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CONTENTS

The Ducci Problem and Related Questions

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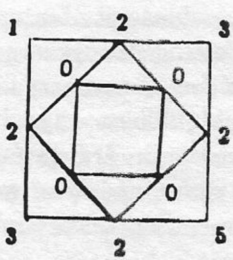
Consider four positive integers placed at the four corners of a square and let us play the following game: Calculate the absolute value of the difference of the two integers at the two endpoints of each side of the square, and use that number to label the midpoint of the side in question. Do this for every side and obtain a set of four positive integers, each at the midpoint of a side of our square. Connect these points pairwise by straight lines to form a new square inside the original one. Now repeat this process for the new square and continue forming squares. Will this go on forever, or will it end, eventually, with a square whose vertices are zeros? Now suppose we play the same game with a triangle instead of a square. Will we go on forming triangles forever? And what about other polygons? ... will this process go on indefinitely or will it end?



The case of the square is often referred to as the Ducci process [8], and there is a sizable literature on it (see the references). Generalizations to other polygons were apparently first discussed in [2]. Here we discuss some aspects of the problem in terms of periodic sequences of real numbers. If α is the sequence $(a_1, a_2, a_3, \ldots)$ of real numbers, define the *period* of α to be the smallest positive integer p such that $a_{p+k} = a_k$ for all $k \geq 1$. Sequences of period $p = 1$ correspond to the constant sequences $(c, c, c, \ldots)$ where c is a real number. In the sequel we will also use the notation $\alpha(k)$ to denote the k-th term a_k of the sequence α. Now define the operator δ as follows:

$$\delta\alpha = (|a_2 - a_1|, |a_3 - a_2|, \ldots)$$

We call δ the ***absolute difference operator***. As in the notation for higher order derivatives, let $\delta^{(k)}\alpha$ denote k successive applications of δ to the sequence α. In this setting, sequences of period $p = 4$ represent squares, and the repeated applications of δ represent the formations of successive squares. Similarly, different periods represent different numbers of sides. The following theorem now answers the questions raised above.

1. *(I) If α is a sequence of real numbers with period $p \geq 3$ and if p is odd, then $\delta^{(k)}\alpha \neq 0$ for all positive integers k.*
(II) If α is a sequence of rational numbers with period p, then $\delta^{(k)}\alpha = 0$ for some $k \geq 1$ depending on α if and only if $p = 2^n$ for some positive integer n.

If p is any positive integer not covered by the above theorem, then $\delta^{(k)}\alpha$ may or may not equal the zero sequence for some *finite* value of k, and the outcome will depend on the choice of the sequence α. Note that the second part of the above theorem is true only if we restrict our attention to the rational numbers. For example, if r is the real root of the polynomial $x^3 - x^2 - x - 1$, and α is the periodic real sequence $(1, r, r^2, r^3, 1, r, r^2, r^3, \ldots)$ of period 4, then $\delta\alpha = (r-1)\alpha$ and therefore, $\delta^{(k)}\alpha \neq 0$ for all k (see [7]).

To prove the above theorem, we begin by stating an interesting property of periodic sequences which we state for the absolute difference operator (it actually holds for more general operators). This may be proved by a straightforward computation:

2. *If α is a periodic real sequence with period $p \geq 2$, then the period of $\delta\alpha$ divides p.*

Now **1**(I) is a consequence of **2** and the following:

3. *Suppose α is a periodic real sequence of period $p \geq 2$. Let n be the number of negative terms among the first p terms of $\Delta\alpha$, the first finite difference of α. Then $p = 2n$ if $\delta\alpha$ is a constant sequence.*

Proof. It is clear that $n \geq 1$. Suppose $\delta\alpha = (c, c, \ldots)$, $c > 0$ is a constant sequence where

$$\alpha = (a_1, a_2, \ldots, a_p, a_1, a_2, \ldots, a_p, \ldots).$$

Then $|a_2 - a_1| = |a_3 - a_2| = \ldots = |a_1 - a_p| = c > 0$, and let $S = \sum_{k=1}^{p} |a_{k+1} - a_k| = pc$. Suppose for $j = 1, 2, \ldots, n$ we have $|a_{k_j+1} - a_{k_j}| = a_{k_j} - a_{k_j+1}$. Then upon writing up all terms of S and simplifying, we obtain $S = \sum_{j=1}^{n}(2a_{k_j} - 2a_{k_j+1}) = 2nc$. So $p = 2n$.

Now for $\alpha \in l^\infty(\mathbf{R})$ (i.e., α is a bounded sequence of real numbers) define $||\alpha||$ to be the l^∞ – *norm* of α; i.e., the supremum of all terms of α. If α is periodic with period p, then

$$||\alpha|| = max(|a_1|, |a_2|, \ldots, |a_p|)$$

. To prove **1**(II), we proceed by showing that repeated applications of δ to a periodic real sequence α tend to reduce $|\alpha|$.

4. *Let α be a periodic real sequence with at least three distinct terms. Then there is an m, $1 \leq m \leq p-1$, such that $||\delta^{(m)}\alpha|| < ||\alpha||$.*

Proof. Let $a = ||\alpha||$. Since $\delta(\alpha+\beta) = \delta\alpha$, where $\beta = (b, b, \ldots)$, for some real number b, is a constant sequence, without loss of generality, we may assume that the smallest term of α is 0. If 0's and a's in α do not occur next to each other, then $||\delta\alpha|| < a$. Otherwise, there are blocks of 0's and a's between terms that are not 0 or a, such that in each block some 0 occurs next to some a. Since δ commutes with the right shift operator, we may assume that the p-th term of α is not 0 or a. Let h denote the number of blocks of 0's and a's within the first p positions and let m_i be the number of elements in the i-th such block. Note that $m_i \geq 2$, $for\ i = 1, 2, \ldots, h \leq p/2$. Let $m = max(m_1, m_2, \ldots, m_h)$. When we apply δ to α, the length of each block of 0's and a's is diminished by one unit, so that after $(m-1)$ applications of δ no 0's occur next to a's. The conclusion follows.

We should note that **4** does not hold for arbitrary $\alpha \in l^\infty(\mathbf{R})$. Let $\alpha = (\alpha(k))$ be defined as follows: $\alpha(k) = 2$ if $k =$

$\binom{n+1}{2} - 4$ for some $n \geq 3$; $\alpha(k) = 1$ if $k = \binom{n+1}{2} - 3$ for some $n \geq 3$; and $\alpha(k) = 0$ for all other $k \geq 1$. Hence, $\|\delta^{(m)}\alpha\| = \|\alpha\| = 2$ for all $m \geq 1$.

As a corollary to **4**, we have:

5. *If α is a periodic sequence of rationals with period $p \geq 2$, then there is k (depending on α) such that $\delta^{(k)}\alpha$ is a sequence consisting entirely of 0's and a's, for some rational number $a \leq \|\alpha\|$.*

Proof. Apply **4** repeatedly to $l\alpha$, where l is the absolute value of the least common multiple of the denominators of the first p terms of α. Since the reductions of $\|l\alpha\|$ take place by integral units, we must eventually end up with no more than one non-zero term in our sequence.

Note that the above corollary is not true if α assumes irrational values (consider the sequence $(0, 1, \pi, 0, 1, \pi, \ldots)$).

Because of **5**, we may restrict our attention to sequences in $\mathbf{Z}_2^{\mathbf{N}}$, where $\mathbf{Z}_2 = \{0, 1\}$, and N is the set of positive integers. $\mathbf{Z}_2$ is a field under addition modulo 2 and ordinary multiplication. Hence $\mathbf{Z}_2^{\mathbf{N}}$ with coordinate-wise addition is a vector space over $\mathbf{Z}_2$ and δ is a linear operator on it. In fact, if $\alpha = (a_1, a_2, \ldots) \in \mathbf{Z}_2^{\mathbf{N}}$, then $\delta\alpha = (a_1 + a_2, a_2 + a_3, \ldots)$. Define the two summation operators $\delta_0^{-1}\alpha$, $\delta_1^{-1}\alpha$ on $\mathbf{Z}_2^{\mathbf{N}}$ as $\delta_0^{-1}\alpha = (0, a_1, a_1 + a_2, a_1 + a_2 + a_3, \ldots)$ and $\delta_1^{-1}\alpha = (1, 1 + a_1, 1 + a_1 + a_2, 1 + a_1 + a_2 + a_3, \ldots)$.

The following result now tells us about the effect of δ_0^{-1} and δ_1^{-1} on the period of a periodic sequence in $\mathbf{Z}_2^{\mathbf{N}}$.

6. *Suppose $\alpha \in \mathbf{Z}_2^{\mathbf{N}}$ has period $p \geq 1$. If $\alpha(k) = 1$ for an even number of k's between 1 and p, then both of the summation operators yield sequences of period p when applied to α. If $\alpha(k) = 1$ for an odd number of k's between 1 and p, then both of the summation operators yield sequences of period $2p$ when applied to α.*

Proof. If $2m$ is the number of k's such that $\alpha(k) = 1$ then $\sum_{k=1}^{p} \alpha(k) = 2m = 0 \ (mod 2)$. If $k = 1$, then $\delta_0^{-1}\alpha(1 + p) = \sum_{i=1}^{p} \alpha(i) = 0 = \delta_0^{-1}\alpha(i)$, and for all $k \geq 2$

$$\delta_0^{-1}\alpha(k+p) = \sum_{i=1}^{k+p-1} \alpha(i) = \sum_{i=p+1}^{k+p-1} \alpha(i)$$

$$= \sum_{i=1}^{k-1} \alpha(i) = \delta_0^{-1}\alpha(k).$$

Hence $\delta_0^{-1}\alpha$ is periodic with period at most p. Also $\delta(\delta_0^{-1}\alpha) = \alpha$, so by **2**, the period of $\delta_0^{-1}\alpha$ is at least p. Thus, p must be the period.

If $2m + 1$ is the number of k's such that $\alpha(k) = 1$, then $\sum_{i=1}^{2p} \alpha(i) = 2(2m + 1) = 0 \ (mod 2)$. As in the even case above, it follows that $\delta_0^{-1}\alpha(k + 2p) = \delta_0^{-1}\alpha(k)$ for all $k \geq 1$. On the other hand, note that

$$\delta_0^{-1}\alpha(p+1) = \sum_{i=1}^{p} \alpha(i) = 2m + 1 = 1,$$

whereas $\delta_0^{-1}\alpha(1) = 0$. Thus $\delta_0^{-1}\alpha$ cannot have period p, so that by **2**, its period must be $2p$.

As for $\delta_1^{-1}\alpha$, note that if q denotes the period of $\delta_0^{-1}\alpha$, then for all $k \geq 1$

continued on page 11

Ducci Problem...
continued from page 7

we have

$$\begin{aligned}\delta_1^{-1}\alpha(k+q) &= 1+\delta_0^{-1}\alpha(k+q)\\ &= 1+\delta_0^{-1}\alpha(k)\\ &= \delta_1^{-1}\alpha(k).\end{aligned}$$

Hence $\delta_1^{-1}\alpha$ also has period q.

We shall need two combinatorial lemmas before proving **1**(II). The first of these is proved by a straight-forward calculation and the second is a simple consequence of

Pascal's identity for binomial coefficients and induction.

7. *The binomial coefficient* $\binom{2^m}{k}$ *is even for all* $m \geq 1$, $k = 1, 2, \ldots, 2^m - 1$.

8. For $\alpha \in \mathbf{Z}_2^{\mathbf{N}}$,

$$\delta^{(n)}\alpha(i) = \sum_{j=0}^{n} \binom{n}{j} \alpha(n-j+i) \ (mod2).$$

Now we are in a position to prove 1(II).

Proof of 1(II). Because of **5**, there is a rational number a and a positive integer k_1 such that $\beta = (1/a)\delta^{(k_1)}\alpha \in \mathbf{Z}_2^{\mathbf{N}}$. Suppose α has period 2^n. Then by **2**, β has period 2^m, $m \leq n$. By **7** and **8**

$$\begin{aligned} \delta^{(2^m)}\beta(i) &= \sum_{j=0}^{2^m} \binom{2^m}{j} \beta(2^m - j + i) \\ &= \beta(2^m + i) + \beta(i) \\ &= 2\beta(i) = 0 \ (mod2) \end{aligned}$$

for all $i \geq 1$. Hence for $k = k_1 + 2^m$, we obtain

$$\begin{aligned} \delta^{(k)}\alpha &= \delta^{(2^m)}(a\beta) \\ &= a\delta^{(2^m)}\beta \\ &= a(0,0,\ldots) = 0. \end{aligned}$$

Conversely, we can obtain a binary tree of sequences in $\mathbf{Z}_2^{\mathbf{N}}$ by successive applications of the two operators δ_0^{-1} and δ_1^{-1} starting from $(0,0,\ldots)$. By **6**, every sequence obtained in this fashion is periodic and its period is a power of 2. It follows that only those sequences whose period is a power of 2 can lead to the zero sequence upon successive applications of the operator δ.

In closing, let us point out that although 1(II) states that for any given α of period 2^n, $n \geq 1$, we obtain the zero sequence after applying δ to α a finite number of times k, it does not imply that there is a bound for k independent of α. Indeed, for the smallest non-trivial case, $p = 2^2$, it has been shown [7] that the value of k can become arbitrarily large depending on the choice of α. On the other hand, Dan Ullman has shown [6] that for "most" sequences α with period $p = 2^2$, one obtains $k \leq 8$.

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