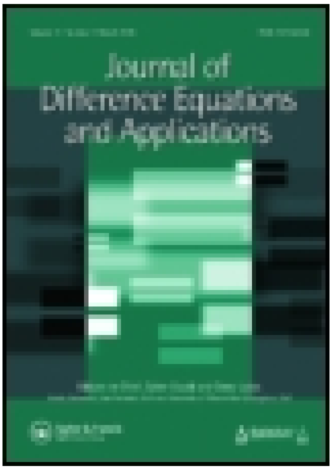

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Folding, cycles and chaos in planar systems

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Folding, cycles and chaos in planar systems

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A typical planar system of difference equations can be folded or transformed into a scalar difference equation of order two plus a passive (non-dynamic) equation. We discuss this method and its application to identify and prove the existence or nonexistence of cycles and chaos in systems of rational difference equations with variable coefficients. These include some systems that converge to autonomous systems and some that do not, e.g. systems with periodic coefficients.

Keywords: folding; cycles; chaos; planar; system; rational

1. Introduction

It is broadly known that in discrete systems periodic and chaotic behaviour may occur for maps of the interval and other one-dimensional manifolds. Planar difference systems, which generalize interval maps to two dimensions are also known to have this feature but they are not as well understood. It is by no means simple to prove whether a given planar map has cycles or exhibits chaos. Certain global results, e.g. the Sharkovski ordering of cycles, are not true for planar maps in general (e.g. the occurrence of a three-cycle does not imply the existence of any other cycle). There are comparatively few methods (e.g. Marotto's snap-back repeller criterion in [9]) that are applicable widely to the study of cycles and chaos in planar systems.

In this article, we use the new method of folding to explore planar systems and their orbits. This method has been in use (though not by this name) in different contexts in the literature. Folding linear systems in both continuous and discrete time is seen in control theory; the 'controllability canonical form' is precisely the folding of a controllability matrix into a linear higher order equation, whether in continuous or discrete time; see e.g. [5,7]. In an entirely different line of research, in [4] a variety of nonlinear differential systems displaying chaotic behaviour are studied and classified by converting them to ordinary differential equations of order 3 that define jerk (or jolt) functions, i.e. time rates of change of acceleration.

These ideas in control theory and in chaotic differential systems are special instances of the same concept, namely, folding systems to equations. In [12], these and similar notions are unified by means of a new algorithmic process for folding difference or differential systems to scalar equations.

In the case of planar systems, folding yields a second-order scalar difference equation whose analysis provides useful information about the orbits of the original system in cases where standard methods are unavailing. We establish the existence or nonexistence of cycles and chaos for various rational planar systems. Furthermore, since in principle

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folding applies to non-autonomous systems in the same way that it does to the autonomous ones, time-dependent parameters are considered in this study. But the systems that we study here exhibit cycles and chaos even with constant parameters.

2. Folding difference systems

The material in this section comes from [12]. A (recursive, or explicit) system of two first-order difference equations is typically defined as

$$\begin{cases} x_{n+1} = f(n, x_n, y_n) \\ y_{n+1} = g(n, x_n, y_n) \end{cases} \quad n = 0, 1, 2, \ldots, \tag{1}$$

where $f, g : \mathbb{N}_0 \times D \to S$ are given functions, $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ is the set of non-negative integers, S a nonempty set and $D \subset S \times S$. If S is a subset of the set $\mathbb{R}$ of the real numbers with the usual topology then (1) is a *planar system*.

An initial point $(x_0, y_0) \in D$ generates a (forward) orbit or solution $\{(x_n, y_n)\}$ of (1) in the state-space $S \times S$ through the iteration of the function

$$(n, x_n, y_n) \to (n + 1, f(n, x_n, y_n), g(n, x_n, y_n)) : \mathbb{N}_0 \times D \to \mathbb{N}_0 \times S \times S,$$

for as long as the points (x_n, y_n) remain in D. If (1) is *autonomous*, i.e. the functions f, g do not depend on the index n then $(x_n, y_n) = F^n(x_0, y_0)$ for every n where F^n denotes the composition of the map $F(u, v) = (f(u, v), g(u, v))$ of $S \times S$ with itself n times.

A second-order, scalar difference equation in S is defined as

$$s_{n+2} = \phi(n, s_n, s_{n+1}), \quad n = 0, 1, 2, \ldots, \tag{2}$$

where $\phi : \mathbb{N}_0 \times D' \to S$ is a given function and $D' \subset S \times S$. A pair of initial values $s_0, s_1 \in S$ generates a (forward) solution $\{s_n\}$ of (2) in S if $(s_0, s_1) \in D'$. If $\phi(n, u, v) = \phi(u, v)$ is independent of n then (2) is autonomous.

An equation of type (2) may be 'unfolded' to a system of type (1) in a standard way; e.g.

$$\begin{cases} s_{n+1} = t_n \\ t_{n+1} = \phi(n, s_n, t_n) \end{cases} \tag{3}$$

In (3), the temporal delay in (2) is converted to an additional variable in the state space. All solutions of (2) are reproduced from the solutions of (3) in the form $(s_n, s_{n+1}) = (s_n, t_n)$ so in this sense, higher order equations may be considered to be special types of systems. In general, (2) may be unfolded in different ways into systems of two equations and (3) is not unique.

Definition 1. *Let S be a nonempty set and consider a function $f : \mathbb{N}_0 \times D \to S$ where $D \subset S \times S$. Then f is* semi-invertible *or* partially invertible *if there are sets $M \subset D$, $M' \subset S \times S$ and a function $h : \mathbb{N}_0 \times M' \to S$ such that for all $(u, v) \in M$ if $w = f(n, u, v)$ then $(u, w) \in M'$ and $v = h(n, u, w)$ for all $n \in \mathbb{N}_0$.*

The function h above may be called a semi-inversion, or partial inversion of f. If f is independent of n then n is dropped from the above notation.

Semi-inversion refers more accurately to the *solvability* of the equation $w - f(n, u, v) = 0$ for v which recalls the implicit function theorem (see [12]). However, a substantial class of semi-invertible functions is supplied (globally) by the following idea.

DEFINITION 2. *(Separability) Let* $(G, *)$ *be a nontrivial group and let* $f : \mathbb{N}_0 \times G \times G \to G$. *If there are functions* $f_1, f_2 : \mathbb{N}_0 \times G \to G$ *such that*

$$f(n, u, v) = f_1(n, u)*f_2(n, v)$$

for all $u, v \in G$ *and every* $n \geq 1$ *then we say that* f *is* separable *on* G *and write* $f = f_1*f_2$ *for short.*

Every affine function $f(n, u, v) = a_n u + b_n v + c_n$ with real parameters a_n, b_n, c_n is separable on $\mathbb{R}$ relative to ordinary addition for all n with, e.g. $f_1(n, v) = a_n u$ and $f_2(n, v) = b_n v + c_n$. Similarly, $f(n, u, v) = a_n u/v$ is separable on $\mathbb{R}\backslash\{0\}$ relative to ordinary multiplication.

Now, suppose that $f_2(n, \cdot)$ is a bijection for every n and $f_2^{-1}(n, \cdot)$ is its inverse for each n; i.e. $f_2(n, f_2^{-1}(n, v)) = v$ and $f_2^{-1}(n, f_2(n, v)) = v$ for all v. A separable function f is semi-invertible if the component function $f_2(n, \cdot)$ is a bijection for each fixed n, since for every $u, v, w \in G$

$$w = f_1(n, u)*f_2(n, v) \Rightarrow v = f_2^{-1}(n, [f_1(n, u)]^{-1}*w),$$

where map inversion and group inversion, both denoted by -1, are distinguishable from the context. In this case, an explicit expression for the semi-inversion h exists globally as

$$h(n, u, w) = f_2^{-1}(n, [f_1(n, u)]^{-1}*w), \tag{4}$$

with $M = M' = G \times G$. We summarize this observation as follows.

PROPOSITION 3. *Let* $(G, *)$ *be a nontrivial group and* $f = f_1*f_2$ *be separable. If* $f_2(n, \cdot)$ *is a bijection for each* n *then* f *is semi-invertible on* $G \times G$ *with a semi-inversion uniquely defined by (4).*

If $a_n \neq 0$ (or $b_n \neq 0$) for all n then the separable function $f(n, u, v) = a_n u + b_n v + c_n$ is semi-invertible as it can readily be solved for u (or v). If a_n, b_n are both zero for infinitely n then f is separable but not semi-invertible for either u or v.

Now, suppose that $\{(x_n, y_n)\}$ is an orbit of (1) in D. If one of the component functions in (1), say, f is semi-invertible then by Definition 1 there is a set $M \subset D$, a set $M' \subset S \times S$ and a function $h : \mathbb{N}_0 \times M' \to S$ such that if $(x_n, y_n) \in M$ then $(x_n, x_{n+1}) = (x_n, f(n, x_n, y_n)) \in M'$ and $y_n = h(n, x_n, x_{n+1})$. Therefore,

$$\begin{aligned} x_{n+2} &= f(n+1, x_{n+1}, y_{n+1}) = f(n+1, x_{n+1}, g(n, x_n, y_n)) \\ &= f(n+1, x_{n+1}, g(n, x_n, h(n, x_n, x_{n+1}))), \end{aligned} \tag{5}$$

and the function

$$\phi(n, u, w) = f(n+1, w, g(n, u, h(n, u, w))) \tag{6}$$

is defined on $\mathbb{N}_0 \times M'$. If $\{s_n\}$ is the solution of (2) with initial values $s_0 = x_0$ and $s_1 = x_1 = f(0, x_0, y_0)$ and ϕ defined by (6) then

$$s_2 = f(1, s_1, g(0, s_0, h(0, s_0, s_1))) = f(1, x_1, g(0, x_0, h(0, x_0, x_1))) = f(1, x_1, g(0, x_0, y_0)) = x_2.$$

By induction, $s_n = x_n$ and thus $h(n, s_n, s_{n+1}) = h(n, x_n, x_{n+1}) = y_n$. It follows that

$$(x_n, y_n) = (s_n, h(n, s_n, s_{n+1})), \tag{7}$$

i.e. the solution $\{(x_n, y_n)\}$ of (1) can be obtained from a solution $\{s_n\}$ of (2) via (7). Thus the following is true.

THEOREM 4. *Suppose that f in (1) is semi-invertible with M, M' and h as in Definition 1. Then each orbit of (1) in M may be derived from a solution of (2) via (7) with ϕ given by (6).*

The following gives a name to the pair of equations that generate the solutions of (1) in the above theorem.

DEFINITION 5. *(Folding) The pair of equations*

$$s_{n+2} = \phi(n, s_n, s_{n+1}) \quad (core), \tag{8}$$

$$y_n = h(n, x_n, x_{n+1}) \quad (passive), \tag{9}$$

where ϕ is defined by (6) is a folding of the system (1). The initial values of the core equation are determined from the initial point (x_0, y_0) as $s_0 = x_0,\ s_1 = f(0, x_0, y_0)$.

We call Equation (9) *passive* because it simply evaluates the function h on a solution of the core Equation (8) – no dynamics or iterations are involved. Also observe that (1) may be considered an unfolding of the second-order Equation (8) that is generally not equivalent to the standard unfolding (3) of that equation.

If one of the component functions in the system is separable then a global result is readily obtained from Theorem 4 using (4).

COROLLARY 6. *Let $(G, *)$ be a nontrivial group and $f = f_1 * f_2$ be separable on $G \times G$. If $f_2(n, \cdot)$ is a bijection for every n then (1) folds to*

$$s_{n+2} = f_1\left(n+1, s_{n+1}, g\left(n, s_n, f_2^{-1}(n, [f_1(n, s_n)]^{-1} * s_{n+1})\right)\right), \tag{10}$$

$$y_n = f_2^{-1}\left(n, [f_1(n, s_n)]^{-1} * s_{n+1}\right). \tag{11}$$

Each orbit $\{(x_n, y_n)\}$ of (1) in $G \times G$ is obtained from a solution $\{s_n\}$ of (10) with the initial values $s_0 = x_0,\ s_1 = f_1(0, x_0) * f_2(0, y_0)$.

The next result is a special case of Corollary 6.

COROLLARY 7. *Let a_n, b_n, c_n be sequences in a ring R with identity and let $g : \mathbb{N}_0 \times R \times R \to R$. If b_n is a unit in R for all n then the semi-linear system*

$$\begin{cases} x_{n+1} = a_n x_n + b_n y_n + c_n \\ y_{n+1} = g(n, x_n, y_n), \end{cases} \tag{12}$$

folds to

$$\begin{aligned} s_{n+2} &= c_{n+1} + a_{n+1} s_{n+1} + b_{n+1} g\left(n, s_n, b_n^{-1}(s_{n+1} - a_n s_n - c_n)\right), \\ y_n &= b_n^{-1}(s_{n+1} - a_n s_n - c_n). \end{aligned} \tag{13}$$

Each orbit $\{(x_n, y_n)\}$ of (12) in R is obtained from a solution $\{s_n\}$ of (13) with the initial values $s_0 = x_0, s_1 = a_0 x_0 + b_0 y_0 + c_0$.

A natural question after folding a system is whether the qualitative properties of the solutions of the core Equation (8) are shared by the orbits of (1). The answer clearly depends on the passive equation so that despite its non-dynamic nature, (9) plays a nontrivial role in the folding. The next result illustrates this feature and is used in the next section.

LEMMA 8. *Assume that the semi-inversion h in (9) has period $p \geq 1$, i.e. p is the least positive integer such that $h(n+p,u,w) = h(n,u,w)$ for all $(n,u,w) \in \mathbb{N}_0 \times M'$. Let $\{s_n\}$ be a solution of (8) with initial values $s_0 = x_0$, $s_1 = f(0,x_0,y_0)$.*

(a) *If $\{s_n\}$ is periodic with period $q \geq 1$ then the corresponding orbit $\{(x_n,y_n)\}$ of (1) is periodic with period equal to the least common multiple* $\mathrm{lcm}(p,q)$.
(b) *If $\{s_n\}$ is non-periodic then $\{(x_n,y_n)\}$ is non-periodic.*

Proof.

(a) Recall that $x_n = s_n$ so that the sequence $\{x_n\}$ of the x-components of $\{(x_n,y_n)\}$ has period q. Also by (9)

$$y_{n+\mathrm{lcm}(p,q)} = h(n+\mathrm{lcm}(p,q), x_{n+\mathrm{lcm}(p,q)}, x_n + 1 + \mathrm{lcm}(p,q)) = h(n,x_n,x_{n+1}) = y_n,$$

since both p and q divide $\mathrm{lcm}(p,q)$. Therefore, the sequence $\{y_n\}$ of the y-components of $\{(x_n,y_n)\}$ has period $\mathrm{lcm}(p,q)$ and it follows that $\{(x_n,y_n)\}$ has period $\mathrm{lcm}(p,q)$.
(b) If $\{(x_n,y_n)\}$ is periodic then so is $\{x_n\}$, which implies that $\{s_n\}$ is periodic.

□

3. Cycles and chaos in a rational system

Various definitions of chaos for non-autonomous systems exist in the literature. Possibly the most familiar form of deterministic chaos, in the sense of Li and Yorke, is defined generally as follows.

DEFINITION 9. *(Li–Yorke chaos) Let $F_n : (X,d) \to (X,d)$ be functions on a metric space for all $n \geq 0$ and define $F_0^n = F_n \circ F_{n-1} \circ \cdots \circ F_0$ i.e. the composition of maps F_0 through F_n. The non-autonomous system (X,F_n) is chaotic if there is an uncountable set $S \subset X$ (the scrambled set) such that for every pair of points $x,y \in S$,*

$$\limsup_{n\to\infty} d\big(F_0^n(x),F_0^n(y)\big) > 0, \qquad \liminf_{n\to\infty} d\big(F_0^n(x),F_0^n(y)\big) = 0.$$

For planar maps, $F_n(u,v) = (f(n,u,v), g(n,u,v))$ on the Euclidean space $\mathbb{R}^2 = \mathbb{R}\times\mathbb{R}$. Despite the similarity of the above definition to the familiar one for interval maps (autonomous one-dimensional systems) proving that a particular non-autonomous system is chaotic in the sense of Definition 9 is a non-trivial task. For a continuous interval map the existence of a 3-cycle is sufficient for the occurrence of Li–Yorke chaos [8] and for a continuously differentiable map of R^N a sufficient condition is the existence of a snap-back repeller [9]. To take advantage of such relatively practical results, we may consider non-autonomous systems that are tied in some way to an autonomous one.

One natural case that is frequently studied in the literature concerns nonautonomous systems where the sequence $\{F_n\}$ converges uniformly to a function F on X so that (X, F) is an autonomous system; see e.g. [1,3] for studies of pertinent issues, including whether the occurrence of chaos in the autonomous system implies, or is implied by the same for the non-autonomous one.

We consider a different approach where a non-autonomous system is tied to an autonomous one through folding. In this section, we study a rational system that folds to an autonomous, first-order difference equation for its core. In this case, the non-autonomous system need not converge to an autonomous one; e.g. the system may have periodic coefficients. The dynamic aspects of the core equation are not affected by the time-dependent parameters which influence the orbits of (1) through the passive equation.

Consider the rational system

$$x_{n+1} = \frac{\alpha_n x_n + \beta_n y_n}{A_n x_n + y_n}, \tag{14a}$$

$$y_{n+1} = \frac{\alpha'_n x_n + \beta'_n y_n}{x_n + B_n y_n}, \tag{14b}$$

where all coefficients are sequences of real numbers. The autonomous version of the above system, i.e.

$$x_{n+1} = \frac{\alpha x_n + \beta y_n}{A x_n + y_n}, \tag{15a}$$

$$y_{n+1} = \frac{\alpha' x_n + \beta' y_n}{x_n + B y_n}, \tag{15b}$$

has been classified as a type (36,36) system in [2] when all coefficients are non-zero (separate number pairs are assigned to special cases where one or more of the coefficients are zeros). System (15) is semi-conjugate to a first-order rational equation via the substitution of $r_n = x_n/y_n$ (or the reciprocal of this ratio); see [11] for a study of semi-conjugate systems. We note that (15) is also a homogeneous system – a generalization of the aforementioned type of semi-conjugacy exists for such systems; see [10].

A comprehensive study of (15) appears in [6] for non-negative coefficients where the positive quadrant of the plane is invariant under the action of the underlying planar map. By analysing the one-dimensional semi-conjugate map, the authors show that exactly one of the following possibilities occurs: (i) every non-negative solution of (15) converges to a fixed point, or (ii) there is a unique positive 2-cycle and every non-negative solution of (15) either converges to this 2-cycle or to a fixed point of the system, or (iii) there exist unbounded solutions.

When all parameters in a rational system are non-negative, the positive quadrant $[0, \infty)^2$ is invariant and the underlying mapping of the system is continuous. In the absence of singularities, linear-fractional equations such as those in (14) tend to behave mildly and not exhibit the type of complex behaviour that is often associated with rapid rates of change. So questions remain about the nature of the orbits of (15) for a wider range of parameters, including negative coefficients. Does the system have cycles of period greater than two? Can it exhibit complex, aperiodic behaviour?

With negative parameters the occurrence of singularities (discontinuity) raises significant existence and boundedness issues for orbits. We use folding to identify special cases where singularities are avoided and some results are obtained about (14) and similar systems.

To fold (14) we first solve (14a) for y_n to find

$$y_n = \frac{x_n(\alpha_n - A_n x_{n+1})}{x_{n+1} - \beta_n} = h(n, x_n, x_{n+1}). \tag{16}$$

Next, using (5) and (14b) we obtain the following *first-order* core equation:

$$x_{n+2} = \frac{\alpha_{n+1}(A_nB_n - 1)x_{n+1}^2 + [\alpha_{n+1}(\beta_n - \alpha_nB_n) + \beta_{n+1}(A_n\beta_n' - \alpha_n')]x_{n+1} + \beta_{n+1}(\alpha_n'\beta_n - \alpha_n\beta_n')}{A_{n+1}(A_nB_n - 1)x_{n+1}^2 + [A_{n+1}(\beta_n - \alpha_nB_n) + (A_n\beta_n' - \alpha_n')]x_{n+1} + \alpha_n'\beta_n - \alpha_n\beta_n'}. \tag{17}$$

The first-order nature of this folding and the semi-conjugacy of planar mapping of the autonomous system (15) to a one-dimensional map are evidently related. However, folding does not require knowledge of a semi-conjugate relation or even of whether such a relation exists.

Equation (17) does not have complex solutions for all choices of parameters. For instance, if $A_n = \beta_n = \beta_n' = 0$ for all n then (17) reduces to the affine equation

$$x_{n+2} = \frac{\alpha_{n+1}}{\alpha_n'} x_{n+1} + \frac{\alpha_{n+1}\alpha_nB_n}{\alpha_n'}, \tag{18}$$

which does not exhibit complex behaviour with constant or even periodic parameters.

To assure the existence of cycles and the occurrence of chaos even in the autonomous case, we consider a different special case where

$$A_n = \alpha_n' = \beta_n = 0, \quad \alpha_n, \beta_n' \neq 0 \text{ for all } n \geq 0. \tag{19}$$

These conditions are not necessary for the occurrence of cycles or chaos but we show that they are sufficient. If conditions (19) hold then (17) reduces to the quadratic equation

$$x_{n+2} = \frac{\alpha_{n+1}}{\alpha_n\beta_n'} x_{n+1}^2 + \frac{\alpha_{n+1}B_n}{\beta_n'} x_{n+1}. \tag{20}$$

To simplify calculations we also assume that there are constants a, b such that for all n,

$$\frac{\alpha_{n+1}}{\alpha_n\beta_n'} = a, \quad \frac{\alpha_{n+1}B_n}{\beta_n'} = b.$$

Since $\alpha_n \neq 0$ for all n, we see that $a \neq 0$. These equalities yield

$$\beta_n' = \frac{\alpha_{n+1}}{a\alpha_n}, \quad B_n = \frac{b}{a\alpha_n}, \tag{21}$$

with α_n unspecified. Under these assumptions, (14) folds to

$$x_{n+2} = ax_{n+1}^2 + bx_{n+1}, \quad y_n = \frac{\alpha_n x_n}{x_{n+1}}. \tag{22}$$

By a change of variables $r_n = x_{n+1}$ the first-order, autonomous core equation above may be written as

$$r_{n+1} = ar_n^2 + br_n, \quad r_0 = x_1 = \frac{\alpha_0 x_0}{y_0}. \tag{23}$$

If $b \neq 0$ then the quadratic equation above exhibits complex behaviour in some invariant interval for a range of parameter values. This behaviour for the x-components occurs regardless of the choice of α_n, and in particular, when $\alpha_n = \alpha$ is constant, i.e. the autonomous case.

Equation (23) is conjugate to the logistic equation $t_{n+1} = bt_n(1 - t_n)$ via the substitution $t_n = -ar_n/b$. It is well known that as b goes from 3 to 4 the solutions of the logistic equation in the invariant interval [0,1] undergo a familiar sequence of bifurcations. In particular, the logistic equation has a period 3 solution when $b > 3.83$ so (23) has periodic solutions of all possible periods due to the Sharkovski ordering and exhibits chaos in [0,1] in the sense of Li and Yorke. In fact, a lower value $b \geq 3.57$ that corresponds to the end of the period-doubling cascade may be taken to be the parameter value for the onset of chaos. These observations lead to the following result.

Theorem 10. *Consider the system (14) subject to (19), i.e. the system*

$$x_{n+1} = \frac{\alpha_n x_n}{y_n}, \tag{24a}$$

$$y_{n+1} = \frac{\beta_n' y_n}{x_n + B_n y_n}. \tag{24b}$$

Assume also that (21) holds with $0 < b < 4$.

(a) *If* (x_0, y_0) *is an initial point such that*

$$\frac{\alpha_0 x_0}{y_0} \in \left(-\frac{b}{a}, 0\right) \quad \text{if} \quad a > 0; \quad \frac{\alpha_0 x_0}{y_0} \in \left(0, -\frac{b}{a}\right) \quad \text{if} \quad a < 0, \tag{25}$$

then the following are true:

(i) *The orbit* $\{(x_n, y_n)\}$ *is well-defined with* $y_n = \alpha_n x_n / x_{n+1}$ *and* $x_n \in (0, -b/a)$ *if* $a < 0$, $x_n \in (-b/a, 0)$ *if* $a > 0$ *for all* $n \geq 0$. *Furthermore, the orbit is bounded if* $\{\alpha_n\}$ *is bounded.*

(ii) *If* $\lim_{n\to\infty} \alpha_n = \alpha \neq 0$ *and the solution* $\{r_n\}$ *of (23) converges to a q-cycle then the orbit* $\{(x_n, y_n)\}$ *converges to a q-cycle.*

(iii) *If* $\{\alpha_n\}$ *converges to zero then the orbit* $\{(x_n, y_n)\}$ *converges to a limit set that is contained in the x-axis. If the solution* $\{r_n\}$ *of (23) converges to a cycle or is chaotic then the orbit has the same behaviour but the limit set itself does not contain a solution of (10).*

(iv) *If* $\{\alpha_n\}$ *converges to a p-cycle and the solution* $\{r_n\}$ *of (23) converges to a q-cycle then the orbit* $\{(x_n, y_n)\}$ *converges to a cycle with period* $\mathrm{lcm}(p, q)$.

(v) *If* $\{\alpha_n\}$ *is bounded and the solution* $\{r_n\}$ *of (23) is chaotic (e.g. if* $3.57 < b < 4$*) then the orbit* $\{(x_n, y_n)\}$ *is chaotic.*

(b) *If* (x_0, y_0) *is such that* $\alpha_0 x_0 / y_0$ *does not satisfy (25) and* $\alpha_0 x_0 / y_0 \neq 0, \pm b/a$ *then the orbit* $\{(x_n, y_n)\}$ *is well-defined and unbounded.*

Proof.

(a) We prove the case $a < 0$ here and leave out the analogous arguments for the case $a > 0$.

(i) Let $\alpha_0 x_0/y_0 = r_0 \in (0, -b/a)$. The critical point of $\mu(r) = ar^2 + br$ at $r = -b/2a$ yields the maximum value $\mu_{\max} = -b^2/4a$. It follows that

$$0 < -\frac{b^3}{4a}\left(1 - \frac{b}{4}\right) = \mu(\mu_{\max}) \leq r_n \leq \mu_{\max} < -\frac{b}{a},$$

for all n sufficiently large. In particular, $x_n = r_{n-1}$ does not approach 0 so $y_n = \alpha_n x_n/x_{n+1}$ is well defined. Furthermore, since

$$|y_n| = \frac{|\alpha_n| x_n}{x_{n+1}} = \frac{r_{n-1}}{r_n}|\alpha_n| \leq \frac{\mu_{\max}}{\mu(\mu_{\max})}|\alpha_n| = \frac{16}{b^2(4-b)}|\alpha_n|,$$

it follows that the orbit $\{(x_n, y_n)\}$ is bounded if $\{\alpha_n\}$ is.

(ii) Since $x_n = r_{n-1}$ for $n \geq 1$ if $\{r_n\}$ converges to a q-cycle in $(0, -b/a)$ then $\{x_n\}$ converges to the same q-cycle (with a phase shift), say, $\lim_{n\to\infty}|x_n - \xi_n| = 0$, where $\{\xi_n\}$ is a q-cycle in the interval $[\mu(\mu_{\max}), \mu_{\max}]$. Then $\xi_{n+q}/\xi_{n+1+q} = \xi_n/\xi_{n+1}$ for all n so $\{\xi_n/\xi_{n+1}\}$ has period q and

$$\begin{aligned}\left|\frac{x_n}{x_{n+1}} - \frac{\xi_n}{\xi_{n+1}}\right| &\leq \frac{1}{x_{n+1}\xi_{n+1}}\left(\xi_{n+1}|x_n - \xi_n| + \xi_n|x_{n+1} - \xi_{n+1}|\right)\\ &\leq \frac{\mu_{\max}}{\mu(\mu_{\max})^2}\left(|x_n - \xi_n| + |x_{n+1} - \xi_{n+1}|\right).\end{aligned}$$

Thus $\{x_n/x_{n+1}\}$ converges to the periodic sequence $\{\xi_n/\xi_{n+1}\}$ with period q. Since

$$\begin{aligned}\left|y_n - \frac{\alpha\xi_n}{\xi_{n+1}}\right| &\leq \left|\frac{\alpha_n x_n}{x_{n+1}} - \frac{\alpha_n\xi_n}{\xi_{n+1}}\right| + \left|\frac{\alpha_n\xi_n}{\xi_{n+1}} - \frac{\alpha\xi_n}{\xi_{n+1}}\right|\\ &\leq |\alpha_n|\left|\frac{x_n}{x_{n+1}} - \frac{\xi_n}{\xi_{n+1}}\right| + |\alpha_n - \alpha|\frac{\mu_{\max}}{\mu(\mu_{\max})}.\end{aligned}$$

it follows that $\{y_n\}$ converges to the sequence $\{\alpha\xi_n/\xi_{n+1}\}$ which has period q. Hence, the orbit $\{(x_n, y_n)\}$ converges to a sequence with period q.

(iii) By (i) above, $\mu(\mu_{\max}) \leq x_n/x_{n+1} \leq \mu_{\max}$ so $\lim_{n\to\infty} y_n = 0$ and the limit set of $\{(x_n, y_n)\}$ is contained in the x-axis. If $\{r_n\}$ converges to a cycle or is chaotic then so is $\{x_n\}$ and the same behaviour is exhibited by $\{(x_n, y_n)\}$ as it approaches the x-axis. The limit set in the x-axis may be finite or infinite depending on whether the limit of $\{x_n\}$ is periodic or not. However, the limit set itself cannot be a solution of the system where $y_n \neq 0$ must hold for all n.

(iv) Suppose that $\{r_n\}$ converges to a q-cycle. Then $\{x_n\}$ converges to a q-cycle $\{\xi_n\}$ in the interval $[\mu(\mu_{\max}), \mu_{\max}]$. As in (ii), $\{x_n/x_{n+1}\}$ converges to the periodic sequence $\{\xi_n/\xi_{n+1}\}$ with period q. If $\{\alpha_n\}$ converges to a sequence

$\{\alpha_n^*\}$ of period p then by Lemma 8 $\{\alpha_n^*\xi_n/\xi_{n+1}\}$ has period $\mathrm{lcm}(p,q)$ and

$$\begin{aligned}\left|y_n - \frac{\alpha_n^*\xi_n}{\xi_{n+1}}\right| &\leq \left|\frac{\alpha_n x_n}{x_{n+1}} - \frac{\alpha_n\xi_n}{\xi_{n+1}}\right| + \left|\frac{\alpha_n\xi_n}{\xi_{n+1}} - \frac{\alpha_n^*\xi_n}{\xi_{n+1}}\right| \\ &\leq |\alpha_n|\left|\frac{x_n}{x_{n+1}} - \frac{\xi_n}{\xi_{n+1}}\right| + |\alpha_n - \alpha_n^*|\frac{\mu_{\max}}{\mu(\mu_{\max})}.\end{aligned}$$

Therefore, $\{y_n\}$ converges to the sequence $\{\alpha_n^*\xi_n/\xi_{n+1}\}$ with period $\mathrm{lcm}(p,q)$. Hence, the orbit $\{(x_n,y_n)\}$ converges to a sequence with period $\mathrm{lcm}(p,q)$.

(v) If $\{r_n\}$ is chaotic then so is $\{x_n\}$. If $\{\alpha_n\}$ is bounded then $\{y_n\}$ is also bounded since $x_n \in [\mu(\mu_{\max}), \mu_{\max}]$ for all $n \geq 0$. Therefore, regardless of the nature of the behaviour of $\{y_n\}$, the orbit $\{(x_n,y_n)\}$ is chaotic in the sense of Definition 9.

(b) If (x_0,y_0) is such that α_0x_0/y_0 does not satisfy (25) and $\alpha_0x_0/y_0 \neq 0, \pm b/a$ then the solution r_n of (23) is unbounded. Thus the sequence $\{x_n\}$ is also unbounded and it follows that the orbit $\{(x_n,y_n)\}$ is unbounded, regardless of the nature of $\{y_n\}$.

□

An example of a system to which the preceding result applies is the following

$$x_{n+1} = \frac{(-1)^n x_n}{y_n}, \quad y_{n+1} = \frac{-y_n}{ax_n + b(-1)^n y_n},$$

where a, b are real numbers such that $a \neq 0$. Note that this system does not converge to an autonomous system though it is periodic. A more comprehensive study of 14 in the future through folding and (17) may reveal additional interesting possibilities.

Remark 11. Certain exceptional solutions of (24) cannot be derived from the folding. In particular, if $x_0 = 0$ and $y_0 \neq 0$ then $x_n = 0$ for all n so that $y_n = \beta_n'/B_n = \alpha_{n+1}/b$. Thus the sequence $\{(0, \alpha_{n+1}/b)\}$ is an orbit of (24) that cannot be obtained from the ratio $\alpha_n x_n/x_{n+1}$ in the passive equation. This orbit is unstable if $b > 1$, a parameter range for which 0 is unstable in (23) but if $0 < b \leq 1$ then it attracts all orbits of the system with α_0x_0/y_0 as given in (25).

4. An inverse problem and more rational systems

Folding a given nonlinear system into a higher order equation does not always simplify the study of solutions. From a practical point of view, a significant gain in terms of simplifying the analysis of solutions is desirable. This was the case in the previous section where the folding had a one-dimensional structure. In this section, we determine and study classes of difference systems that fold to equations of order 1 or 2 with known properties. We start with one of the two equations of the system, say, the one given by f along with a known function ϕ that defines a second-order equation with desired properties. Then a function g is determined with the property that the system with components f and g folds to an equation of order 2 defined by ϕ.

This process is indeed an inverse of folding in the sense that the resulting system with f and g is a (non-standard) *unfolding* of the equation of order 2 that is defined by the function ϕ. In the autonomous case, if $f(u,v) = v$ then $g = \phi$ and we obtain a standard unfolding.

Using a rational function f the above unfolding process leads, in particular, to a rediscovery of the rational system discussed in the previous section as a non-standard unfolding of the first-order logistic equation. By unfolding other first or second order difference equations in this way, we discover other rational systems that are not homogeneous but which can still be analysed using the same method.

Suppose that a function f satisfies Definition 1. By (6) the following

$$f(n+1, w, g(n, u, h(n, u, w))) = \phi(n, u, w)$$

is a function of n, u, w. Since f is semi-invertible, once again from Definition 1 we obtain

$$g(n, u, h(n, u, w)) = h(n+1, w, \phi(n, u, w)). \tag{26}$$

Now, suppose that $\phi(n, u, w)$ is prescribed on a set $\mathbb{N}_0 \times M'$ where $M' \subset S \times S$ and we seek g that satisfies (26). Assume that a subset M of D exists with the property that $f(\mathbb{N}_0 \times M) \times \phi(\mathbb{N}_0 \times M') \subset M'$ For $(n, u, v) \in \mathbb{N}_0 \times M$ define

$$g(n, u, v) = h(n+1, f(n, u, v), \phi(n, u, f(n, u, v))). \tag{27}$$

In particular, if $v \in h(\mathbb{N}_0 \times M^{'})$ then g above satisfies (27). These observations establish the following result.

Theorem 12. *Let f be a semi-invertible function with h given by Definition 1. Furthermore, let ϕ be a given function on $\mathbb{N}_0 \times M'$. If g is given by (27) then (1) folds to the difference equation $s_{n+2} = \phi(n, s_n, s_{n+1})$ plus a passive equation.*

In separable cases, explicit expressions are possible with the aid of (4). Note that semi-linear systems are included in the next result.

Corollary 13. *Let $(G, *)$ be a nontrivial group and $f(n, u, v) = f_1(n, u)*f_2(n, v)$ be separable on $G \times G$ with f_2 a bijection. If ϕ is a given function on $\mathbb{N}_0 \times G \times G$ and g is given by*

$$g(n, u, v) = f_2^{-1}\Big(n+1, \big[f_1(n+1, f_1(n, u)*f_2(n, v))\big]^{-1}*\phi(n, u, f_1(n, u)*f_2(n, v))\Big),$$

then (1) folds to the difference equation $s_{n+2} = \phi(n, s_n, s_{n+1})$ plus a passive equation.

The next result yields a class of systems that actually reduce to first-order difference equations.

Corollary 14. *Assume that f, h satisfy the hypotheses of Theorem 12 and let $\phi(n, \cdot)$ be a function of one variable for each n. If*

$$g(n, u, v) = h(n+1, f(n, u, v), \phi(n, f(n, u, v)))$$

then (1) folds to the difference equation $s_{n+2} = \phi(n, s_{n+1})$ with order 1 plus a passive equation.

We may use the above corollary to rediscover the rational system discussed in the previous section. Let $f(n, u, v) = \alpha_n u/v$ as in (14) subject to (19). With $\phi(n, u, w) =$

$aw^2 + bw$ that defines (22) we obtain, using Corollary 14

$$g(n,u,v) = \frac{\alpha_{n+1}\alpha_n u}{v[a(\alpha_n u)^2/v^2 + ba_n u/v]} = \frac{\alpha_{n+1}v}{a\alpha_n u + bv} = \frac{(\alpha_{n+1}/a\alpha_n)v}{u + (b/a\alpha_n)v}.$$

Using the substitutions (21) we obtain the homogeneous system (24). The next result yields a class of systems that fold to second-order difference equations whose orbits are determined by first-order equations.

COROLLARY 15. *Assume that f, h satisfy the hypotheses of Theorem 12 and let $\phi(n,\cdot)$ be a function of one variable for each n. If*

$$g(n,u,v) = h(n+1, f(n,u,v), \phi(n,u)), \tag{28}$$

then (1) folds to the difference equation $s_{n+2} = \phi(n, s_n)$ whose even terms and odd terms are (separately) solutions of the first-order equation

$$r_{n+1} = \phi(n, r_n), \tag{29}$$

i.e. $s_{2k} = \phi(2k-2, s_{2k-2})$ and $s_{2k+1} = \phi(2k-1, s_{2k-1})$ for all $k \geq 1$, $s_0 = x_0$ and $s_1 = f(0, x_0, y_0)$.

As an application, consider the function $f(n,u,v) = \alpha_n u/v$ again but now with $\phi(n,u,w) = au^2 + bu$. Then Corollary 15 yields

$$g(n,u,v) = \frac{\alpha_{n+1}\alpha_n u}{v(au^2 + bu)} = \frac{\alpha_{n+1}\alpha_n}{v(au+b)},$$

which results in the system

$$x_{n+1} = \frac{\alpha_n x_n}{y_n}, \tag{30a}$$

$$y_{n+1} = \frac{\alpha_n \alpha_{n+1}}{(ax_n + b)y_n}. \tag{30b}$$

The core of its folding is $s_{n+2} = as_n^2 + bs_n$ a second-order equation of type seen in Corollary 15. The even- and odd-indexed terms are generated by a conjugate of the logistic map so an analysis similar to that of the previous section may be carried out for the rational system (30).

The fact that, despite similarities, the folding of (30) has order 2 whereas that of (24) has order 1 has some interesting consequences about the corresponding systems and their orbits. To be more precise, consider the autonomous version of (24) with $\alpha_n = \alpha$ for all n i.e. the system

$$x_{n+1} = \frac{\alpha x_n}{y_n}, \tag{31a}$$

$$y_{n+1} = \frac{\beta y_n}{x_n + \gamma y_n}, \tag{31b}$$

where $\beta = 1/a$ and $\gamma = b/a\alpha$ which we compare to the autonomous version of (30)

$$x_{n+1} = \frac{\alpha x_n}{y_n}, \tag{32a}$$

$$y_{n+1} = \frac{\beta}{(x_n + \gamma)y_n} \tag{32b}$$

with $\alpha_n = \alpha$, $\beta = \alpha^2/a$ and $\gamma = b/a$.

The graph of a single typical orbit of (31) that satisfies the hypotheses in Part (v) of Theorem 10 appears in Figure 1. We note that the chaotic orbit is in the positive quadrant since $y_n > 0$ for all n in this case.

The one-dimensional manifold that contains the orbit is the curve $y = \alpha/(ax + b)$ which is calculated using (22) as follows:

$$y_n = \frac{\alpha x_n}{x_{n+1}} = \frac{\alpha x_n}{ax_n^2 + bx_n} = \frac{\alpha}{ax_n + b}.$$

The graph of a single typical orbit of (32) is shown in Figure 2. The spread of the orbit in the plane reflects the higher order of the folding in this case.

Other facts worth mentioning with regard to (31) and (32) are that the latter is not homogeneous, and semi-conjugacy to a known map is not known for it. Furthermore, the second equation of (32) is not linear-fractional.

The next result concerns systems that fold to autonomous affine equations of order 2.

COROLLARY 16. *Assume that f, h satisfy the hypotheses of Theorem 12 and let $\phi(u, w) = au + bw + c$ be an affine function where $|a| + |b| > 0$. If*

$$g(n, u, v) = h(n + 1, f(n, u, v), au + bf(n, u, v) + c),$$

then (1) folds to the difference equation $s_{n+2} = as_n + bs_{n+1} + c$ plus a passive equation.

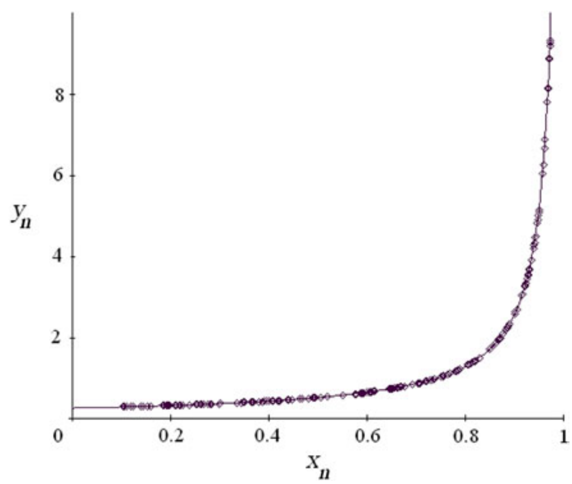

Figure 1. A typical orbit of the homogeneous system.

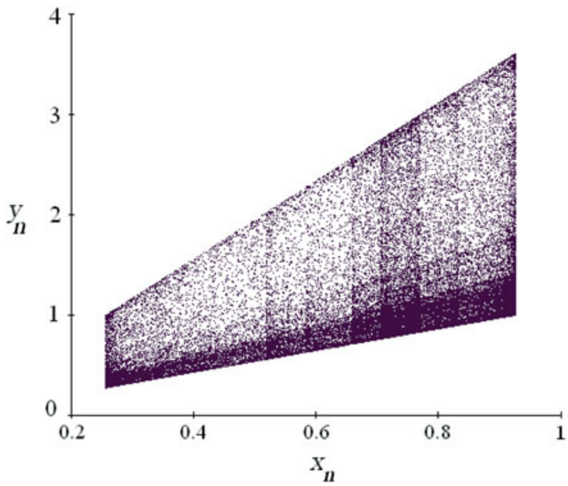

Figure 2. A typical orbit of the non-homogeneous system.

As an application of the above corollary, consider $f(n,u,v) = \alpha_n u/v$ together with $\phi(u,w) = au + bw + c$. Then by Corollary 16

$$g(n,u,v) = \frac{\alpha_{n+1}\alpha_n u}{v[au + b(\alpha_n u/v) + c]} = \frac{\alpha_{n+1}\alpha_n u}{auv + cv + b\alpha_n},$$

corresponding to the following rational system

$$x_{n+1} = \frac{\alpha_n x_n}{y_n}, \tag{33a}$$

$$y_{n+1} = \frac{\alpha_n \alpha_{n+1} x_n}{\alpha_n b x_n + (a x_n + c) y_n}. \tag{33b}$$

In the special case where $\alpha_n = \alpha$ is a constant and $a = 0$ the above system takes the form

$$x_{n+1} = \frac{\alpha x_n}{y_n}, \tag{34a}$$

$$y_{n+1} = \frac{\beta x_n}{x_n + \gamma y_n}, \tag{34b}$$

where $\beta = \alpha/b$ and $\gamma = c/\alpha b$. This homogeneous system, which folds to the affine first-order equation (18) does not generate complex behaviour, in contrast to (31), which is also homogeneous. On the other hand, if $b = 0$ and $\alpha_n = \alpha$ then (16) reduces to the autonomous system

$$x_{n+1} = \frac{\alpha x_n}{y_n}, \tag{35a}$$

$$y_{n+1} = \frac{\beta x_n}{(x_n + \gamma) y_n}, \tag{35b}$$

where $\beta = \alpha^2/a$ and $\gamma = c/a$. This system may be compared to (32) since (35) is not homogeneous, semi-conjugacy to a known map is not known for it and the second equation of the system is not linear-fractional. But in contrast to (32), system (35) folds to a second-order affine difference equation and thus, cannot generate complex behaviour. In fact, general formulas for the orbits of (34) and (35) can be easily obtained in closed form if desired.

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