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RESEARCH ARTICLE

ON EXTENDING CONTINUOUS FUNCTIONS ON DENSE SUBSEMIGROUPS

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Let S be a dense subsemigroup of a semitopological semigroup T. In this note we consider the following *extension problem*: Given a property N of functions on semigroups, find conditions on S and T such that every bounded, continuous, (complex-valued) function on S with property N extends continuously to a function on T with the same property.

Several authors have considered special cases of this problem. For example, it has been shown that if N is the property of almost periodicity, weak almost periodicity, or strong almost periodicity, then the extension problem has a positive solution if T contains an identity and is topologically right simple (e.g., if T is a group) [3].

It is interesting (and fruitful) to break up the extension problem into the following somewhat more basic problems:

(a) Find conditions on S and T such that every bounded continuous function on S with property N extends to a continuous function (not necessarily possessing property N) on T.

(b) Find conditions on S and T such that if the restriction of a continuous function f on T has property N, then f has property N.

Note that positive solutions to both (a) and (b) imply a positive solution to the extension problem. In this paper we consider only those properties N which define m-admissible algebras of functions (defined below). If $F(S)$ and

$F(T)$ denote such algebras on S and T, respectively, then (a) and (b) may be rephrased as follows:

(a′) Find conditions on S and T such that $F(S) \subset C(T)|_S$.

(b′) Find conditions on S and T such that $F(S) \cap C(T)|_S \subset F(T)|_S$.

We prove below a general theorem which addresses (b′) and we use this theorem, together with various positive solutions of (a′), to prove some special cases of the extension problem.

PRELIMINARIES

In this section we summarize some of the main ideas concerning semigroup compactifications. A detailed account of the theory may be found in [2] or [3].

A topological space and semigroup X is is said to be *right topological* if the mappings $x \mapsto xy : X \mapsto X$, $y \in X$, are continuous. If the mappings $x \mapsto yx : X \mapsto X$, $y \in X$, are also continuous, then X is said to be *semitopological*. A *compactification* of a semitopological semigroup S is a pair (ϕ, X), where X is a compact, Hausdorff, right topological semigroup and $\phi : S \mapsto X$ is a continuous homomorphism such that $\overline{\phi(S)} = X$ and the mappings $x \mapsto \phi(s)x : X \mapsto X$, $s \in S$, are continuous. When convenience dictates, we shall omit reference to the mapping ϕ and call X a compactification of S. A continuous function π from a compactification (ϕ, X) of S to a compactification (ψ, Y) of S is said to be a *homomorphism* if $\pi \circ \phi = \psi$. Note that such a mapping preserves multiplication and is onto. A homomorphism which is one–to–one is called an *isomorphism*.

A compactification of S which possesses a certain property P (such as that of being a topological group) will be called a *P–compactification* of S. A P–compactification (ϕ, X) of S is said to be *universal* if for any P–compactification (ψ, Y) of S there exists a continuous homomorphism from (ϕ, X) onto (ψ, Y).

Let (ϕ, X) be a compactification of S and let $\phi^* : C(X) \mapsto C(S)$ denote the dual mapping $f \mapsto f \circ \phi$. Then the C^*–subalgebra $F(S) := \phi^*(C(X))$ has the following properties:

(a) $F(S)$ is *translation invariant*; i.e., $R_s F(S) \cup L_s F(S) \subset F(S)$ for all $s \in S$, where $R_s f(t) := f(ts)$ and $L_s f(t) := f(st)$, $t \in S$, $f \in C(S)$.

(b) $F(S)$ is *left m–introverted*; i.e., $T_\mu F(S) \subset F(S)$ for all μ in the spectrum of $F(S)$, where T_μ is defined by $T_\mu f(s) = \mu(L_s f)$, $f \in F(S)$, $s \in S$.

Conversely, let $F(S)$ be a C^*–subalgebra of $C(S)$ with properties (a) and (b) and containing the constant functions (such an algebra is called *m–admissible*). Let X denote the spectrum of $F(S)$ with the weak* topology, and let $\phi : S \mapsto X$ be the *evaluation mapping* defined by $\phi(s)(f) = f(s)$, $f \in F(S)$, $s \in S$. Then (ϕ, X) is a compactification of S such that $F(S) = \phi^*(C(X))$, where multiplication on X is defined by $xy = x \circ T_y$. (ϕ, X) is called the *canonical $F(S)$–compactification* of S. We shall frequently denote this compactification by $S^{F(S)}$, or simply by S^F. Here are some important examples (see Chapters 3 and 4 of [3]): S^{AP} is the universal topological semigroup compactification of S, where $AP(S)$ is the algebra of almost periodic functions on S. S^{WAP} is the universal semitopological semigroup compactification of S, where $WAP(S)$ is the algebra of weakly almost periodic functions on S. S^{SAP} is the universal topological group compactification of S, where $SAP(S)$ is the algebra of strongly almost periodic functions on S. S^{LC} is the compactification of S which is universal with respect to the property that the mapping $(s, x) \mapsto \phi(s)x : S \times S^{LC} \mapsto S^{LC}$ is continuous, where $LC(S)$ is the algebra of left norm continuous functions on S. Finally, S^{LMC} is the universal (right topological) semigroup compactification of S, where $LMC(S)$ is the algebra of left multiplicatively continuous functions on S (equivalently, $LMC(S)$ is the largest m–admissible subalgebra of $C(S)$).

RESULTS

Lemma 1. *Let A be an m–admissible subalgebra of $C(S)$ such that $A \subset C(T)|_S$ and let $B = \{\overline{f} : f \in A\}$, where $\overline{f}$ denotes the continuous extension of f to T. Then there exists a topological isomorphism θ of the compactification T^B onto the compactification S^A such that*

$$\theta(y)(f) = y(\overline{f}) \qquad (y \in T^B,\ f \in A). \tag{1}$$

Proof. Since the mapping $f \mapsto \overline{f} : A \mapsto B$ is an isomorphism of C^*–algebras, the mapping θ, as defined by (1), is a homeomorphism. It remains to show that θ is a homomorphism, i.e., that

$$\theta(xy)(f) = \theta(x)\theta(y)(f) \tag{2}$$

for all $x, y \in T^B$ and all $f \in A$. By (1) and the definition of multiplication in T^B, the left side of (2) equals $(xy)(\overline{f}) = x(T_y\overline{f})$ and the right side of (2) equals $x(\overline{T_{\theta(y)}f})$. Hence it suffices to show that $(T_y\overline{f})|_S = T_{\theta(y)}f$ or, equivalently, that $y(L_s\overline{f}) = y(\overline{L_s f})$ for all $s \in S$. But this is immediate from the identity $L_s\overline{f} = \overline{L_s f}$.

Lemma 2. *Let $F(S)$ be an m–admissible subalgebra of $C(S)$ and define $B := \{g \in C(T) : g|_S \in F(S)\}$.*

(i) B is a right translation invariant C^*–subalgebra of $C(T)$ such that $L_S B \subset B$.

(ii) If $F(S)$ is right m–introverted (i.e., the mapping $s \mapsto \mu(R_s f)$ is a member of $F(S)$ for each μ in the spectrum of $F(S)$ and each $f \in F(S)$), then B is left translation invariant.

(iii) If B is left translation invariant and contained in $LMC(T)$, then B is m–admissible.

Proof. (i) B is easily seen to be a C^*–subalgebra of $C(T)$ and the identity

$$(L_s g)|_S = L_s(g|_S), \quad s \in S,\ g \in C(T)$$

shows that $L_S B \subset B$. let μ be in the spectrum of B and choose a net $\{s_\alpha\}$ in S such that for all $g \in B$

$$\mu(g) = \lim_\alpha g(s_\alpha).$$

This is possible since S is dense in T and the evaluation functionals are dense in the spectrum of B. We may assume that there exists $\nu \in F(S)^*$ such that for all $f \in F(S)$

$$\nu(f) = \lim_\alpha f(s_\alpha).$$

Then for any $g \in B$ and $s \in S$, we have

$$T_\mu g(s) = \mu(L_s g) = \lim_\alpha g(s s_\alpha) = \lim_\alpha g|_S(s s_\alpha) = \nu(L_s(g|_S)) = T_\nu(g|_S)(s),$$

i.e.,

$$(T_\mu g)|_S = T_\nu(g|_S). \tag{3}$$

Right translation invariance of B follows from (3) upon taking $\mu = \epsilon(t)$ and noting that $T_{\epsilon(t)} = R_t$, where ϵ is the evaluation mapping.

(ii) Let $g \in B$, $t \in T$ and let $\{s_\alpha\}$ be a net in S converging to t. Then $L_{s_\alpha}(g|_S) \in F(S)$, and since $F(S)$ is right m–introverted, there exists a subnet

$\{s_\beta\}$ such that the net $\{L_{s_\beta}(g|_S)\}$ converges pointwise on S to some function $f \in F(S)$. It follows that $(L_t g)|_S = f \in F(S)$, i.e., $L_t g \in B$.

(iii) If B is contained in $LMC(T)$ then $T_\mu g$ is continuous; hence (3) implies that $T_\mu B \subset B$, i.e., B is left m–introverted.

Remark. In connection with part (iii) of Lemma 2 it should be pointed out that in general B may not be contained in $LMC(T)$. For example, if S and T denote, respectively, the topological groups of rational and real numbers under the usual topology and addition, then $B = \{g \in C(T) : \ g|_S \in LMC(S)\} \not\subset LMC(T)$ [6].

In the following theorem and its corollaries we consider only those properties P which are invariant under homomorphisms of compactifications and for which universal P–compactifications exist. Examples of such properties and a general existence theorem for universal P–compactifications appear in Chapter 3 of [3].

Theorem. *Let (ψ, X) and (ϕ, Y) be universal P–compactifications of S and T, respectively, and let $F_P(S) = \psi^* C(X)$ and $F_P(T) = \phi^* C(Y)$. Then*

$$F_P(S) \cap C(T)|_S = F_P(T)|_S \tag{4}$$

if and only if the C^*–algebra

$$B := \{g \in C(T) : g|_S \in F_P(S)\} \tag{5}$$

is contained in $LMC(T)$ and is left translation invariant (in which case it is m–admissible).

Proof. If (4) holds then $B = F_P(T)$, hence B is contained in $LMC(T)$.

Conversely, assume that B is left translation invariant and contained in $LMC(T)$. By Lemma 2, B is m–admissible. Let

$$A := F_P(S) \cap C(T)|_S. \tag{6}$$

Note that A is m–admissible, since it is the image of the m–admissible algebra B under the restriction mapping $f \mapsto f|_S$. Since $A \subset F_P(S)$, S^A is a continuous homomorphic image of the compactification X of S (Theorem 3.1.9, [3]) and hence has property P. Let $\theta : T^B \mapsto S^A$ be the mapping of Lemma 1. Then T^B has property P, and because (ϕ, Y) is the universal P-compactification of T, $B \subset \phi^* C(Y) = F_P(T)$. Therefore $A \subset F_P(T)|_S$. For the reverse inclusion, note that by Lemma 1, the compactifications Y and $S^{F_P(T)|_S}$ are isomorphic, hence $S^{F_P(T)|_S}$ has property P. By the universality of X, $F_P(T)|_S \subset F_P(S)$. This completes the proof of the theorem.

Corollary 1. *If $F_P(S)$ is right m-introverted (e.g., if T is abelian) and $F_P(S) \subset LMC(T)|_S$, then*

$$F_P(S) = F_P(T)|_S.$$

Proof. The hypothesis implies that $B \subset LMC(T)$, where B is the algebra in (5). Hence the corollary follows from Lemma 2 and the theorem.

Corollary 2. *If $F_P(S) = F_P(T)|_S$ then*

$$F_Q(S) = F_Q(T)|_S \tag{7}$$

for all stronger properties Q for which $F_Q(S)$ is right m-introverted.

Proof. If Q is stronger than P, then $F_Q \subset F_P$. Hence (7) follows from Corollary 1.

Remark. Equation (7) need not hold for properties Q which are not stronger than P. For example, if S and T are the topological groups of rational and real numbers, respectively, then $WAP(S) = WAP(T)|_S$ (Corollary 4 below) but $LMC(S) \neq LMC(T)|_S$ [6].

The following corollary generalizes Proposition 4 of [1].

Corollary 3. *If $F_P(S) \subset WAP(S)$ then*

$$F_P(S) \cap C(T)|_S = F_P(T)|_S.$$

Hence if also $F_P(S) \subset C(T)|_S$ then $F_P(S) = F_P(T)|_S$.

Proof. We show that the algebra B in (5) satisfies the hypotheses of the theorem. Since $F_P(S)$ is right m-introverted (by virtue of its being a subset of $WAP(S)$), Lemma 2 implies that B is left translation invariant. It remains to show that $B \subset LMC(T)$. To this end let $f \in B$, let μ be a member of the spectrum of B, and let $\{s_\alpha\}$ be a net in S such that $g(s_\alpha) \to \mu(g)$

for all $g \in B$. Let A be as in (6). Since $f|_S \in A \subset WAP(S)$ and A is translation invariant (in fact, m–admissible), we may suppose that $\{(R_{s_\alpha} f)|_S\}$ converges weakly to some function $f_\mu \in A$. Fix $t \in T$. The map $h :\mapsto \overline{h}(t)$ is a continuous linear functional on A so $(R_{s_\alpha} f)(t) \to \overline{f_\mu}(t)$. On the other hand, $(R_{s_\alpha} f)(t) = L_t f(s_\alpha) \to \mu(L_t f) = T_\mu f(t)$. Therefore $T_\mu f = \overline{f_\mu} \in C(T)$, i.e., $f \in LMC(T)$.

Corollary 4. *Suppose that T contains a right identity and is topologically right simple. Let $F_P(S) \subset WAP(S)$. Then*

$$F_P(S) = F_P(T)|_S.$$

Proof. First note that since T is topologically right simple so is S. Hence $WAP(S) \subset LC(S)$ [4]. Also, by Theorem 3.2 of [5], $LC(S) \subset C(T)|_S$. Therefore $WAP(S) \subset C(T)|_S$, and the conclusion follows from Corollary 3.

Corollary 5. *Suppose that T is topologically left simple and topologically right simple. If $F_P(S) \subset WAP(S)$, then*

$$F_P(S) = F_P(T)|_S.$$

Proof. By Lemma 4.2 of [5] $WAP(S) \subset C(T)|_S$. The conclusion now follows from Corollary 3.

Corollary 6. *Suppose that T is topologically simple. If $F_P(S) \subset SAP(S)$, then*

$$F_P(S) = F_P(T)|_S.$$

Proof. By the proof of Theorem 4.6 of [5], $SAP(S) \subset C(T)|_S$. Now apply Corollary 3.

Corollary 7. *Suppose that T is topologically right simple and contains a right identity. Then $LC(S) = LC(T)|_S$ if and only if the C^*–algebra $\{f \in C(T) : f|_S \in LC(S)\}$ is m–admissible.*

Proof. By Theorem 3.2 of [5] $LC(S) \subset C(T)|_S$. The conclusion now follows from the above theorem.

Remark. The statement that $F_P(S) = F_P(T)|_S$ is easily seen to be equivalent to the assertion that $S^{F_P(S)}$ is isomorphic to the universal P–compactification of T. (In fact, Lemma 1 provides the necessity.) This observation allows us, for example, to restate Corollary 4 as follows:

Let T be topologically right simple and contain a right identity. Then S and T have the same universal semitopological semigroup P-compactifications.

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