A derivative test for uniform continuity

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Using the uniform continuity theorem for compact intervals together with the mean value theorem, a derivative-based criterion is obtained with which to establish the uniform continuity of functions on non-compact intervals.

From the mean value theorem it readily follows that *if the derivative of the function f is bounded on an interval I, then f is Lipschitz, hence uniformly continuous on I* [1]. Since I is not necessarily closed or bounded here, the italicized statement represents an extension of the uniform continuity theorem (for differentiable functions) to arbitrary intervals. The statement is also analogous to the one for continuous functions from elementary calculus, namely, that *every function which is differentiable on I, is continuous on I*. However, unlike its continuous analogue, the uniformly continuous version is not applicable to elementary functions such as $\sqrt[3]{x}$ that are not Lipschitz on, say $(-\infty, \infty)$. This problem can be overcome if the uniform continuity theorem itself is also applied in addition to the mean value theorem to get past a compact set of troublesome points. The resulting derivative test applies with ease to virtually all elementary functions of calculus on their intervals of uniform continuity, including the class of functions such as $\sqrt[3]{x^2}$ that are continuous and piecewise smooth on real intervals. Thus, when presented in instruction as a corollary of the two basic theorems on which it is based, the test here takes the form of an immediately useful application of those two theorems.

In this note, we consider only real-valued functions of a real variable. Also an interval is defined in its most general form as a non-empty, connected subset of the real numbers (thus single points as well as the set of all real numbers are also intervals in this note).

Lemma. Let f be continuous on an interval I, and let I_1 and I_2 be subintervals of I. If f is uniformly continuous on each of I_1 and I_2, then f is uniformly continuous on $I_1 \cup I_2$.

Proof. If either subinterval is contained in the other, then the lemma is trivially true. So assume that there is a number a in the relative complement $I_1 \backslash I_2$ and also there is $b \in I_2 \backslash I_1$ with $a < b$. This assumption implies that I_1 has a right end point b_1 and likewise I_2 has a left end point a_2. It is clear that $a \leq a_2 \leq b$ and $a \leq b_1 \leq b$ where at least one of the inequalities is strict in each case. Let $\varepsilon > 0$ be given and consider the following cases:

Case 1. $a_2 < b_1$ or $b_1 < a_2$; for $i = 1, 2$ we may choose δ_i such that if $x, y \in I_i$ and $|x - y| < \delta_i$, then $|f(x) - f(y)| < \varepsilon$. Define $\delta = \min\{|b_1 - a_2|, \delta_1, \delta_2\}$. Then $\delta = \delta(\varepsilon) > 0$, and if $x, y \in I_1 \cup I_2$ with $|x - y| < \delta$, then x and y are both in I_1 or both in I_2. It follows that $|f(x) - f(y)| < \varepsilon$.

Case 2. $a < a_2 = b_1 < b$; let $c = a_2 = b_1$. Since the closed interval $[a, b]$ is compact, the uniform continuity theorem implies the existence of $\delta_0 = \delta_0(\varepsilon) > 0$ with the property that if $x, y \in [a, b]$ and $|x - y| < \delta_0$ then $|f(x) - f(y)| < \varepsilon$. Define $\delta = \min\{c - a, b - c, \delta_0, \delta_1, \delta_2\}$. Clearly $\delta = \delta(\varepsilon) > 0$, and if $x, y \in I_1 \cup I_2$ with $|x - y| < \delta$ then both x and y must be contained in at least one of the subintervals $[a, b]$, I_1 or I_2. Once again $|f(x) - f(y)| < \varepsilon$.

Case 3. Either $a < a_2 = b_1 = b$ or $a = a_2 = b_1 < b$; since the two sub-cases are similar, we consider only the latter. Since $a \in I_1$, there are two possibilities: $I_1 = \{a\}$ or there is $c \in I_1 \backslash I_2$ such that $c < a$. In the latter situation apply Case 2 to the interval $[c, b]$. If, on the other hand, $I_1 = \{a\}$, then $I_1 \cup I_2$ is just the left closure of I_2. Thus, since f is continuous on $I_1 \cup I_2 \subset I$ and uniformly continuous on I_2, it follows that f is uniformly continuous on the union [1]. QED

Remark. Continuity of f on an ambient interval I is crucial to the validity of the above lemma (as well as the following theorem). For instance, the function $|x|/x$ is continuous (in fact, differentiable) on its domain $(-\infty, 0) \cup (0, \infty)$ and uniformly continuous on each of the subintervals $(-\infty, 0)$ and $(0, \infty)$ separately. However, this function is not uniformly continuous on the union of the two subintervals.

Theorem. Let f be continuous on an interval I. Assume that the derivative f' exists and is bounded on I except possibly on a compact set $K \subset I$. Then f is uniformly continuous on I.

Proof. In the light of the mean value and the uniform continuity theorems, we may assume that I is non-compact and that K is non-empty. Suppose then that a and b are the least and the greatest elements of K, respectively. Thus $K \subset [a, b] \subset I$, and the closed interval $[a, b]$ partitions I into three intervals: $[a, b]$ itself, and two other intervals I_1 and I_2 on either side of $[a, b]$ (since I is assumed not compact, then at least one of the latter two intervals is non-empty). By the mean value theorem, f is uniformly continuous on I_1 and I_2. Since by the uniform continuity theorem f is uniformly continuous on $[a, b]$, the lemma above implies that f is uniformly continuous on I. QED

Examples. The theorem can be used to rapidly establish the uniform continuity for most of the functions encountered in elementary calculus over suitable intervals. Examples are $|x|$ and $x^{1/2n-1}$, $n = 1, 2, 3, \ldots$ which are uniformly continuous on $(-\infty, \infty)$ with $K = \{0\}$ and $K = [-1, 1]$, respectively. The functions $x^{1/2n}$, $n = 1, 2, 3, \ldots$ are likewise uniformly continuous on their common domain $[0, \infty]$ with K equal to, say $[0, 1]$. In the same manner, we can use the theorem on more general functions such as $x^{m/n}$, where $m < n$ and m and n are relatively prime.

Also, since the composition of two uniformly continuous functions is again a uniformly continuous function, we conclude that if f is uniformly continuous (though not necessarily differentiable) over a non-empty set S of real numbers, then so are the functions $f^{1/2n-1}$, $n = 1, 2, 3, \ldots$. And if the range of f is contained in $[0, \infty)$, then the same is true of the functions $f^{1/2n}$, $n = 1, 2, 3, \ldots$. Clearly, identical statements may be made if f is a real-valued, uniformly continuous function on a general metric space. Of course, the above theorem also applies to composite functions which have no natural expression as a composition of other uniformly continuous functions on a given set. Familiar examples include $\exp(-x^2)$ and $(1+x^2)^{-1}$ on $(-\infty, \infty)$.

Remark. The compactness of K in the statement of the theorem may be weakened somewhat (as in the case of a periodic, piecewise linear function on the reals), but it cannot be replaced with 'boundedness'. A familiar counterexample would be $\sin(1/x)$, which is bounded and differentiable on $(0, \infty)$ and has a bounded derivative on (ε, ∞) for each $\varepsilon > 0$. However, $\sin(1/x)$ is not uniformly continuous on $(0, \infty)$.

References

[1] PROTTER, M. H., and MORREY, C. B., 1991, *A First Course in Real Analysis*, 2nd Edn (New York: Springer-Verlag), pp. 81, 337.

Nowhere differentiability of the coordinate functions of the Von Koch curve

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In 1904, Helge von Koch constructed a continuous curve which does not have a tangent anywhere. Based on a parametrization of this remarkable curve, which was found by this author recently, it will be proved analytically that its coordinate functions are nowhere differentiable.

The first sentence in K. Knopp's paper [1] on a unified generation of the curves of Peano, Osgood and von Koch says that 'Zu den besten Beispielen einer Kurve ohne Tangente gehört wohl ohne Zweifel dasjenige von v. Koch' ('Among the best examples of a curve without tangent is without doubt the one by von Koch'.) Von Koch found this curve in 1904 and proved geometrically that it does not have a tangent anywhere [2–4]. Knopp called von Koch's proof cumbersome and lacking in lucidity. He supplied a much simpler proof in [1] but his, as well as von Koch's proof are of a geometric nature and are based on the observation that for a curve to have a tangent at one of its points, all sufficiently small chords which contain that point or emanate from it, have to form an arbitrary small angle with each other.

Von Koch did not find an explicit representation (parametrization) of his curve but Cesàro and Knopp did [1, 4]. However, neither made use of it to obtain the stronger result that the coordinate functions of this curve are nowhere differentiable and it appears that nobody else did either. We will prove the non-differentiability of the coordinate functions analytically, based on a parametrization which we found