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Existence of Solutions for Certain Singular Difference Equations

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Crash sets are initial sets of points from which the trajectories of singular difference equations reach inifinite discontinuities after a finite number of steps; therefore, a solution does not exist in such cases. For injective maps of the line possessing at least one fixed point, the structure of crash sets is determined. For second order equations whose vector maps of the plane are injective, a general algorithm for recursively constructing crash sets is obtained, together with results on the structure of those sets in a special case.

Keywords: Infinite discontinuity; Monotonic maps; Crash set

AMS Subject Classification: 39A10

1 INTRODUCTION

Discrete initial value problems may not have solutions when infinite discontinuities (or singularities) exist. This fact is discussed in several recent works on difference equations; see, e.g. [1–3]. Yet, there appears to be very little systematic study done on this problem; indeed, it is noted in [1] that "The problem of *existence of solutions* for difference equations is of paramount importance but so far has been systematically neglected."

The aim of this paper is to consolidate and extend various apparently unrelated results, particularly those in [1–3], in the context of monotonic map iterates in dimensions one and two. Given that solutions of delay (or higher order) difference equations can be expressed as map iterations, for monotonic (or injective) maps the natural thing to consider is the images of the singularity set under successive iterations of the inverse mapping. We are interested in the structure of the set consisting of all these inverse images. In this paper we consider first and second order equations. In the first order case, we obtain general results for monotonic maps and apply those results to

$$x_{n+1} = \frac{a}{x_n^p} + 1, \tag{1}$$

where a is a nonzero real number and p is a positive *odd rational*, i.e., $p = (2i-1)/(2j-1)$ where i, j are positive integers. The equation studied in [2] is of this type. We also study the existence of solutions for second order equations of type

$$x_{n+1} = \frac{a}{x_{n-1}^q} - \frac{1}{x_n^p}, \tag{2}$$

where p, q are odd rationals, at least one of which is positive. The equations in [1] and [3] are of this type.

2 MONOTONIC MAPS WITH POLES

The mapping on the right hand side of Eq. (1) is a member of the class of functions on the real line $\mathbb{R}$ defined by the four properties:

(A1) f is continuous on its domain $\mathbb{R} - \{0\}$;
(A2) f is injective (or one-to-one) on $\mathbb{R} - \{0\}$;
(A3) $0 \in f(\mathbb{R} - \{0\})$;
(A4) $\lim_{x \to 0} |f(x)| = \infty$.

Property (A4) makes origin a "pole type" singularity for f. Property (A2) implies that the left and right limits of f at the origin must both have infinite magnitudes, but with opposite signs. Also by (A1) and (A2), f consists of two continuous pieces on the two sides of 0, and these pieces are either both increasing or both decreasing. Hence, we may refer to f as an "increasing map" or a "decreasing map" in this sense.

Injectivity forces a unique pole, which may be assumed to be at the origin, because a map having two or more poles cannot be injective on the complement of its set of poles. Property (A3) ensures nontriviality;

for in the absence of (A3), f is just a continuous map of $\mathbb{R}-\{0\}$ into itself. If f^{-1} denotes the inverse map, then (A3) is equivalent to assuming $f^{-1}(0)$ exists and is nonzero.

The iterates of f at some specific point $x_0 \in \mathbb{R}-\{0\}$ make up the orbit $\{x_0, f(x_0), f^2(x_0), \ldots\}$ where f^n denotes the composition of f with itself n times. This orbit is the solution of the discrete initial value problem,

$$x_{n+1} = f(x_n), \quad x_0 \in \mathbb{R}-\{0\}. \tag{3}$$

Such an orbit cannot be defined if $f^n(x_0)=0$ (equivalently, $x_0 = f^{-n}(0)$) for some positive integer n; we refer to the backward trajectory $\{f^{-n}(0)\}$ where $f^{-n}=(f^{-1})^n$ as the *crash set* of f. We now list some relevant elementary properties of monotonic maps as a lemma.

LEMMA 1 (a) *The limits* $\lim_{x\to\infty} f(x)=\alpha$, $\lim_{x\to-\infty} f(x)=\beta$ *both exist and* $\alpha\beta>0$ (*hence, we may assume* $\alpha, \beta > 0$).

(b) *Suppose that f is decreasing. Then* $\beta \le \alpha$, f^{-1} *is decreasing with*

$$\lim_{y\to\alpha^+} f^{-1}(y) = \infty, \qquad \lim_{y\to\beta^-} f^{-1}(y) = -\infty.$$

Further, $f^{-1}(0)<0$ *and there are precisely two fixed points, one in the interval* $(f^{-1}(0), 0)$ *and another in* (α, ∞).

(c) *Suppose that f is increasing. Then* $\alpha \le \beta$ *and* f^{-1} *is increasing with*

$$\lim_{y\to\alpha^-} f^{-1}(y) = \infty, \qquad \lim_{y\to\beta^+} f^{-1}(y) = -\infty.$$

Further, $0<f^{-1}(0)<\alpha$ *and if S is the (possibly empty) set of fixed points of f, then* $S \subset (f^{-1}(0), \alpha)$.

THEOREM 1 (a) *Assume that f is decreasing and let $\bar{x}$ be the fixed point in* $(f^{-1}(0), 0)$. *Then the sequence* $\{f^{-n}(0)\}$ *is contained in the interval* $[f^{-1}(0), f^{-2}(0)]$, *and converges either to $\bar{x}$ or to a 2-cycle.*

(b) *Let f be increasing and assume that S is nonempty with* $\bar{x} = \inf(S) > f^{-1}(0)$. *Then the sequence* $\{f^{-n}(0)\}$ *is contained in* $[f^{-1}(0), \bar{x})$ *and converges monotonically to $\bar{x}$.*

Proof (a) Note that since f^{-1} maps $(-\infty, 0)$ into itself,

$$f^{-2}(0) = f^{-1}(f^{-1}(0)) \in (f^{-1}(0), 0).$$

It follows inductively that $f^{-n}(0) \in [f^{-1}(0), 0)$ for all positive integers n. In particular, $f^{-1}(0) < f^{-3}(0)$ and since f^{-2} is increasing,

$$f^{-5}(0) = f^{-2}(f^{-3}(0)) > f^{-3}(0).$$

By induction we conclude that

$$f^{-1}(0) < f^{-3}(0) < \cdots < f^{-2k-1}(0) < \cdots$$

Next, for each integer $k \geq 1$,

$$f^{-2k-2}(0) = f^{-1}(f^{-2k-1}(0)) < f^{-1}(f^{-2k+1}(0)) = f^{-2k}(0),$$

so that the even iterates form a decreasing sequence. Further, if $f^{-2k}(0) > \bar{x}$, then $f^{-2k-2}(0) > f^{-2}(\bar{x}) = \bar{x}$ and since $f^{-2}(0) > f^{-1}(\bar{x}) = \bar{x}$ we see that the even iterates are bounded below by $\bar{x}$. On the other hand, for each k,

$$f^{-2k-1}(0) = f^{-1}(f^{-2k}(0)) < f^{-1}(\bar{x}) = \bar{x},$$

so that the sequence of odd iterates of f^{-1} is bounded above by $\bar{x}$. If the sequence $\{f^{-2k}(0)\}$ does *not* converge to $\bar{x}$, then it must converge to some $c > \bar{x}$. Hence, given the uniqueness of $\bar{x}$ in $[f^{-1}(0), 0)$, $\{c, f^{-1}(c)\}$ is a 2-cycle in $[f^{-1}(0), 0)$ to which $\{f^{-n}(0)\}$ converges. Note that due to continuity,

$$c = \sup\{u \in [f^{-1}(0), 0): f^{-1}(u) = f(u)\}, \tag{4}$$

so that this limit cycle is in fact the largest possible 2-cycle in $[f^{-1}(0), 0)$.

(b) By the minimality of $\bar{x}$, $f^{-1}(y) > y$ for $0 < y < \bar{x}$. Therefore,

$$f^{-1}(0) < f^{-1}(f^{-1}(0)) = f^{-2}(0) < f^{-1}(f^{-2}(0)) = f^{-3}(0) < \cdots < \bar{x}$$

and $\{f^{-n}(0)\}$ must increase to $\bar{x}$.

COROLLARY 1 *Suppose that f has a fixed point. Then $\{f^{-n}(0)\}$ is bounded, infinite and has at most two limit points.*

The next corollary uses the following result from [4].

LEMMA 2 *Let f be decreasing. A fixed point $\bar{x}$ of f is asymptotically stable (respectively, unstable) if and only if there is $\delta > 0$ such that $f^{-1}(x) > f(x)$ (respectively, $f^{-1}(x) < f(x)$) for $\bar{x} - \delta < x < \bar{x}$.*

COROLLARY 2 *The sequence $\{f^{-n}(0)\}$ has two limit points if f is decreasing and its negative fixed point $\bar{x}$ is asymptotically stable. In particular, if f is continuously differentiable at $\bar{x}$ with $-1 < f'(\bar{x}) < 0$, then $\{f^{-n}(0)\}$ has two limit points.*

Proof Suppose $(\bar{x})$ is asymptotically stable. Then by Lemma 2 there is $\delta > 0$ such that $f^{-1}(x) > f(x)$ for $\bar{x} - \delta < x < \bar{x}$. On the other hand,

$$f^{-1}(f^{-1}(0)) = f^{-2}(0) < 0 = f(f^{-1}(0)), \tag{5}$$

so there is a point in $(f^{-1}(0), \bar{x})$ at which f equals f^{-1}. Hence, the number c defined by (4) exists and by Theorem 1 $\{f^{-n}(0)\}$ has the two limit points c and $f^{-1}(c)$.

3 THE MAPPING $(a/x^p) + 1$

In this section we consider rational functions of type

$$f(x) = \frac{a}{x^p} + 1, \quad a \neq 0, \; p = \frac{2i-1}{2j-1}, \; i,j \in \{1, 2, 3, \ldots\},$$

where the odd rationality of p ensures monotonicity. It is easy to see that

$$f^{-1}(y) = \left(\frac{a}{y-1}\right)^{1/p},$$

so that $f^{-1}(0) = -a^{1/p}$ and f^{-1} has a pole at $y = 1$.

THEOREM 2 (a) *The sequence $\{f^{-n}(0)\}$ of backward iterates converges to a 2-cycle $\{c, -(a/(1-c))^{-1/p}\}$ if and only if $p < 1$ and*

$$a(1-p)^{p+1} > p^p. \tag{6}$$

The number $c \in (-a^{1/p}, -(a/(a^{1/p}+1))^{1/p})$ is an extremal solution of the equation $f^{-1}(u) = f(u)$.

(b) *For* $a > 0$, *the sequence* $\{f^{-n}(0)\}$ *converges to* $\bar{x} \in (-a^{1/p}, -(a/(a^{1/p}+1))^{1/p})$ *if* $p \geq 1$, *or if* $p < 1$ *and*

$$a(1-p)^{p+1} \leq p^p. \tag{7}$$

The fixed point $\bar{x}$ *is the unique negative solution of* $f^{-1}(u) = f(u)$ *in this case.*

(c) *The sequence* $\{f^{-n}(0)\}$ *increases to* $\bar{x} \in ((-a)^{1/p}, p/(p+1)]$ *if*

$$-p^p \leq a(p+1)^{p+1} < 0. \tag{8}$$

The fixed point $\bar{x}$ *is the smaller of at most two positive solutions of* $f^{-1}(u) = f(u)$ *in this case.*

Proof (a) From (6) we see that $a > 0$. Suppose that $p < 1$ and (6) holds. To prove the existence of a 2-cycle, Corollary 2 may be used if it is shown that $f'(\bar{x}) > -1$, where $\bar{x}$ is the negative solution of

$$0 = \phi_1(u) \doteq u^{p+1} - u^p - a. \tag{9}$$

Since $f(\bar{x}) = \bar{x}$, direct calculation shows that $f'(\bar{x}) > -1$ is equivalent to the inequality

$$(p-1)\bar{x} > p, \tag{10}$$

or equivalently,

$$\bar{x} < \frac{-p}{1-p}. \tag{11}$$

If $u < 0$, then

$$\phi_1'(u) = (p+1)u^p - pu^{p-1} = u^{p-1}[(p+1)u - p] < 0.$$

Since $\phi_1(\bar{x}) = 0$, from (11) we conclude that $f'(\bar{x}) > -1$ is equivalent to

$$0 > \phi_1\left(\frac{-p}{1-p}\right) = \frac{p^p}{(1-p)^{p+1}} - a,$$

which is the same as (6).

Conversely, since 2-cycles do not exist when $a<0$ (i.e., when f is increasing), we may suppose that $a>0$ and either $p<1$ and (7) holds, or that $p \geq 1$. In either case, it suffices to show that $f(u)>f^{-1}(u)$ for $u<\bar{x}$, or equivalently,

$$\phi(u) \doteq f(u) - f^{-1}(u) = au^{-p} + 1 - a^{1/p}(u-1)^{-1/p} > 0, \quad \text{for } u < \bar{x}.$$

with

$$\phi(-a^{1/p}) = \phi(f^{-1}(0)) = -f^{-2}(0) > 0, \quad \phi(\bar{x}) = 0. \qquad (12)$$

To this end, consider

$$\phi'(u) = \frac{-ap}{u^{p+1}} + \frac{a^{1/p}}{p(u-1)^{1+1/p}}.$$

Direct calculation shows that $\phi'(u)=0$ if and only if

$$\sigma(u) \doteq u^p - bu + b = 0, \quad b \doteq a^{(p-1)/(p+1)} p^{2p/(p+1)} \qquad (13)$$

and also that σ and ϕ' have opposite signs everywhere.

Now, if $p=1$ then $b=1$, so it follows that $\sigma(u)=1$ for all $u<0$; in particular, $\phi'(u)$ is negative for $u<\bar{x}$ and by (12) ϕ has no zeros less than $\bar{x}$ in this case.

If $p \neq 1$, then $\sigma'(u)=pu^{p-1}-b$ has a unique negative solution

$$\mu \doteq -\left(\frac{b}{p}\right)^{1/(p-1)}.$$

It is easily verified that if $p<1$ then μ represents a local minimum for σ and when $p>1$ then μ is a local maximum.

Suppose first that $p<1$. Then

$$\min(\sigma) \geq 0, \qquad (14)$$

if and only if

$$\begin{aligned} 0 &\leq \sigma(\mu), \\ &= b^{-p/(1-p)}[p^{1/(1-p)} - p^{p/(1-p)}] + b, \\ &= b^{-p/(1-p)} p^{p/(1-p)}(p-1) + b. \end{aligned}$$

This can be restated as

$$b^{1/(1-p)} \geq p^{p/(1-p)}(1-p). \tag{15}$$

Using the definition of b in (13), we find that (15) is equivalent to

$$a^{(p-1)/(p+1)}p^{2p/(p+1)} \geq p^p(1-p)^{1-p},$$

which may be easily reduced to

$$a \leq \frac{p^p}{(1-p)^{p+1}} \tag{16}$$

with equality holding if and only if we have equality in (14). If strict inequality holds in (16), then σ and hence, ϕ' have no zeros less than $\bar{x}$; therefore, ϕ is strictly decreasing and by (12), ϕ has no zeros less than $\bar{x}$. If, on the other hand, equality holds in (16), then,

$$\sigma(\mu) = \min(\sigma) = 0$$

and a straightforward calculation shows that $\mu = -p/(1-p)$. Further,

$$f\left(\frac{-p}{1-p}\right) = 1 - a\left(\frac{1-p}{p}\right)^p = 1 - \frac{1}{1-p} = \frac{-p}{1-p},$$

so that

$$\bar{x} = \frac{-p}{1-p} = \mu.$$

It follows that $\bar{x}$ is the unique negative zero of ϕ', so once again ϕ has no zeros less than $\bar{x}$.

Now, we consider the last remaining case, $p > 1$. In this case, μ is the unique local maximum of σ, and $\sigma(0) = b > 0$ so that σ has a unique negative zero $x' < \mu$. Further, since $f(\bar{x}) = \bar{x} = f^{-1}(\bar{x})$, direct calculation shows

$$\phi'(\bar{x}) = \frac{-p(\bar{x}-1)}{\bar{x}} + \frac{\bar{x}}{p(\bar{x}-1)}.$$

It follows that $\phi'(\bar{x}) \geq 0$ if and only if $p \leq (p-1)\bar{x}$, which is not possible. Hence, $\phi'(\bar{x}) < 0$, i.e., $\sigma(\bar{x}) > 0$, which implies that $\bar{x} > x'$; therefore, $\phi'(u) < 0$ for $u \in (x', \bar{x})$. Since x' is the unique zero of σ, hence also of ϕ', we see that ϕ increases before x' and decreases on $(x', \bar{x})$. This fact, together with (12) imply that once again ϕ has no zeros less than $\bar{x}$, and the proof of (a) – and (b) – is completed.

To prove (c) it must be shown that (8) implies that f has at least one positive fixed point. As noted earlier, ϕ_1 is decreasing when $u < p/(p+1)$ and since

$$\phi_1\left(\frac{p}{p+1}\right) = \frac{-p^p}{(p+1)^{p+1}} - a \leq 0 \tag{17}$$

by (8), we conclude that there is one positive fixed point $\bar{x} \leq p/(p+1)$ in this case (and possibly another one greater than $p/(p+1)$). This completes the proof.

Corollary 3 *Let $p \geq 1$. Then $\{f^{-n}(0)\}$ converges (not in a finite number of steps) to a unique limit point if and only if $a \geq -p^p/(p+1)^{p+1}$, $a \neq 0$.*

Proof If $a < -p^p/(p+1)^{p+1}$, then by (17) f has no fixed points. Hence, if $\{f^{-n}(0)\}$ converges, then the limit must be α; this is not possible, since if $f^{-k}(0)$ is very close to α, then $|f^{-(k+1)}(0)|$ is very large. The converse is obvious from Theorem 2.

We close this section with a result whose proof uses the ideas in [2].

Theorem 3 *For $a < -p^p/(p+1)^{p+1}$, the crash set of (1) is finite, if and only if $f^{-1}(0)$ is a zero of one of the functions $g_n(u)$ defined recursively for $n \geq 2$ by*

$$g_n = [g_{n-1} - u g_{n-2}^{1/p}]^{1/p}, \quad g_0 \doteq 1, \; g_1 \doteq (1-u)^{1/p}. \tag{18}$$

Proof First, we show that for $n \geq 1$,

$$f^{-(n+1)}(0) = \frac{u_0[g_{n-1}(u_0)]^{1/p}}{g_n(u_0)}, \quad u_0 \doteq -a^{1/p}. \tag{19}$$

Since $f^{-1}(0) = -a^{1/p}$ and $f^{-2}(0) = -a^{1/p}/(1 + a^{1/p})^{1/p}$, (19) holds for $n = 1$. Suppose, inductively, that (19) holds for some $k \geq 1$. Then

$$f^{-1}(f^{-(k+1)}(0)) = \frac{a^{1/p}}{(u_0[g_{k-1}(u_0)]^{1/p}/g_k(u_0)) - 1)^{1/p}} = \frac{u_0[g_k(u_0)]^{1/p}}{(-u_0[g_{k-1}(u_0)]^{1/p} + g_k(u_0))^{1/p}},$$

so that by (18) the claim is true for $k+1$. Next, (18) implies that for each n, g_n and g_{n+1} have no zeros in common; for otherwise, going back inductively we reach the contradiction that g_0 has a zero. Now, by (19) $f^{-(n+2)}(0) = 0$ if and only if $g_n(f^{-1}(0)) = 0$ for some $n \geq 1$ (in which case $g_{n+1}(f^{-1}(0)) \neq 0$). This completes the proof.

Remark In the case $p = 1$ discussed in [2], each function g_n in Theorem 3 is a polynomial of degree no more than $(n+1)/2$.

4 THE SECOND ORDER EQUATION

Consider the second order difference equation

$$x_{n+1} = f(x_n, x_{n-1}), \quad n = 1, 2, 3, \ldots$$

where the initial values are specified as the real numbers x_0 and x_{-1}. We assume that the mapping f has the following properties in the sequel:

(B1) $f: \mathbb{R}^2 \to \mathbb{R}$ is continuous at each point of the Euclidean plane $\mathbb{R}^2$ not in a singularity set S of zero area;
(B2) For all $(x, y) \in S$, $|f(x, y)| = \infty$;
(B3) For each real number x, $f(x, y_1) = f(x, y_2)$ implies $y_1 = y_2$.

Let F be a standard vectorization of f, i.e., $F(x, y) = (f(x, y), x)$. Then properties (B1), (B2) have their obvious interpretations for F while (B3) implies that F is injective or monotonic.

LEMMA 3 (a) *There exists a unique function g such that for all x, z, if z is in the range of f, then*

$$f(x, g(z, x)) = z.$$

(b) $F^{-1}(u, v) = (v, g(u, v))$ *for all* u, v;

(c) *Let* $(\xi_1(t), \xi_2(t))$, $t \in R$ *be a curve in the singularity set* S. *Define a family* $\{\psi_n\}$ *of curves recursively by*

$$\psi_0 \doteq \xi_1, \quad \psi_1 \doteq \xi_2, \quad \psi_{n+1}(t) \doteq g(\psi_{n-1}(t), \psi_n(t)), \quad n = 1, 2, 3, \ldots$$

Then for each positive integer n,

$$F^{-n}(\xi_1(t), \xi_2(t)) = (\psi_n(t), \psi_{n+1}(t)). \tag{20}$$

Proof (a) For each given pair of numbers x and z, by (B3) there is a unique y such that $f(x, y) = z$. We define $g(z, x) = y$.

(b) Suppose that $F^{-1}(u, v) = (\phi_1(u, v), \phi_2(u, v))$. Then, solving the equation $F(F^{-1}(u, v)) = (u, v)$, we see that $\phi_1(u, v) = v$ and by Part (a), $\phi_2(u, v) = g(u, v)$ for all u, v.

(c) By Part (b), (20) holds for $n = 1$. If true for $n = k$, then by (b) and the definition of the ψ_n,

$$\begin{aligned} F^{-(k+1)}(\xi_1(t), \xi_2(t)) &= F^{-1}(\psi_k(t), \psi_{k+1}(t)) \\ &= (\psi_{k+1}(t), g(\psi_k(t), \psi_{k+1}(t))) \\ &= (\psi_{k+1}(t), \psi_{k+2}(t)) \end{aligned}$$

as required.

Part (c) of Lemma 3 gives an algorithm for recursively computing the curves $F^{-n}(\xi_1(t), \xi_2(t))$ for each n, provided that an explicit expression for g is found. This is the case for Eq. (2) where

$$f(x, y) = \frac{a}{y^q} - \frac{1}{x^p}.$$

Assuming p and q are odd rationals and none of a, p, q are zero, solving $z = f(x, y)$ for y gives

$$y = g(z, x) = \left(\frac{a}{z + x^{-p}}\right)^{1/q}.$$

We use the algorithm of Lemma 3 to construct graphs of $F^{-n}(\xi_1(t), \xi_2(t))$ in later sections of this paper.

For later reference, note that

$$F(x,y) = (ay^{-q} - x^{-p}, x),$$
$$F^{-1}(u,v) = \left(v, a^{1/q}(u + v^{-p})^{-1/q}\right). \tag{21}$$

To save space, we assume in the sequel that $a > 0$ and $p > 0$; the cases $a < 0$ or $p < 0$ can be handled similarly to the cases we consider below. Thus, in the sequel we always assume the following without further mention:

$$a > 0,\ p > 0,\ q \neq 0,\ \text{and}\ p, q\ \text{are odd rationals.} \tag{22}$$

This leaves two cases to consider: $q < 0$ and $q > 0$. The equation in [1] is of the latter type (with $p = q = 1$) while that in [3] is of the former type, which is seen more clearly in their equivalent vectorization,

$$F(x,y) = \left(\frac{a}{y^{-q}} - \frac{1}{x^p}, \frac{1}{x}\right), \quad (-q > 0).$$

with $q = -1, p = 1$.

THE CASE $q < 0$

In this case, we assume $q < 0$ and for convenience define

$$r \doteq \frac{-1}{q} > 0.$$

Under conditions (22), Eq. (2) cannot have a solution if $x_{n+1} = 0$ for some n, for then x_{n+2} cannot be defined. Hence, the following definition.

DEFINITION *The* crash set Z *of* (2) *is the set of all pairs* (u, v) *in the plane where* $F^n(u,v) \in \{0\} \times \mathbb{R}$; *i.e.*,

$$Z = \bigcup_{n=0}^{\infty} F^{-n}(\{0\} \times \mathbb{R}).$$

where F^0 *is taken as the identity function. Note that* $F^{-1}(Z) \subset Z$.

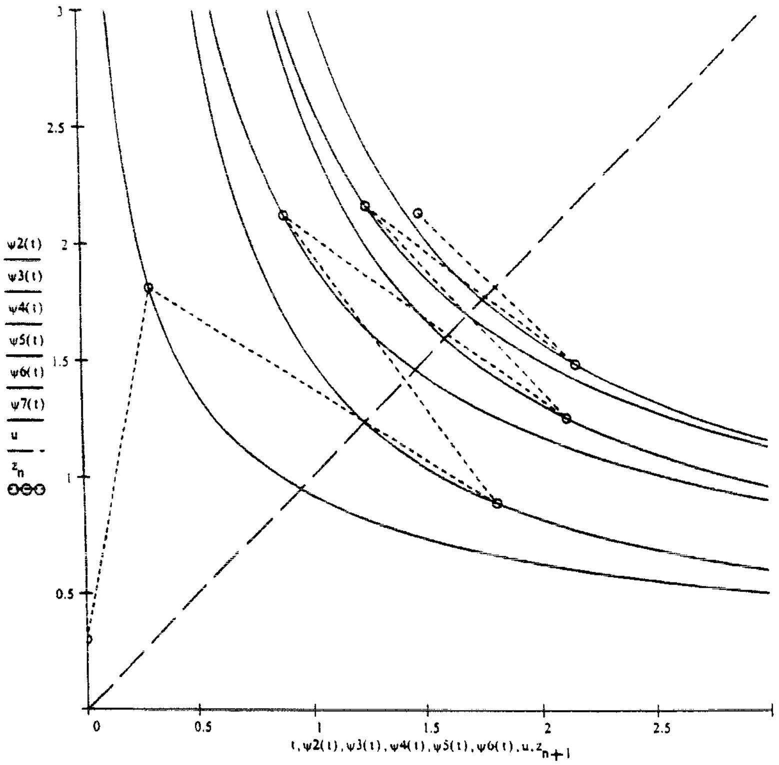

FIGURE 1 Initial part of a crash set and sample trajectory; $q=-9/7$, $p=5/7$, $a=1.1$.

In the notation of the previous subsection $(\xi_1(t), \xi_2(t))=(0, t)$ is actually the singularity set S. Figure 1 shows $F^{-n}(0, t)$ for $n=1, \ldots, 6$; given Lemma 4(a) below, only the portion for $t>0$ is shown. These curves are constructed recursively on a computer using the algorithm of Lemma 3(c).

We now proceed to determine some properties of Z. The next result improves upon Lemma 3 by giving a *first order* algorithm for the F^{-n} and thus making the analysis somewhat easier. Interestingly, this approach fails in the case $q>0$.

LEMMA 4 *Let mappings ζ_n be defined recursively as follows:*

$$\zeta_1(u) = a^{-r}u^{-pr}, \quad u \neq 0$$

and for $n \geq 1$,

$$\zeta_{n+1}(u) = a^{-r}[\zeta_n^{-1}(u) + u^{-p}]^r. \tag{23}$$

Then the following statements are true:

(a) $\zeta_n(-u) = -\zeta_n(u)$ *for all* u *and all* n *(i.e.,* ζ_n *is odd for all* n*);*
(b) *For each* n, ζ_n *is a decreasing function and* $\zeta_n(0, \infty) = (0, \infty)$;
(c) *For each* n, $\lim_{u \to \infty} \zeta_n(u) = 0$, *and* $\lim_{u \to 0^+} \zeta_n(u) = \infty$;
(d) *For each* n, ζ_n *and* ζ_n^{-1} *are continuous on* $(0, \infty)$;
(e) $\zeta_n(u) < \zeta_{n+1}(u)$ *for all* $n \geq 1$ *and each* $u > 0$, *with inequality reversed for* $u < 0$;
(f) $F^{-n}(\{0\} \times \mathbb{R}) = G(\zeta_n)$, *where* $G(\zeta_n)$ *denotes the graph of* ζ_n.

Proof Note that ζ_1 has properties mentioned in (a)–(d) and

$$\zeta_1^{-1}(v) = a^{-1/p} v^{-1/pr}, \quad v \neq 0.$$

By (23)

$$\zeta_2(u) = a^{-r}[a^{-1/p} u^{-1/pr} + u^{-p}]^r$$

and it is easy to check that ζ_2 also satisfies (a)–(d); further, (e) is clearly satisfied for $n = 1$ since $a^{-1/p} u^{-1/pr} > 0$ for all $u > 0$ and oddness implies the reverse inequality for $u < 0$. Property (f) is also satisfied for $n = 1$ since $(u, v) \in F^{-1}(\{0\} \times \mathbb{R})$ if and only if

$$a v^{1/r} - u^{-p} = 0, \quad u \in \mathbb{R} - \{0\},$$

which is true if and only if $v = a^{-r} u^{-pr}$; i.e., if and only if $(u, v) \in G(\zeta_1)$.

Next, suppose inductively that (a)–(e) hold for ζ_n, $n \leq k$. Then, ζ_k^{-1} has the properties listed in (a) and (b), so it follows from (23) that ζ_{k+1} satisfies (a) and (b). Property (c) is satisfied because

$$\zeta_k(\zeta_k^{-1}(u)) = u,$$

so if $u \to \infty$ then $\zeta_k^{-1}(u)$ must converge to zero by (d) and the induction hypothesis. Hence, by (23), $\lim_{u \to \infty} \zeta_{k+1}(u) = 0$. A similar argument establishes the case where $u \to 0^+$. To prove (d), we see that ζ_{k+1}

is continuous by (23). Since

$$\zeta_{k+2}(\zeta_{k+1}(u)) = a^{-r}[u + \zeta_{k+1}(u)^{-p}]^r, \quad u \neq 0,$$

it follows that $\zeta_{k+2} \circ \zeta_{k+1}$ is continuous. Thus, ζ_{k+2} is continuous, implying also the continuity of ζ_{k+1}^{-1} because

$$\zeta_{k+1}^{-1}(u) = a\zeta_{k+2}(u)^{1/r} - u^{-p}, \quad u \neq 0.$$

As for (e), let $u > 0$ and note that it is enough to show that

$$\zeta_n(\zeta_n(u)) < \zeta_{n+1}(\zeta_n(u)), \quad u > 0, \tag{24}$$

for $n = k + 1$. Using (23), inequality (24) reduces to

$$\zeta_k^{-1}(\zeta_{k+1}(u)) < u,$$

which is equivalent to $\zeta_{k+1}(u) > \zeta_k(u)$ for all $u > 0$. Since this last inequality is true by the induction hypothesis, (e) is established.

Finally, regarding (f), we have $(u, v) \in F^{-(k+1)}(\{0\} \times \mathbb{R})$ if and only if $F(u, v) \in G(\zeta_k)$. This latter statement means

$$av^{1/r} - u^{-p} = w, \quad u = \zeta_k(w) \tag{25}$$

for some $w \neq 0$. Conditions (25) are equivalent to

$$av^{1/r} - u^{-p} = \zeta_k^{-1}(u),$$

which is true if and only if

$$v = a^{-r}[\zeta_k^{-1}(u) + u^{-p}]^r = \zeta_{k+1}(u),$$

that is, $(u, v) \in G(\zeta_{k+1})$. This completes the proof of the lemma.

LEMMA 5 *For each positive integer n, let s_n be the unique positive solution of the equation*

$$\zeta_n(u) = u, \quad u > 0. \tag{26}$$

(a) *The sequence* $\{s_n\}$ *is increasing and for each* n,

$$s_n < s_{n+1} < a^{-r}(s_n + s_n^{-p})^r. \tag{27}$$

(b) *Let* $\gamma(u) \doteq a^{-r}(u+u^{-p})^r$, *the expression on the right hand side of* (27). *Then* $\zeta_{n+1}(s_n) = \gamma(s_n)$ *for all* n, *and*

$$F^{-1}(u,u) = (u, \gamma(u)), \quad u \neq 0, \tag{28}$$

i.e., the curve γ *is the image of the identity under* F^{-1}.

(c) *Let* $G^+(\zeta_n) = \{(u, \zeta_n(u)) \colon u > s_n\}$ *be the part of the graph of* ζ_n *over* (s_n, ∞) *and let* $G^-(\zeta_n) = \{(u, \zeta_n(u)) \colon u < s_n\}$ *be the part over the interval* $(0, s_n)$, *where* $s_n > 0$. *Then*

$$F^{-1}(G^+(\zeta_n)) = G^-(\zeta_{n+1}), \quad F^{-1}(G^-(\zeta_n)) = G^+(\zeta_{n+1}).$$

Proof (a) For each n, Eq. (26) has a unique positive solution by Lemma 4(a); hence, s_n is well defined. Now, the function,

$$\sigma_n(u) \doteq \zeta_n(u) - u$$

is decreasing for each n and by Lemma 4(d), $\sigma_n < \sigma_{n+1}$. In particular,

$$\sigma_{n+1}(s_n) > \sigma_n(s_n) = 0 = \sigma_{n+1}(s_{n+1}),$$

so that $s_n < s_{n+1}$ and $\{s_n\}$ is increasing. Using this fact and (23), we can also see that

$$s_{n+1} = \zeta_{n+1}(s_{n+1}) < \zeta_{n+1}(s_n) = a^{-r}(s_n + s_n^{-p})^r, \tag{29}$$

which establishes (27).

(b) The first claim is already proved in (29). As for the second, from (21),

$$F^{-1}(u,v) = (v, a^{-r}(u + v^{-p})^r), \quad v \neq 0, \tag{30}$$

which upon setting $v = u$ gives (28).

(c) This is a straightforward consequence of (30) and (23).

LEMMA 6 (a) *If $r < 1$, then Eq.* (2) *has a unique positive fixed point $\bar{x} \in (c, \infty)$ where*

$$c = \left[\frac{r(p+1)}{a(pr+1)}\right]^{r/(1-r)}. \tag{31}$$

(b) *Let $r = 1$. If $a \leq 1$ then* (2) *does not have any fixed points. If $a > 1$, then* (2) *has a unique positive fixed point,*

$$\bar{x} = (a-1)^{-1/(p+1)}. \tag{32}$$

(c) *Let $r > 1$. If*

$$a > r(p+1)(r-1)^{-(r-1)/r(p+1)}(pr+1)^{(pr+1)/r(p+1)}, \tag{33}$$

then (2) *has two positive fixed points, one on each side of c as defined by* (31). *If equality holds in* (33), *then there is a unique positive fixed point $\bar{x} = c$. If the reverse inequality holds in* (33), *then there are no fixed points.*

Proof From (2) we find that the fixed points are the solutions of the equation

$$\phi(u) \doteq au^{p+1/r} - u^{p+1} - 1 = 0.$$

Since ϕ is an even function, we need only concern ourselves with the positive zeros of ϕ. Clearly, when $r = 1$, there is a unique positive fixed point given by (32), thus proving (b). If $r < 1$, then an examination of the derivative

$$\phi'(u) = u^p[a(p+1/r)u^{1/r-1} - (p+1)]$$

shows the number c given by (31) to be a unique local minimum. Since $\phi(0) = -1$ and $\lim_{u\to\infty} \phi(u) = \infty$, it follows that there is a unique fixed point $\bar{x} \in (c, \infty)$ and the proof of (a) is completed. Finally, to prove (c), we observe that the number c given by (31) is a unique local maximum when $r > 1$. Thus, there are 2, 1, or no positive zeros for ϕ, according to whether the quantity

$$\phi(c) = a^{r(p+1)/(r-1)}\left[\frac{r-1}{r(p+1)}\right]\left[\frac{pr+1}{r(p+1)}\right]^{(pr+1)/(r-1)} - 1$$

is positive, zero or negative, respectively. These three cases readily translate into (33) and its associated cases.

THEOREM 4 (a) *If Z is the crash set of* (2), *then*

$$Z = (\{0\} \times \mathbb{R}) \cup \bigcup_{n=1}^{\infty} G(\zeta_n).$$

In particular, if

$$S \doteq \{(x,y)\colon x = 0, \text{ or } |y| \geq a^{-r}|x|^{-pr}\},$$

then

$$Z \subset S_0 \doteq S \cap \{(x,y)\colon xy \geq 0\}.$$

(b) *If either* (i) $r < 1$, *or* (ii) $r = 1$ *and* $a > 1$, *then for all positive integers n*

$$Z \subset S_1^+ \cup S_2^+ \cup S_1^- \cup S_2^-,$$

where, if $\bar{x}$ is the unique fixed point of (2)

$$S_1^+ = S_0 \cap [0,\bar{x}) \times (0,\infty), \quad S_1^- = S_0 \cap (-\bar{x}, 0] \times (-\infty, 0),$$
$$S_2^+ = S_0 \cap [0,\infty) \times (0,\bar{x}), \quad S_2^- = S_0 \cap (-\infty, 0] \times (-\bar{x}, 0).$$

Proof By Lemma 5, Z is a countable collection of graphs of monotonic odd functions ζ_n defined on $\mathbb{R} - \{0\}$, so the truth of assertions in (a) follows. To prove (b), given the origin symmetry of the curves ζ_n, it is only necessary to prove that the positive half of Z, namely, the part consisting of the positive halves of the ζ_n, is contained in $S_1^+ \cup S_2^+$. This is established if we only show that the sequence $\{s_n\}$ of Lemma 5 is bounded if either (i) or (ii) hold. From (27) we have $s_n < \gamma(s_n)$ which is equivalent to

$$\phi(s_n) < 0, \quad n = 1, 2, 3, \ldots \tag{34}$$

with ϕ as defined in the proof of Lemma 6. Since either (i) or (ii) requires ϕ to be negative on $(0, \bar{x})$ and positive on $(\bar{x}, \infty)$, we must conclude from (34) that $s_n \in (0, \bar{x})$ for all n. This completes the proof.

The next theorem refines some of the preceding results in a special case; it also complements Theorem 2 in [3].

THEOREM 5 *Let* $r = p = 1$. *Then,*

(a) *The crash set* $Z = \{(x, y)\colon x = 0 \text{ or } xy = \alpha_n, n = 1, 2, 3, \ldots\}$, *where*

$$\alpha_n \doteq \sum_{k=1}^{n} a^{-k} = \frac{1 - a^{-n}}{a - 1} \quad (\alpha_n = n \text{ if } a = 1). \tag{35}$$

(b) *For all* $n \geq 1, a \neq 1$,

$$F^{-(2n-1)}(0, y) = \left(A_{n-1}y, \frac{1 - a^{-2n+1}}{(a-1)A_{n-1}y}\right), \tag{36}$$

where

$$A_0 \doteq 1, \qquad A_n = \prod_{k=1}^{n} \frac{1 - a^{-2k}}{1 - a^{-2k+1}}. \tag{37}$$

If $a = 1$, *then for* $n \geq 2$

$$F^{-(2n-3)}(0, y) = \left(\frac{2^{2n} n!^2 y}{(2n)!}, \frac{(2n+1)(2n)!}{2^{2n} n!^2 y}\right). \tag{38}$$

(c) *If* $a \leq 1$, *then* Z *partitions the positive quadrant of the plane; i.e., if* $x, y \geq 0$, *then the point* (x, y) *is between* ζ_{n-1} *and* ζ_n, *or on one of these curves, for some* $n \geq 1$, *where we define* ζ_0 *to be the union of the two coordinate axes. The same conclusion is valid for points* (x, y) *with* $x, y \leq 0$.

(d) *If $a>1$ then the function sequence $\{\zeta_n\}$ converges monotonically at each point to ζ_∞ where*

$$\zeta_\infty(u) \doteq \frac{\alpha_\infty}{u}, \quad \alpha_\infty \doteq \lim_{n\to\infty} \alpha_n = \frac{1}{a-1}.$$

(e) *If $\{s_n\}$ is the sequence defined in Lemma 5, then $s_n = \alpha_n^{1/2}$; hence, $\{s_n\}$ converges monotonically to $\bar{x} = \alpha_\infty^{1/2}$ if $a>1$. If $a \le 1$, then $\{s_n\}$ is unbounded.*

Proof (a) Since $\zeta_1(u) = \zeta_1^{-1}(u) = 1/au$, (23) yields $\zeta_2(u) = (a^{-1} + a^{-2})/u$ and this is clearly equal to its own inverse, too. So suppose inductively that $\zeta_k(u) = \alpha_k/u$ where α_k is given by (35). Then by (23)

$$\zeta_{k+1}(u) = a^{-1}(\alpha_k/u + 1/u) = (a^{-1} + a^{-1}\alpha_k)/u = \alpha_{k+1}/u$$

as desired. Now, (a) follows from Lemma 4.

(b) By the defintion of F^{-1}, equality (36) is obviously true for $n=1$. Next, observe that by (23), (30) and (a)

$$F^{-1}(u, \zeta_n(u)) = (\zeta_n(u), \zeta_{n+1}(\zeta_n(u))) = (\alpha_n/u, \alpha_{n+1}u/\alpha_n)$$

for all $u \ne 0$ and all n. Thus repeated applications of F^{-1} result in the following development:

$$(0,y) \to (y, \alpha_1/y) \to (\alpha_1/y, \alpha_2 y/\alpha_1) \to (\alpha_2 y/\alpha_1, \alpha_3\alpha_1/\alpha_2 y) \to \cdots$$

Now, using (37) and a straightforward induction argument, we obtain (36). This also proves (38) since $a=1$ implies $\alpha_n = n$.

(c) This is clear, since $\zeta_n(u) = \alpha_n/u$ by (a) and if $a \le 1$ then $\alpha_n \to \infty$ as $n \to \infty$.

(d) If $a>1$ then $\alpha_n \to 1/(a-1) = \alpha_\infty$ as $n\to\infty$. The statements in (e) are now clear.

COROLLARY 4 *Let $a>1$.*

(a) *The limit curve ζ_∞ is invariant under F^{-1}, every neighborhood of the each of the two fixed points $\pm\alpha_\infty^{1/2}$ contains points of Z, and if*

$(x, y) \in Z$ with $x \neq 0$, then (x, y) is between the two curves $1/au$ and $1/(a-1)u$, or possibly on the former curve.

(b) *The trajectories $\{F^{-(2n-1)}(0, y)\}$ and $\{F^{-2n}(0, y)\}$ converge to unequal points on ζ_∞ for all but two values of y. Hence, all but two backward trajectories converge to 2-cycles determined by y. The two exceptional backward trajectories converge to the two fixed points $\pm(a-1)^{-1/2}$; see Fig. 2.*

Proof (a) By (28)

$$F^{-1}(u, \zeta_\infty(u)) = (\alpha_\infty/u, a^{-1}(u + u/\alpha_\infty))$$
$$= (\alpha_\infty/u, u) = (\alpha_\infty/u, \zeta_\infty(\alpha_\infty/u)),$$

i.e., ζ_∞ is invariant under F^{-1}. The remaining assertions in (d) are clear from Theorem 5.

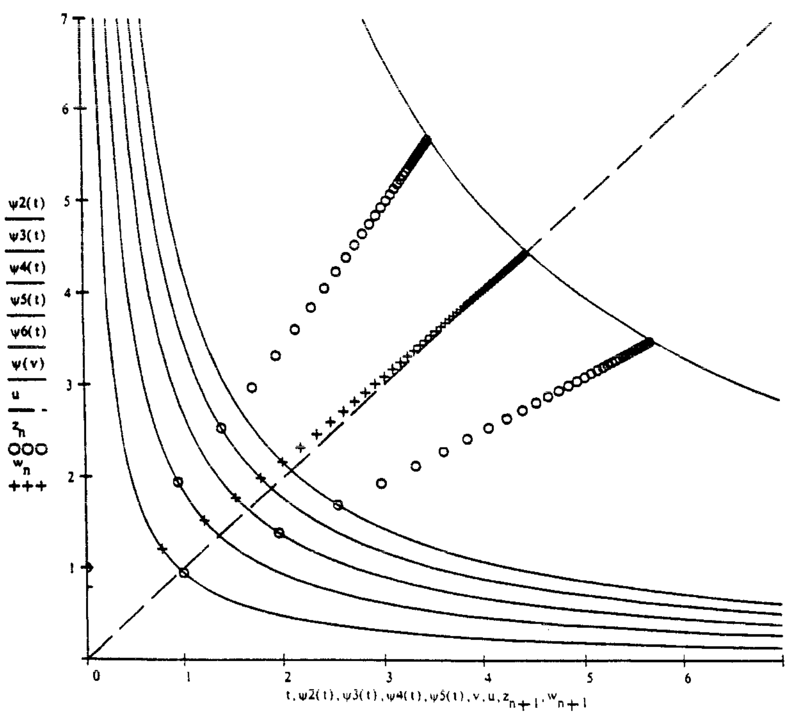

FIGURE 2 The exceptional trajectory (+), a sample one and the limit curve; $a = 1.05$.

(b) Since $F^{-2n} = F^{-1} \circ F^{-(2n-1)}$, Theorem 5 implies that

$$F^{-2n}(0,y) = \left(\frac{1 - a^{-2n+1}}{(a-1)A_{n-1}y}, A_n y\right).$$

By Theorem 5 and earlier results, the sequence $\{A_n\}$ has a limit,

$$L \doteq \lim_{n\to\infty} \prod_{k=1}^{n} \frac{1 - a^{-2k}}{1 - a^{-2k+1}}.$$

Thus, the sequences of the first coordinates of F^{-2n} and $F^{-(2n-1)}$ converge to Ly and $1/(a-1)Ly$, respectively. The only values of y for which

$$Ly = \frac{1}{(a-1)Ly}$$

are $y^{\pm} = \pm 1/(a-1)^{1/2}L = \pm\bar{x}/L$. Note also that $\pm\bar{x} = Ly^{\pm}$, thus establishing the final assertion of the corollary.

The Case $q > 0$

This case is more difficult than the case $q < 0$ because our ζ-curves are no longer invertible and there is no analog of (23). To obtain some information about the crash set Z, we begin by solving Eq. (2) for x_{n-1},

$$x_{n-1} = \left(\frac{ax_n^p}{x_{n+1}x_n^p + 1}\right)^{1/q}. \tag{39}$$

Hence, the first two ζ curves in this case may be defined as follows.

Definition *For every $u \in \mathbb{R}$, define*

$$\zeta_1(u) = a^{1/q}u^{p/q}, \qquad \zeta_2(u) = \left(\frac{au^p}{a^{-1/p}u^{p+p/q} + 1}\right)^{1/q}.$$

Note that ζ_1 is strictly increasing on $\mathbb{R}$, and its inverse map is given by

$$\zeta_1^{-1}(u) = a^{-1/p}u^{q/p}. \tag{40}$$

Additional properties of ζ_1 and ζ_2 are listed in the next lemma whose elementary proof is omitted.

LEMMA 7 (a) *ζ_2, ζ_1 are odd functions (i.e., with origin symmetry).*

(b) *$\zeta_2(u) < \zeta_1(u)$ for $u > 0$ and $\zeta_2(0) = \zeta_1(0) = 0$.*

(c) *Derivatives $\zeta_2'(0) = \zeta_1'(0) = 0$, $a^{1/q}$, or ∞ depending on whether $p > q$, $p = q$, or $p < q$, respectively.*

(d) *ζ_2 achieves a maximum value at the single point,*

$$\mu = (a^{1/q} p^2 / q)^{p/(p^2+q)} > 0.$$

(e) $\lim_{u \to \infty} \zeta_2(u) = 0$.

The next definition is based on the preceding lemma and on Eq. (39). Because of origin symmetry, we need consider only $u \geq 0$.

DEFINITION *Let $\omega = \zeta_2(\mu)$ be the maximum value of ζ_2. For $0 \leq u \leq \omega$ define*

$$\zeta_3^\alpha(u) = \left(\frac{au^p}{u^p \zeta_{2,\alpha}^{-1}(u) + 1} \right)^{1/q}, \qquad \zeta_3^\beta(u) = \left(\frac{au^p}{u^p \zeta_{2,\beta}^{-1}(u) + 1} \right)^{1/q},$$

where $\zeta_{2,\alpha}^{-1}(u), \zeta_{2,\beta}^{-1}(u)$ are inverse maps of, respectively,

$$\zeta_{2,\alpha}(u) = \zeta_2(u), \quad 0 \leq u \leq \mu,$$
$$\zeta_{2,\beta}(u) = \zeta_2(u), \quad \mu \leq u < \infty.$$

LEMMA 8 (i) $\zeta_3^\alpha(0) = \zeta_3^\beta(0) = 0$ *and* $\zeta_3^\alpha(\omega) = \zeta_3^\beta(\omega) = [a\omega^p/(\omega^p \mu + 1)]^{1/q}$;

(ii) *ζ_3^β is an increasing function on its domain* $[0, \omega]$;

(iii) $0 < \zeta_3^\beta(u) < \zeta_3^\alpha(u) < \zeta_2(u)$ *for* $0 < u < \omega$;

(iv) *The graph $\zeta_3^\alpha \cup \zeta_3^\beta$ is the boundary of the compact region,*

$$M^+ = \{(x, y) \colon 0 \leq x \leq \omega,\ \zeta_3^\beta(x) \leq y \leq \zeta_3^\alpha(x)\}$$

and the area of M^+ is less than $q(p+q)^{-1} a^{1/q} \omega^{1+p/q}$.

Proof Statement (i) is clear from the definitions, and (ii) is an easy consequence of the decreasing nature of the inverse map $\zeta_{2,\beta}^{-1}$. As for (iii), the first inequality from left is easy to verify from the definitions. The second inequality from the left is an immediate consequence of the inequalities

$$\zeta_{2,\alpha}^{-1}(u) < \mu < \zeta_{2,\beta}^{-1}(u), \quad 0 < u < \omega,$$

which are true by the previous lemma. The last inequality in (iii) is equivalent to

$$\zeta_{2,\alpha}^{-1}(u) > a^{-1/p}u^{q/p} = \zeta_1^{-1}(u) \tag{41}$$

for $0 < u < \omega$. For each u in this range, $v = \zeta_{2,\alpha}^{-1}(u)$ is the unique number in the interval $(0, \mu)$ where $\zeta_2(v) = u$. Inserting this in both sides of (41), and applying ζ_1 to both sides we have

$$\zeta_1(v) > \zeta_1(\zeta_1^{-1}(\zeta_2(v))) = \zeta_2(v), \tag{42}$$

which is true for all positive v. Since (42) is equivalent to (41), the proof of (iii) is complete. The first assertion in statement (iv) is now obvious from statements (i) and (iii), and the final assertion just states that the area of M^+ is smaller than the area under the graph of ζ_1 over $[0, \omega]$, namely, smaller than the integral $\int_0^\omega \zeta_1(u)\,du$.

Remarks (1) In the special case $p = q = 1$, the functions $\zeta_3^\alpha, \zeta_3^\beta$ can be explicitly determined; they are given by the formula

$$\frac{2au}{2 + a^2 \pm \sqrt{a^4 - 2au^2}},$$

with the "+" giving ζ_3^β. Using these explicit formulas, we can also determine the area of S^+ precisely as

$$\int_0^{a\sqrt{a}/2} \frac{au\sqrt{a^4 - 2au^2}}{1 + a^2 + au^2}\,du = \left(1 + \frac{a^2}{2}\right)\ln(1 + a^2) - a^2.$$

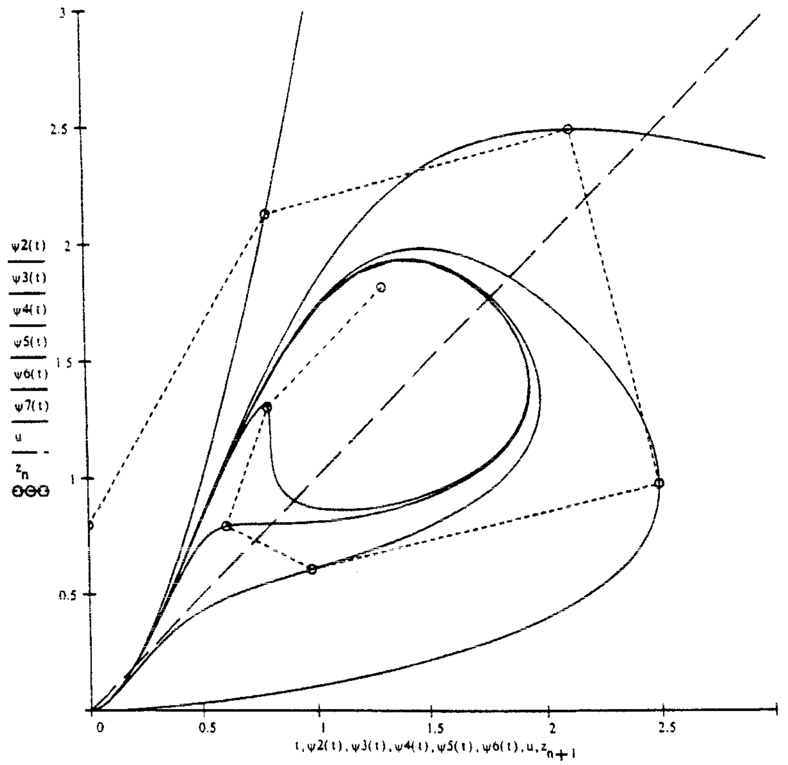

FIGURE 3 Initial part of a crash set and sample trajectory; $q=7/9$, $p=9/7$, $a=2.4$.

(2) In the case $q>0$, $(\xi_1(t), \xi_2(t))=(0, t)$ and the singularity set S is the union of the two coordinate axes. Figure 3 shows $F^{-n}(0, t)$ for $n=1, \ldots, 6$; given Lemma 7(a), only the portion for $t>0$ is shown. These curves are constructed recursively on a computer using the algorithm of Lemma 3(c). Note that for $n \geq 3$, the curves $F^{-n}(0, t)$ are closed.

DEFINITION *Let $A=\mathbb{R}\times\{0\}\cup\{0\}\times\mathbb{R}$ denote the union of the two coordinate axes in $\mathbb{R}^2$ and define*

$$M^- = \{(x, y)\colon -\omega \leq x \leq 0, \ -\zeta_3^{\alpha}(-x) \leq y \leq -\zeta_3^{\beta}(-x)\}.$$

We also define $M=A\cup G(\zeta_1)\cup G(\zeta_2)\cup M^+\cup M^-$.

THEOREM 6 *Equation* (2) *has a solution for each initial point* $(x_0, x_{-1}) \notin M$; *i.e., the crash set* $Z \subset M$.

Proof We prove that if $x_{m+1} = 0$ for some $m \geq 0$ then $(x_0, x_{-1}) \in A$. We consider the case $x_m > 0$, in which case, by (39) $x_n > 0$ for $1 \leq n \leq m$. The case $x_m < 0$ is proved analogously. By construction, for $j = 0, 1, 2$, $(x_{m-j+1}, x_{m-j}) \in \zeta_j$ if $m \geq j$, and $(x_{m-2}, x_{m-3}) \in \zeta_3^\alpha \cup \zeta_3^\beta$ if $m \geq 3$. So suppose that $m \geq 4$. Having shown that $(x_{m-j+1}, x_{m-j}) \in M^+$ for $j = 3$, assume inductively that the same is true for $j = 3, \ldots, k$ where $k < m$. Then

$$x_{m-k} \leq \zeta_3^\alpha(x_{m-k+1}) \leq \zeta_2(x_{m-k+1}) \leq \omega \tag{43}$$

and using (39), it is easy to see that $(x_{m-k}, x_{m-k-1}) \in M^+$, i.e.,

$$\zeta_3^\beta(x_{m-k}) \leq x_{m-k-1} \leq \zeta_3^\alpha(x_{m-k}),$$

if and only if

$$\zeta_{2,\alpha}^{-1}(x_{m-k}) \leq x_{m-k+1} \leq \zeta_{2,\beta}^{-1}(x_{m-k}). \tag{44}$$

To prove (44), we note that $x_{m-k+1} \leq \omega$ and consider two possible cases.

Case 1 $\mu \leq \omega$ If $0 < x_{m-k+1} < \mu$, then $\zeta_{2,\alpha}(x_{m-k+1}) = \zeta_2(x_{m-k+1}) \geq x_{m-k}$ by (43). Thus, $x_{m-k+1} \geq \zeta_{2,\alpha}^{-1}(x_{m-k})$ due to the increasing nature of $\zeta_{2,\alpha}$; this, together with the fact that $\mu \leq \zeta_{2,\beta}^{-1}(x)$ for all $x \leq \omega$ establish (44). Next, if $\mu \leq x_{m-k+1} \leq \omega$, then (43) again implies that $\zeta_{2,\beta}(x_{m-k+1}) = \zeta_2(x_{m-k+1}) \geq x_{m-k}$ which since $\zeta_{2,\beta}$ is decreasing, yields $x_{m-k+1} \leq \zeta_{2,\beta}^{-1}(x_{m-k})$. Therefore, as $\zeta_{2,\alpha}^{-1}(x) \leq \mu$, we conclude that (44) holds again.

Case 2 $\mu > \omega$ Since $x_{m-k+1} \leq \omega < \mu$, we can argue as in the first part of Case 1 to establish (44).

It follows that every point (x_{m-k}, x_{m-k-1}) of the backward orbit starting from (x_{m+1}, x_m), must be in M^+. Now setting $k = m$, we see in particular that $(x_0, x_{-1}) \in M^+$.

5 CONCLUSIONS, FUTURE DIRECTIONS

This paper has left some questions unanswered. Significant among them are the following:

(1) For the first order case, if f is increasing and $\{f^{-n}(0)\}$ is not finite, is $\{f^{-n}(0)\}$ dense outside of a compact set?

(2) In the second order case, when $q < 0$, a unique fixed point $\bar{x} > 0$ exists and $\{s_n\}$ is bounded, should $\bar{x}$ be a limit point of the crash set? If so, will $F^{-n}(0, t)$ actually converge to $(\bar{x}, \bar{x})$ as $n \to \infty$ for all $t > 0$?

(3) When $q > 0$, under what conditions will a unique fixed point $\bar{x} > 0$ be a limit point of the crash set?

The above questions and many related ones (e.g., extending the study to equations of orders 3 and higher) deserve answers, if we are to gain a deeper understanding of monotonic, singular difference equations.

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