} \ge \frac{1}{2}$$

for all $(u,v) \in (0,\infty)^2$. But when $u + v = \|(u,v)\|_1$ is large enough, $b/(u + v) + 1/2 < 1$.

(2) Theorem 1 may be false if f cannot be properly defined on the boundary of the cone $[0,\infty)^m$. For example, consider the first order equation

$$x_n = \frac{1}{x_{n-1}^p}, \tag{6}$$

where $p, x_0 > 0$. Here, $f(u) = u^{-p}$ is continuous on $(0,\infty)$ and strictly decreasing, approaching 0 as $u \to \infty$. The general solution of (6) is given by

$$x_n = x_0^{(-p)^n}.$$

For $0 < p < 1$, the unique fixed point $\bar{x} = 1$ is globally asymptotically stable, so in particular, (6) is permanent. However, for $p = 1$, while every solution is bounded (and of period 2 if $x_0 \neq 1$), (6) is no longer permanent; and if $p > 1$, then every solution of (6) with $x_0 \neq 1$ is unbounded.

4. APPLICATIONS

With the aid of the results of the preceding section, we obtain in this section general permanence and boundedness theorems for various classes of nonlinear difference equations. The first result concerns a generalization of Eq. (2).

COROLLARY 1. *Let* $g: [0,\infty)^m \to [0,\infty)$ *be bounded and assume further that*

$$\limsup_{\|\mathbf{u}\|_\infty \to \infty} g(\mathbf{u}) < 1.$$

Then the equation

$$x_n = x_{n-k} g(x_{n-1}, \ldots, x_{n-m}), \qquad 1 \le k \le m \tag{7}$$

is permanent.

Proof. Defining

$$f(u_1, \ldots, u_m) \doteq u_k g(u_1, \ldots, u_m) \tag{8}$$

we see immediately that

$$\limsup_{\|\mathbf{u}\|_\infty \to \infty} \frac{f(\mathbf{u})}{\|\mathbf{u}\|_\infty} \le \limsup_{\|\mathbf{u}\|_\infty \to \infty} g(\mathbf{u}) < 1$$

so the proof concludes by applying Theorem 1. ∎

Remarks.

(1) With regard to Eq. (2), or when $k = 1$ in (7), the simple conditions imposed on g in Corollary 1 are quite different from hypotheses (H_1)–(H_4) stated in [6, p. 36], although there is a significant overlap. The latter hypotheses imply that (2) is permanent without requiring that g be bounded. The same hypotheses, however, also put several restrictions on g (e.g., that g be continuous and nonincreasing in $u_2, \ldots, u_m$, that (2) have a unique positive fixed point $\bar{x}$, etc.) that we do not assume in Corollary 1.

(2) Do the aforementioned (H_1)–(H_4) imply the hypotheses of Theorem 1(I)? It turns out that they do imply the boundedness of the function f in (8) on the compact sets and the finiteness of the limit supremum in Theorem 1; however, the said limit supremum may exceed 1 under the max-norm since the following (finite) quantity

$$\limsup_{x \to \infty} \left\{ g(\mathbf{u}) \colon \mathbf{u} \in (x, \infty) \times [0, \bar{x}]^{m-1} \right\}$$

may exceed 1 under (H_1)–(H_4). Evidently, utilizing the particular product form of (2) as is done in [6] (i.e., more substantially than in Corollary 1) is required for proving permanence under (H_1)–(H_4).

(3) The Baumol–Wolff model of "productivity growth" [1, p. 355], if generalized by introducing additional lags or delays into the basic first order model, represents an interesting economic application of Eq. (2).

The hypotheses in Corollary 1 as well as those in [6] are both general enough to be admissible; hence, the attractivity results for (2) in [6] and the oscillation results in [9] provide immediate insights into the global behavior of the extended Baumol–Wolff equations.

COROLLARY 2. *Let functions* $f_i\colon [0,\infty) \to [0,\infty)$, $i = 1,\dots,m$, *be given and assume that there exist constants* $a_i \in [0,1)$ *and* $b_i \in [0,\infty)$ *such that* $f_i(x) \le a_i x + b_i$ *for all* $x \ge 0$. *Then the equation*

$$x_n = \max\{f_1(x_{n-1}),\dots,f_m(x_{n-m})\}$$

is permanent.

Proof. Let $\mathbf{a} = (a_1,\dots,a_m)$, $\mathbf{b} = (b_1,\dots,b_m)$, and define

$$f(u_1,\dots,u_m) \doteq \max\{f_1(u_1),\dots,f_m(u_m)\}.$$

Note that

$$f(\mathbf{u}) \le \|\mathbf{a}\|_\infty \max\{u_1,\dots,u_m\} + \|\mathbf{b}\|_\infty = \|\mathbf{a}\|_\infty\|\mathbf{u}\|_\infty + \|\mathbf{b}\|_\infty.$$

Now Lemma 3 may be applied to conclude the proof. ∎

COROLLARY 3. *Let functions* $f_i\colon [0,\infty) \to [0,\infty)$, $i = 1,\dots,m$, *be given and assume that for some* $j \in \{1,2,\dots,m\}$, *there are constants* $a \in [0,1)$ *and* $b \in [0,\infty)$ *such that* $f_j(x) \le ax + b$ *for all* $x \ge 0$. *Then the equation*

$$x_n = \min\{f_1(x_{n-1}),\dots,f_m(x_{n-m})\}$$

is permanent.

Proof. Define $f(u_1,\dots,u_m) \doteq \min\{f_1(u_1),\dots,f_m(u_m)\}$ and note that

$$f(u_1,\dots,u_m) \le f_j(u_j) \le au_j + b.$$

The proof now concludes by applying Theorem 1(II). ∎

COROLLARY 4. *Let functions* $f_i\colon [0,\infty) \to [0,\infty)$, $i = 1,\dots,m$, *satisfy the following conditions*:

(a) *There are nondecreasing functions* $g_i\colon [0,\infty) \to [0,\infty)$ *such that* $f_i(x) \le g_i(x)$ *for all* $x \ge 0$;

(b) $\limsup_{x\to\infty} x^{-1}\prod_{i=1}^m g_i(x) < 1$.

Then the equation

$$x_n = \prod_{i=1}^{m} f_i(x_{n-i}) \tag{9}$$

is permanent.

Proof. Define $f(\mathbf{u}) = f(u_1, \dots, u_m) \doteq \prod_{i=1}^m f_i(u_i)$ and note that

$$f(\mathbf{u}) \le \prod_{i=1}^m g_i(u_i) \le \prod_{i=1}^m g_i(\|\mathbf{u}\|_\infty),$$

where the last inequality holds because $u_i \le \|\mathbf{u}\|_\infty$ and g_i is nondecreasing for every $i = 1, \dots, m$. Now condition (b) implies that

$$\limsup_{\|\mathbf{u}\|_\infty \to \infty} \frac{f(\mathbf{u})}{\|\mathbf{u}\|_\infty} < 1.$$

Hence, by Theorem 1, Eq. (9) is permanent. ∎

COROLLARY 5. *Let functions* f_i: $[0, \infty) \to [0, \infty)$, $i = 1, \dots, m$, *satisfy the following conditions*:

(a) $f_i(x) \le a_i x^{p_i} + b_i$ *for all* $x \ge 0$, *where* $a_i > 0$, $b_i \ge 0$, *and* $0 \le p_i \le 1$;

(b) *Either* $p \doteq \sum_{i=1}^m p_i < 1$, *or* $p = 1$ *and* $\prod_{i=1}^m a_i < 1$.

Then Eq. (9) *is permanent.*

Proof. The functions $a_i x^{p_i} + b_i$ are increasing on $[0, \infty)$ and

$$\prod_{i=1}^m (a_i x^{p_i} + b_i) = x^p \prod_{i=1}^m a_i + \rho(x),$$

where $\rho(x)$ consists of a sum of powers of x with each power strictly less than p. Since either of the conditions in (b) implies condition (b) in Corollary 4, the proof is completed by an application of the latter boundary. ∎

Remark. Under suitable restrictive hypotheses, Corollary 5 may hold even if some of the exponents p_i are negative while others may exceed 1; see Theorem 4.1 in [2] for a second-order example.

COROLLARY 6. *Let* f_i: $[0, \infty) \to [0, \infty)$, $i = 1, \dots, m$, *be functions for which there are constants* $a_i, b_i \in [0, \infty)$, *such that* $f_i(x) \le a_i x + b_i$ *on* $[0, \infty)$. *If* $\sum_{i=1}^m a_i < 1$, *then the equation*

$$x_n = \sum_{i=1}^m f_i(x_{n-i})$$

is permanent.

Proof. Observe that

$$f(u_1, \ldots, u_m) \doteq \sum_{i=1}^{m} f_i(u_i) \le b + \sum_{i=1}^{m} a_i u_i, \qquad b \doteq \sum_{i=1}^{m} b_i$$

and apply Theorem 1 to complete the proof. ∎

The (omitted) proof of the next corollary is similar to the proof of Corollary 6.

COROLLARY 7. *Let* $g\colon [0, \infty) \to [0, \infty)$ *satisfy* $g(x) \le \alpha x + \beta$ *on* $[0, \infty)$ *for constants* $\alpha, \beta \in [0, \infty)$. *The equation*

$$x_n = \sum_{i=1}^{m} a_i x_{n-i} + g\left(\sum_{i=1}^{m} b_i x_{n-i} \right), \qquad a_i, b_i \in [0, \infty) \tag{10}$$

is permanent if $\sum_{i=1}^{m}(a_i + \alpha b_i) < 1$.

Remarks. Stability of the fixed points of Eq. (10) has been studied in [4, 5] (or see [6]). This type of equation appears in a model of whale populations [3].

Permanence of (10) is proved in [6] under several specialized hypotheses, then used in establishing global stability of a positive fixed point $\bar{x}$ whose unique existence is guaranteed by the same hypotheses that are used in proving permanence. These latter hypotheses, in particular, imply the following: $g(x) \le \alpha x + \beta$ with $\beta = \sup_{0 \le x \le \bar{x}} g(x)$ and $\alpha = 1 - a$, where $a \doteq \sum_{i=1}^{m} a_i < 1$; also implied is $\sum_{i=1}^{m}(a_i + \alpha b_i) = 1$. The last equality shows that under additional hypotheses on f, Corollary 7 holds even if the sum $\sum_{i=1}^{m}(a_i + \alpha b_i)$ equals 1. On the other hand, when the last sum is less than 1, Corollary 7 shows that permanence exists without additional restrictive hypotheses.

COROLLARY 8. *Let* $g\colon (-\infty, \infty) \to [0, \infty)$ *satisfy* $g(x) \le \alpha \max\{x, 0\} + \beta$ *on* $(-\infty, \infty)$ *for constants* $\alpha, \beta \in [0, \infty)$. *Then the equation*

$$x_n = \sum_{i=1}^{m} a_i x_{n-i} + g\left(b x_{n-j} - c x_{n-k}\right), \qquad c, b, a_i \in [0, \infty),\ 1 \le i, j, k \le m \tag{11}$$

is permanent if $\alpha b + \sum_{i=1}^{m} a_i < 1$.

Proof. Define the function

$$f(u_1, \ldots, u_m) \doteq \sum_{i=1}^{m} a_i u_i + g(bu_j - cu_k)$$

and observe that

$$\begin{aligned} f(u_1, \ldots, u_m) &\leq \sum_{i=1}^{m} a_i u_i + \alpha \max\{bu_j - cu_k, 0\} + \beta \\ &\leq \beta + \alpha b u_j + \sum_{i=1}^{m} a_i u_i. \end{aligned}$$

Therefore, Theorem 1 implies that (11) is permanent if $\alpha b + \sum_{i=1}^{m} a_i < 1$. ∎

Remark. Like Corollary 7, in special cases it is possible to improve the parameter range in Corollary 8. For instance, the second order equation

$$x_n = a x_{n-1} + g(x_{n-1} - x_{n-2}) \tag{12}$$

is a special case of (11), so according to Corollary 8 permanence obtains for (12) if, in particular, $\alpha + a < 1$. However, it was shown in [10] that (12) is permanent if only $\alpha, a < 1$. This increase in parameter range (known so far only for the second order case) came at the price of requiring g to be nondecreasing in addition to satisfying the linear bound of Corollary 8.

Generalizations of (12) and variants of (11) provide natural mathematical settings for discussing the global behavior of ‘‘accelerator-based’’ models of the macroeconomic business cycle; see [9, 10].

COROLLARY 9. *Assume that* $f_i, g_i\colon [0, \infty) \to [0, \infty)$, $i = 1, \ldots, m$, *are functions satisfying*

$$\frac{f_i(x)}{b + g_i(x)} \leq a_i x + b_i$$

on $[0, \infty)$ *for constants* $a_i, b_i \in [0, \infty)$ *and* $b \in (0, \infty)$. *If* $\sum_{i=1}^{m} a_i < 1$ *then the equation*

$$x_n = \frac{\sum_{i=1}^{m} f_i(x_{n-i})}{b + \sum_{i=1}^{m} g_i(x_{n-i})} \tag{13}$$

is permanent.

Proof. Note that

$$f(u_1,\ldots,u_m) \doteq \frac{\sum_{i=1}^{m} f_i(u_i)}{b + \sum_{i=1}^{m} g_i(u_i)}$$

$$\le \sum_{i=1}^{m} \frac{f_i(u_i)}{b + g_i(u_i)} \le \sum_{i=1}^{m} b_i + \sum_{i=1}^{m} a_i u_i$$

so that once again Theorem 1 applies. ∎

Remark. If the functions f_i, g_i are polynomials, then solutions of (13) are called rational recursive sequences. In particular, if the f_i, g_i are polynomials of degree 1 or 0, then Corollary 9 proves a theorem of Ladas and Kocic [6, p. 61], who also use the methodology presented in Lemma 1 and Theorem 1(II), though not at as general a level as we have done here. One advantage of the generalization is that the second part of the aforementioned Kocic–Ladas Theorem need not be proved independently, as both parts of that theorem are included in Corollary 9.

We close with an application of the results obtained here to the oscillation theory. For convenience, we quote the main result in [9] as a lemma here. We state for reference that a bounded solution $\{x_n\}$ of (1) is said to *oscillate persistently* if the sequence $\{x_n\}$ has at least two distinct limit points.

LEMMA 4. *Assume that the function* $f\colon [0,\infty)^m \to [0,\infty)$ *is continuous and satisfies the following conditions*:

(a) *The equation* $f(x,\ldots,x) = x$ *has a finite number of solutions* $0 < \bar{x}_1 < \cdots < \bar{x}_k < \infty$;

(b) *For every* $j = 1,\ldots,k$, $f(\bar{x}_j,\ldots,\bar{x}_j, x) \ne \bar{x}_j$ *if* $x \ne \bar{x}_j$;

(c) *For* $i = 1,\ldots,m$, *the partial derivatives* $\partial f/\partial x_i$ *exist continuously at* $(\bar{x}_j,\ldots,\bar{x}_j)$, *and every root of the characteristic polynomial* $\lambda^m - \sum_{i=1}^{m} \partial f/\partial x_i(\bar{x}_j,\ldots,\bar{x}_j)\lambda^{m-i}$ *has modulus greater than* 1 *for each* $j = 1,\ldots,k$.

Then all bounded solutions of (1) *except the trivial solutions* $\bar{x}_j$, $j = 1,\ldots,k$ *oscillate persistently*.

The next corollary gives a sufficient condition for the nontrivial applicability of Lemma 4.

COROLLARY 10. *In addition to the assumptions of Lemma* 4, *suppose that* f *satisfies one of the sublinearity conditions in Theorem* 1. *Then all nontrivial solutions of* (1) *eventually oscillate persistently within a fixed absorbing interval*.

In the second order case, the linearized instability condition (c) in Lemma 4 is somewhat easier to characterize. Once again, using a result from [9], we have the following.

COROLLARY 11. *Assume that* $f: [0, \infty)^2 \to [0, \infty)$ *is continuous and satisfies the following conditions*:

(a) *The equation* $f(x, x) = x$ *has a finite number of solutions* $0 < \bar{x}_1 < \cdots < \bar{x}_k < \infty$;

(b) *For every* $j = 1, \ldots, k$, $f(\bar{x}_j, x) \neq \bar{x}_j$ *if* $x \neq \bar{x}_j$;

(c) *Both* $\partial f / \partial x$ *and* $\partial f / \partial y$ *exist continuously at* $(\bar{x}_j, \bar{x}_j)$ *for all* $j = 1, \ldots, k$ *with*

$$\left| \frac{\partial f}{\partial y}(\bar{x}_j, \bar{x}_j) \right| > 1, \qquad \left| \frac{\partial f}{\partial y}(\bar{x}_j, \bar{x}_j) - 1 \right| > \left| \frac{\partial f}{\partial x}(\bar{x}_j, \bar{x}_j) \right|.$$

(d) *Either there are constants* $a, b, c \geq 0$ *with* $a + b < 1$ *such that* $f(x,y) \leq ax + by + c$ *for all* $x, y \geq 0$, *or*

$$\limsup_{x+y \to \infty} \frac{f(x, y)}{x + y} < \frac{1}{2}.$$

Then the second order nonlinear equation:

$$x_n = f(x_{n-1}, x_{n-2}), \qquad n = 1, 2, 3, \ldots, x_{-1}, x_0 \in [0, \infty)$$

is permanent and all of its nontrivial solutions eventually oscillate persistently in a fixed absorbing interval.

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