# Global Attractivity, Oscillations and Chaos in A Class of Nonlinear, Second Order Difference Equations

**Hassan Sedaghat**

*Department of Mathematics, Virginia Commonwealth University*
*Richmond, Virginia, 23284-2014, USA*

**Abstract** – The asymptotic properties of a class of nonlinear second order difference equations are studied. Sufficient conditions that imply the types of behavior mentioned in the title are discussed, in some cases within the context of the macroeconomic business cycle theory. We also discuss less commonly seen types of behavior, such as the equilibrium being simultaneously attracting and unstable, or the occurrence of oscillations away from a unique equilibrium.

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Few known difference equations display the wide range of dynamic behaviors that the equation

$$x_{n+1} = cx_n + f(x_n - x_{n-1}), \quad 0 \leq c \leq 1, \quad n = 0, 1, 2, \ldots \tag{1}$$

exhibits with even a limited selection of function types for $f$ (assumed continuous throughout this paper). The nonlinear, second order difference equation (1) has its roots in the early macroeconomic models of the business cycle. Indeed, a version of (1) in which $f(t) = \alpha t + \beta$ is a linear-affine function first appeared in Samuelson [10]. Various nonlinear versions of (1) subsequently appeared in the works of many other authors, notably in Hicks [3] and Puu [9]. For more details, some historical remarks and additional references see Sedaghat [13]. The mapping $f$ in (1) includes as a special case the sigmoid-type map first introduced into business cycle models (in continuous time) in Goodwin [2]. These classical models provide an intuitive context for the interpretation of the many varied results about (1).

In this paper we discuss several mathematical results that have been obtained about the asymptotic behavior of (1). These results include sufficient conditions for the global attractivity of the fixed point and conditions that imply the occurrence of persistent oscillations of solutions of (1). Historically, the latter, endogenously driven oscillatory behavior was one of the main attractions of (1) in the economic literature. The case $c = 1$ which in Puu [9] models full consumption of savings, is substantially different from $0 \leq c < 1$; we discuss both cases in some detail.

We also show that under certain conditions, solutions of (1) exhibit strange and complex behavior. These conditions include a case where the fixed point is globally attracting yet *unstable*. Also seen as possible is the occurrence of persistent, *off-equilibrium* oscillations; i.e., oscillations which do not occur about a fixed point. We also state various conjectures and open problems pertaining to (1). The essential background required for understanding the results of this paper is minimal beyond elementary real analysis and some mathematical maturity. However, some readers may benefit from a look at helpful existing texts and monographs such as Elaydi [1], Kocic and Ladas [5], LaSalle [6], Sedaghat [13].

# 1 Oscillations

In this first section of the paper, we consider the problem of oscillations for solutions of (1). In addition to being of interest mathematically, from a historical point of view this was the main attraction of business cycle models based on (1). In fact, in those economic models the kind of non-decaying, nonlinear oscillation that is discussed next was of particular interest.

## 1.1 Persistent oscillations

Consider the general $n$-th order autonomous difference equation

$$x_{n+1} = F(x_n, x_{n-1}, \ldots, x_{n-m+1}) \tag{2}$$

This clearly includes (1) as a special case with $m = 2$.

**Definition 1**. (*Persistent oscillations*) A bounded solution $\{x_n\}$ of (2) is said to be *persistently oscillating* if the set of limit points of the sequence $\{x_n\}$ has two or more elements.

Persistent oscillations are basically a nonlinear phenomenon because nonlinearity is essential for the occurrence of robust or structurally stable persistent oscillations. Indeed, if $F$ is linear then its persistently oscillating solutions can occur only when a root of its characteristic polynomial has magnitude one; i.e., for linear maps persistent oscillations do not occur in a structurally stable fashion. Next we quote a fundamental result on persistent oscillations; for a proof which uses standard tools such as the implicit function theorem and the Hartman-Grobman theorem, see Sedaghat [13].

**Theorem 1**. *Assume that $F$ in Eq.*(2) *satisfies the following conditions*:

(a) *The equation $F(x,\ldots,x) = x$ has a finite number of real solutions $\bar{x}_1 < \ldots < \bar{x}_k$;*

(b) *For $i = 1,\ldots,m$, the partial derivatives $\partial F_i \doteq \partial F/\partial x^i$ exist continuously at $\bar{X}_j = (\bar{x}_j,\ldots,\bar{x}_j)$, and every root of the characteristic polynomial*

$$\lambda^m - \sum_{i=1}^{m} \partial F_i(\bar{X}_j)\lambda^{m-i}$$

*has modulus greater than 1 for each $j = 1,\ldots,k$;*

(c) *For every $j = 1,\ldots,k$, $F(\bar{x}_j,\ldots,\bar{x}_j,x) \neq \bar{x}_j$ if $x \neq \bar{x}_j$.*

*Then all bounded solutions of* (2) *except the trivial solutions $\bar{x}_j$, $j = 1,\ldots,k$, oscillate persistently. If only* (a) *and* (b) *hold, then all bounded solutions that do not converge to some $\bar{x}_j$ in a finite number of steps oscillate persistently.*

The next result is the second-order (and sharper) version of Theorem 1.

**Corollary 1**. *Consider Eq.*(2) *with $m = 2$ and $F = F(x,y)$. Assume that the following conditions hold*:

(a) *The equation $F(x,x) = x$ has a finite number of solutions $\bar{x}_1 < \ldots < \bar{x}_k$;*

(b) *$F_x = \partial F/\partial x$ and $F_y = \partial F/\partial y$ both exist continuously at $(\bar{x}_j,\bar{x}_j)$ for all $j = 1,\ldots,k$, with*:

$$|F_y(\bar{x}_j,\bar{x}_j)| > 1\,, \qquad |F_y(\bar{x}_j,\bar{x}_j) - 1| > |F_x(\bar{x}_j,\bar{x}_j)|\ .$$

(c) *For every $j = 1,\ldots,k$, $F(\bar{x}_j,y) \neq \bar{x}_j$ if $y \neq \bar{x}_j$.*

*Then all non-trivial bounded solutions oscillate persistently.*

**Definition 2**. (*Absorbing intervals*) Equation (2) has an *absorbing interval* $[a, b]$ if for every set $x_0, x_{-1}, \ldots x_{-m+1}$ of initial values, the corresponding solution $\{x_n\}$ is eventually in $[a, b]$; that is, there is a positive integer $N = N(x_0, \ldots x_{-m+1})$ such that $x_n \in [a, b]$ for all $n \geq N$. We may also say that $F$ (or its standard vectorization) has an absorbing interval. In the special case where $a > 0$, (2) is said to be *permanent*.

**Remarks**. 1. If $F$ in (2) is bounded, then obviously (2) has an absorbing interval. Also, if (2) has an absorbing interval, then obviously every solution of (2) is bounded. The converses of these statements are false; the simplest counter-examples are provided by linear maps which are typically unbounded. A straightforward consideration of eigenvalues shows that *if $F$ is linear, then an absorbing interval exists if and only if the origin is attracting.* On the other hand, if all eigenvalues have magnitude at most one with at least one eigenvalue having magnitude one, then every solution is bounded although there can be no absorbing intervals. An example of a nonlinear mapping that has no absorbing intervals, yet all of its solutions are bounded is the well-known Lyness map $F(x, y) = (a + x)/y$, $a > 0$; also see Theorem 6 below.

2. An absorbing interval need not be invariant, as trajectories may leave it and then re-enter it (to eventually remain there); see Corollary 6 below and the Remark following it. Also an invariant interval may not be absorbing since some trajectories may never reach it.

3. The importance of the concept of permanence in population biology (commonly referred to as "persistence" there) has led to a relatively larger body of results than is available for absorbing intervals in general. These results are also of interest in social science models where the state variable is often required to be positive and in some cases, also bounded away from zero. See Kocic and Ladas [5] and Sedaghat [13] for more examples and details.

Our next result requires the following lemma which we quote from Sedaghat [11]. Lemma 1 refers to the first order equation

$$v_{n+1} = f(v_n), \quad v_1 = x_1 - x_0 \tag{3}$$

which with the given initial value relates naturally to (1).

**Lemma 1**. *Let $f$ be non-decreasing.*

(a) *If* $\{x_n\}$ *is a non-negative solution of* (1), *then*

$$x_n \le c^{n-1}x_0 + \sum_{k=1}^{n} c^{n-k} v_k$$

*for all* $n$, *where* $\{v_n\}$ *is a solution of* (3).

(b) *If* $\{x_n\}$ *is a non-positive solution of* (1), *then*

$$x_n \ge c^{n-1}x_0 + \sum_{k=1}^{n} c^{n-k} v_k \ .$$

**Theorem 2.** *Let* $f$ *be non-decreasing and bounded from below on* $\mathbb{R}$, *and let* $c < 1$. *If there exists* $\alpha \in (0,1)$ *and* $u_0 > 0$ *such that* $f(u) \le \alpha u$ *for all* $u \ge u_0$, *then* (1) *has a nontrivial absorbing interval. In particular, every solution of* (1) *is bounded.*

*Proof*. If we define $w_n \doteq f(x_n - x_{n-1})$ for $n \ge 1$, then it follows inductively from (1) that

$$x_n = c^{n-1}x_1 + c^{n-2}w_1 + \ldots + cw_{n-2} + w_{n-1}. \tag{4}$$

for $n \ge 2$. Let $L_0$ be a lower bound for $f(u)$, and without loss of generality assume that $L_0 \le 0$. As $w_k \ge L_0$ for all $k$, we conclude from (4) that

$$x_n \ge c^{n-1}x_1 + \left(\frac{1-c^{n-1}}{1-c}\right) L_0$$

for all $n$, and therefore, $\{x_n\}$ is bounded from below. In fact, it is clear that there is a positive integer $n_0$ such that for all $n \ge n_0$,

$$x_n \ge L \doteq \frac{L_0}{1-c} - 1.$$

We now show that $\{x_n\}$ is bounded from above as well. Define $z_n \doteq x_{n+n_0} - L$ for all $n \ge 0$, so that $z_n \ge 0$ for all $n$. Now for each $n \ge 1$ we note that

$$\begin{aligned} z_{n+1} &= cx_{n+n_0} + f(x_{n+n_0} - x_{n+n_0-1}) - L \\ &= cz_n + f(z_n - z_{n-1}) - L(1-c) \ . \end{aligned}$$

Define $g(u) \doteq f(u) - L(1-c)$, and let $\delta \in (\alpha, 1)$. It is readily verified that $g(u) \le \delta u$ for all $u \ge u_1$ where

$$u_1 \doteq \max\left\{u_0, \frac{-L(1-c)}{\delta - \alpha}\right\} \ .$$

If $\{r_n\}$ is a solution of the first order problem

$$r_{n+1} = g(r_n) \ , \qquad r_1 = z_1 - z_0$$

then since $f$ is bounded from below by $L_0 - (1-c)L = 1-c$, we have

$$r_n = g(r_{n-1}) \geq 1 - c$$

for all $n \geq 2$. Thus $\{r_n\}$ is bounded from below. Also, if $r_k \geq u_1$ for some $k \geq 1$, then

$$r_{k+1} = g(r_k) \leq \delta r_k < r_k \ .$$

If $r_{k+1} \geq u_1$ also, then $\delta r_k \geq r_{k+1} \geq u_1$ and since $g$ is non-decreasing,

$$r_{k+2} = g(r_{k+1}) \leq g(\delta r_k) \leq \delta^2 r_k \ .$$

It follows inductively that

$$r_{k+l} \leq \delta^l r_k$$

as long as $r_{k+l} \geq u_1$. Clearly there is $m \geq k$ such that $r_m < u_1$. Then

$$r_{m+1} = g(r_m) \leq g(u_1) \leq \delta u_1 < u_1$$

by the definition of $u_1$. By induction $r_n < u_1$ for all $n \geq m$. Now Lemma 1(a) implies that for all such $n$,

$$\begin{aligned} z_n &\leq c^{n-1}z_0 + c^{n-1}r_1 + \cdots + c^{n-m+1}r_{m-1} + \textstyle\sum_{k=m}^{n} c^{n-k} r_k \\ &< c^{n-m+1}(z_0 c^{m-2} + \cdots + r_{m-1}) + u_1 \textstyle\sum_{k=0}^{n-m} c^k \\ &= c^{n-m+1}K_0 + u_1(1-c)^{-1}(1-c^{n-m+1}) \ . \end{aligned}$$

Thus there exists $n_1 \geq m$ such that

$$z_n \leq \frac{u_1}{1-c} + 1$$

for all $n \geq n_1$. Hence, for all $n \geq n_0 + n_1$ we have $x_n \in [L, M]$ where

$$M \doteq \frac{u_1}{1-c} + 1 - L \ .$$

It follows that $[L, M]$ is an absorbing interval.

There is also the following more recent result, the first part of which is from Kent and Sedaghat [4]. The second part is an immediate consequence of Corollary 4.2.11 in Sedaghat [13]. In contrast to Theorem 1, $f$ is not assumed to be increasing in the next theorem.

**Theorem 3**. (a) *Let $c < 1$ and assume that constants $0 \leq a < 1$ and $b > 0$ exist such that $a \neq (1 - \sqrt{1-c})^2$ and*

$$|f(t) - at| \leq b \quad \text{for all } t.$$

*Then* (1) *has a non-trivial absorbing interval. In particular, all solutions of* (1) *are bounded.*

(b) *If there are constants $a, b \geq 0$ such that $a < 1 - c$ and*

$$|f(t)| \leq a|t| + b \quad \text{for all } t$$

*then* (1) *has a non-trivial absorbing interval. In particular, all solutions of* (1) *are bounded.*

**Remark**. It is noteworthy that both Theorems 2 and 3 exclude non-increasing functions, except when $f$ is bounded (at least from above). This is not a coincidence; for example, if $f(t) = -at$ then (1) is linear and all solutions are unbounded if

$$\frac{c+1}{2} < a < 1.$$

The next corollary concerns the persistent oscillations of trajectories of (1).

**Corollary 2**. *In addition to the conditions stated in either Theorem 2 or Theorem 3, assume that $f$ is continuously differentiable at the origin with $f'(0) > 1$. Then for all initial values $x_0, x_{-1}$ that are not both equal to the fixed point $\bar{x} = f(0)/(1-c)$, the corresponding solution of* (1) *oscillates persistently, eventually in an absorbing interval $[L, M]$.*

*Proof.* To verify condition (b) in Corollary 1, we note that

$$F_x(\bar{x}, \bar{x}) = c + f'(0), \quad F_y(\bar{x}, \bar{x}) = -f'(0)$$

which together with the fact that $f'(0) > 1 > c$ imply the inequalities in (b).

As for condition (c) in Corollary 1, since $f$ is strictly increasing in a neighborhood of 0, if there is $y$ such that

$$\bar{x} = F(\bar{x}, y) = c\bar{x} + f(\bar{x} - y)$$

then $f(0) = f(\bar{x} - y)$, so that $\bar{x} - y = 0$, as required.

We now consider an application to the Goodwin-Hicks model of the business cycle. This model is represented by the following generalization of Samuelson's linear equation

$$Y_n = cY_{n-1} + I(Y_{n-1} - Y_{n-2}) + A_0 + C_0 + G_0 \tag{5}$$

where $I : \mathbb{R} \to \mathbb{R}$ is a non-decreasing *induced investment function*. The terms $Y_n$ give the output (GDP or national income) in period $n$ and the constants $A_0, C_0, G_0$ are, respectively, the *autonomous investment,* the *minimum consumption* and *government input*. We assume in the sequel that

$$A_0 + C_0 + G_0 \geq 0.$$

The number $c$ here is the "marginal propensity to consume" or MPC. It gives the fraction of output that is consumed in the current period. If we define the function

$$f(t) \doteq I(t) + A_0 + C_0 + G_0, \quad t \in \mathbb{R},$$

we see that (5) is a special case of (1). The next definition gives more precise information about that function.

**Definition 3**. A *Goodwin investment function* is a mapping $G \in C^1(\mathbb{R})$ that satisfies the following conditions:

(i) $G(0) = 0$ and $G(t) + A_0 + C_0 + G_0 \geq 0$ for all $t \in \mathbb{R}$;

(ii) $G'(t) \geq 0$ for all $t \in \mathbb{R}$ and $G'(0) > 0$;

(iii) There are constants $t_0 > 0$, $0 < a < 1$ such that $G(t) \leq at$ for all $t \geq t_0$.

The next result is an immediate consequence of Corollary 2. It gives specific criteria for persistent oscillations of output trajectories, as is expected of a business cycle.

**Corollary 3**. (Persistent oscillations) *Consider the equation*

$$y_n = cy_{n-1} + G(y_{n-1} - y_{n-2}) + A_0 + C_0 + G_0, \quad 0 \leq c < 1, \tag{6}$$

*where $G$ is a Goodwin investment function. If $G'(0) > 1$, then all non-trivial solutions of* (6) *oscillate persistently, eventually in the absorbing interval $[L, t_1/(1-c)+1]$, where $t_1 \geq t_0$ is large enough that $G(t)+A_0+C_0+G_0 \leq at$ for $t \geq t_1$ if $G(t) \leq at$ for $t \geq t_0$ and where*

$$L = \lim_{t \to -\infty} G(t) + A_0 + C_0 + G_0 \geq 0.$$

## 1.2 Other oscillatory behavior

Here we approach the oscillation problem for (1) at a more general level, without requiring that the oscillatory behavior to be persistent. We begin with the following lemma; it gives conditions that imply a more familiar type of oscillatory behavior than that seen in the preceding sub-section. Note that if $tf(t) \geq 0$ for all $t$, then by continuity $f(0) = 0$ and the origin is the unique equilibrium of (1).

**Lemma 2**. *If $tf(t) \geq 0$ for all $t$ then every eventually non-negative and every eventually non-positive solution of* (1) *is eventually monotonic.*

*Proof.* Suppose that $\{x_n\}$ is a solution of (1) that is eventually non-negative, i.e., there is $k > 0$ such that $x_n \geq 0$ for all $n \geq k$. Either $x_n \geq x_{n-1}$ for all $n > k$ in which case $\{x_n\}$ is eventually monotonic, or there is $n > k$ such that $x_n \leq x_{n-1}$. In the latter case,

$$x_{n+1} = cx_n + f(x_n - x_{n-1}) \leq cx_n \leq x_n$$

so that by induction, $\{x_n\}$ is eventually non-increasing, hence monotonic. The argument for an eventually non-positive solution is similar and omitted.

**Theorem 4.** *Let $0 \leq c < 1$.*

(a) *If $tf(t) \geq 0$ for all $t$, then* (1) *has no solutions that are eventually periodic with period two or three.*

(b) *Let $b = \left(1 - \sqrt{1-c}\right)^2$. If $\beta \geq \alpha > b$ and $\alpha|t| \leq |f(t)| \leq \beta|t|$ for all $t$, then every solution of* (1) *oscillates about the origin.*

*Proof.* Let $\{x_n\}$ be a solution of (1). We claim that for all $k \geq 1$,

$$\begin{array}{lcl} x_k > 0 > x_{k+1} & \text{implies} & x_{k+2} < 0 \\ x_k < 0 < x_{k+1} & \text{implies} & x_{k+2} > 0 \end{array}$$

For suppose that $x_k > 0 > x_{k+1}$ for some $k \geq 1$. Then

$$x_{k+2} = cx_{k+1} + f(x_{k+1} - x_k) \leq x_{k+1} < 0.$$

The argument for the other case is similar and omitted. Now by Lemma 2, if a solution $\{x_n\}$ eventually has period 2, then for all sufficiently large $n$, there is $x_n > 0$, $x_{n+1} \leq 0$ and $x_{n+2} = x_n > 0$. But this contradicts the above claim. Hence, no solution of (1) can eventually have period two.

Similarly, if $\{x_n\}$ eventually has period 3, then by Lemma 2 for large enough $n$ there is $x_n > 0$ and $x_{n+1} \leq 0$. Assume first that $x_{n+1} < 0$. Then by the above claim, $x_{n+2} < 0$. But $x_{n+3} = x_n > 0$ so $x_{n+4} \geq cx_n > 0$, contradicting the fact that $x_{n+4} = x_{n+1} \leq 0$. Finally, suppose that $x_{n+1} = 0$. Then $x_{n+2} < 0$ again (for otherwise $x_n = x_{n+3} = 0$ which is a contradiction) and the argument is the same as before.

(b) See Kent and Sedaghat [4] for a proof.

It should be noted here that periods 4 and greater are possible for various choices of $f$ when $tf(t) \geq 0$ although periods 4 and 5 occur rarely in numerical simulations compared with periods 6 and greater.

# 2 The Case c=1

We now consider the case $c = 1$. This case is substantially different from the case $0 \leq c < 1$ and it is informative to contrast these two cases. Also Puu's equation below reduces to this case. First, we note that with $c = 1$, Eq.(1) may be put in the form

$$x_{n+1} - x_n = f(x_n - x_{n-1}) \tag{7}$$

From this it is evident that the standard vectorization of (7) is semiconjugate to the real factor $f$ relative to the link map $H(x, y) \doteq x - y$; see Sedaghat [13]. Further, the solutions $\{x_n\}$ of (1) are none other than the sequences of partial sums of solutions $\{v_n\}$ of (3), since the difference sequence $\{\Delta x_n\}$ satisfies (3).

**Theorem 5**. *Let $\{x_n\}$ be a solution of the second order equation* (7) *and let $\{v_n\}$ be the corresponding solution of the first order equation* (3).

(a) *If $v^*$ is a fixed point of* (3) *then $x_n = x_0 + v^* n$ is a solution of* (7).

(b) *If $\{v_1, \ldots, v_p\}$ is a periodic solution of* (3) *with period $p$, then*

$$x_n = x_0 - \omega_n + \overline{v} n \tag{8}$$

*is a solution of* (7) *with $\overline{v} = p^{-1} \sum_{i=1}^{p} v_i$ the average solution, and*

$$\omega_n = \overline{v}\rho_n - \sum_{j=0}^{\rho_n} v_j \ , \qquad (v_0 \doteq 0)$$

*where $\rho_n$ is the remainder resulting from the division of $n$ by $p$. The sequence $\{\omega_n\}$ is periodic with period at most $p$.*

*Proof*. Part (a) follows immediately from the identity

$$x_n = x_0 + \sum_{i=1}^{n} v_i \tag{9}$$

which also establishes the fact that solutions of the second order equation are essentially the partial sums of solutions of the first order equation.

To prove (b), observe that in (9), after every $p$ iterations we add a fixed sum $\sum_{i=1}^{p} v_i$ to the previous total. Therefore, since $n$ may generally take on any one of the values $pk + \rho_n$, where $0 \leq \rho_n \leq p-1$, we have

$$x_n = x_0 + k\sum_{i=1}^{p} v_i + \sum_{j=0}^{\rho_n} v_j \ . \tag{10}$$

Now substituting $k = n/p - \rho_n/p$ in (10) and rearranging terms we obtain (8). Also $\omega_n$ is periodic since $\rho_n$ is periodic, and the period of $\omega_n$ cannot exceed $p$, since $\omega_{pk} = 0$ for each non-negative integer $k$.

**Corollary 4**. *If* $\{v_n\}$ *is periodic with period* $p \geq 1$*, then the sequence* $\{x_n - \overline{v}n\}$ *is also periodic with period at most* $p$*. In particular,* $\{x_n\}$ *is periodic (hence bounded) if and only if* $\overline{v} = 0$.

The next result in particular shows that unlike the case $c < 1$, under conditions implying boundedness of all solutions, (1) *typically does not have an absorbing interval.*

**Theorem 6**. *Assume that there exists a constant* $\alpha \in (0, 1)$ such that $|f(u)| \leq \alpha\,|u|$ *for all* $u$*. Then every* $\{v_n\}$ *converges to zero and every* $\{x_n\}$ *is bounded and converges to a real number that is determined by the initial conditions* $x_0, x_1$.

*Proof*. Note that $|v_{n+1}| = |f(v_n)| \leq \alpha\,|v_n|$ for all $n \geq 1$. It follows inductively that $|v_n| \leq \alpha^n\,|v_1|$, and hence,

$$\sum_{k=1}^{n} |v_k| \leq |v_1| \sum_{k=1}^{n} \alpha^k \leq \frac{\alpha\,|v_1|}{1-\alpha}$$

which implies that the series $\sum_{n=1}^{\infty} |v_n|$ converges. It follows at once that $\{v_n\}$ must converge to zero and that $\{x_n\}$ is bounded and in fact converges to the real number $x_0 + \sum_{n=1}^{\infty} v_n$.

# 3 Complex Behavior

Under the conditions of Corollary 3, the unique equilibrium

$$\bar{x} = \frac{A_0 + C_0 + G_0}{1 - c}$$

of (6) is repelling (or expanding) but it is not a snap-back repeller (see Marotto [8] or Sedaghat [13] for a definition). This is due to Condition (c) in Corollary 1. Indeed, with a Goodwin function numerical simulations tend to generate quasi-periodic rather than chaotic trajectories. However, if we do not assume that $f$ is increasing, more varied and complex types of behavior are possible. This is the case in Puu's model, which we describe next.

## 3.1 Chaos and Puu's model

The number $s = 1 - c \in (0, 1]$ is called the *marginal propensity to save*, or MPS for short. In each period $n$, a percentage of income $sY_n$ is saved in the Samuelson-Hicks-Goodwin models and is never consumed in future periods - hence, savings are said to be "eternal." At the opposite extreme, we have the case where the savings of a given period are consumed entirely within the next period (Puu [9] Chapter 6). Puu suggested an investment function in the form of a cubic polynomial $Q$ seen in the following type of difference equation

$$y_n = (1 - s)y_{n-1} + sy_{n-2} + Q(y_{n-1} - y_{n-2}) \tag{11}$$

where $Q$ is the cubic polynomial $Q(t) \doteq at(1 - bt - t^2)$, $b > 0$, $a > s$. Puu took $b = 0$ (which makes $Q$ symmetric with respect to the origin); but as we will see later, this restriction is problematic (see the remarks on growth and viability below). Equation (11) may alternatively be written as follows

$$y_n = y_{n-1} + P(y_{n-1} - y_{n-2}) \tag{12}$$

in which we call the (still cubic) function

$$P(t) \doteq Q(t) - st = t(a - s - abt - at^2)$$

*Puu's (asymmetric) investment function.* Note that (12) is of the form (1) with $c = 1$ as in the preceding section. Therefore, each solution $\{y_n\}$ of (12) is expressible as the series $y_n = y_0 + \sum_{k=0}^{n-1} z_k$ where $\{z_n\}$ is a solution of the first order initial value problem

$$z_n = P(z_{n-1}), \quad z_0 \doteq y_1 - y_0. \tag{13}$$

Each term $z_n$ is just the forward difference $y_{n+1} - y_n$, and a solution of (13) gives the sequence of output or income differences for (12). In order to study the dynamics of Equations (12) and (13), we gather some basic information about $P$. Using elementary calculus, it is easily found that the real function $P$ has two critical points

$$\xi^{\pm} = \frac{1}{3}\left[-b \pm \sqrt{b^2 + 3\left(1 - \frac{s}{a}\right)}\right]$$

with $\xi^- < 0 < \xi^+$. Similarly, $P$ has three zeros, one at the origin and two more given by

$$\zeta^{\pm} = \frac{1}{2}\left[-b \pm \sqrt{b^2 + 4\left(1 - \frac{s}{a}\right)}\right].$$

Further, if $a > s + 1$, then for all $b > 0$, $P$ has three fixed points, one at the origin and two more given by

$$t^{\pm} = \frac{1}{2}\left[-b \pm \sqrt{b^2 + 4\left(1 - \frac{s+1}{a}\right)}\right].$$

**Remarks**. (*Growth and Viability Criteria*) Assume that the following inequalities hold:

$$0 < P^2(\xi^-) \leq P(\xi^+). \tag{14}$$

Then it is not hard to see that $P(\xi^-) < \zeta^-$ and that the interval

$$I \doteq [P(\xi^-), \max\{\zeta^+, P(\xi^+)\}]$$

is invariant under $P$. We refer to inequalities (14) as the *viability criteria* for Puu's model as they prevent undesirable outcomes such as negative income. See Figure 1.

Next, suppose that

$$P(\xi^+) \leq \zeta^+, \quad \text{or equivalently,} \quad P^2(\xi^+) \geq 0. \tag{15}$$

In this case, the right half $I^+ \doteq [0, \zeta^+]$ of $I$ is invariant under $P$, and it follows that the income sequence $\{y_n\}$ is eventually increasing. For this

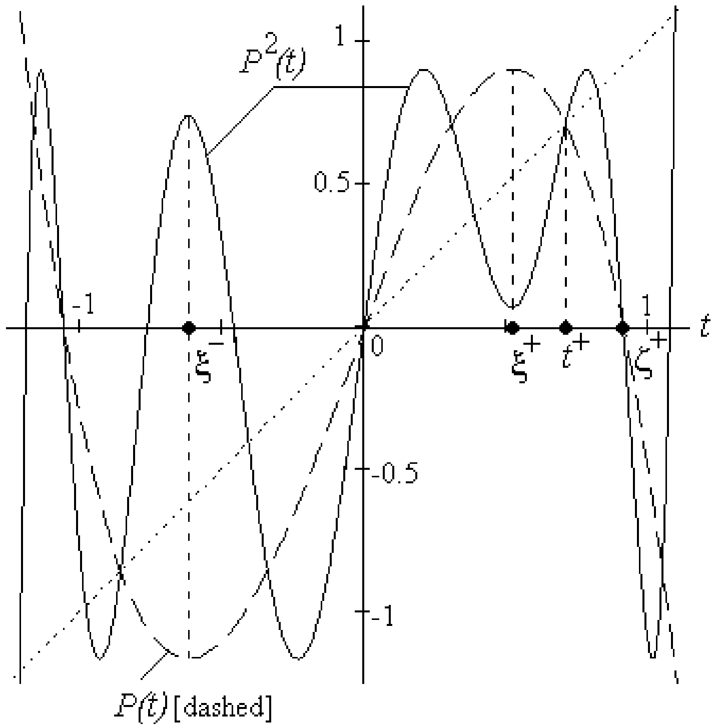


Figure 1: A viable Puu investment function

reason, we refer to condition (15) as the *steady growth condition.* The function depicted in Figure 1 satisfies both the steady growth and the viability criteria.

The proofs of (a) and (b) in the next corollary follow from Theorem 5. For a proof of the rest and some examples, see Sedaghat [13].

**Corollary 5**. (Steady growth) *Assume that inequalities* (14) *and* (15) *hold. Also suppose that* $a > s+1$ *and* $y_0 - y_{-1} \in I^+$. *Then the following statements are true*:

(a) *If* $P'(t^+) < 1$, *then each non-constant solution* $\{y_n\}$ *of* (12) *is increasing and the difference* $|y_n - t^+ n|$ *approaches a constant as* $n \to \infty$.

(b) *If* $P'(t^+) > 1$ *and* $\{v_1, \ldots, v_k\}$ *is a limit cycle of* (13), *then each non-constant solution* $\{y_n\}$ *of* (12) *is increasing and the difference* $|y_n - \bar{v}n|$

*approaches a periodic sequence* $\{\omega_n\}$ *of period at most* $k$*, where*

$$\bar{v} \doteq \frac{1}{k}\sum_{i=1}^{k} v_i, \quad \omega_n \doteq \alpha + \bar{v}\rho_n - \sum_{i=0}^{\rho_n} v_i, \ (v_0 \doteq 0)$$

*with* $\alpha$ *a constant, and* $\rho_n$ *the remainder resulting from the division of* $n$ *by* $k$*.*

(c) *If* $P$ *has a snap-back repeller (e.g., if it has a 3-cycle) then each non-constant solution* $\{y_n\}$ *of* (12) *is increasing, the corresponding difference sequence* $\{\Delta y_n\}$ *is bounded, and for an uncountable set of initial values, chaotic.*

**Remarks**. 1. The preceding result shows that unlike the Hicks-Goodwin model, Puu's model (and hence, (1) with $c = 1$) is capable of generating endogenous growth (i.e., without external input). Under the conditions of Corollary 5(c), this growth occurs at an unpredictable rate. The implication that the existence of a 3-cycle implies chaotic behavior was first established in the well-known paper Li and Yorke [7]. The existence of 3-cycles implies the existence of snap-back repellers [8]. Figure 2 shows a situation where the fixed point is a snap-back repeller because it is unstable yet a nearby point $t_0$ moves into it.

Chaotic behavior may be observed in the output trajectory $\{y_n\}$ itself and not just in its rate sequence. For instance, if inequalities (14) hold but (15) does not, then $I$ is invariant but not $I^+$. Hence, $\Delta y_n$ is negative (and positive) infinitely often, and sustained growth for $\{y_n\}$ either does not occur, or if it occurs over longer stretches of time, it will not be steady or strict. See Sedaghat [13] for an example of this situation.

## 3.2 Strange behavior

Going in a different direction, note that by (i) and (ii) in Definition 3 a Goodwin function can exist only if $A_0 + C_0 + G_0 > 0$. To study the consequences of the equality $A_0 + C_0 + G_0 = 0$, we replace (ii) in Definition 3 by:

(ii)′ $H$ is non-decreasing everywhere on $\mathbb{R}$, and it is strictly increasing on an interval $(0, \delta)$ for some $\delta > 0$;

Here we are using $H$ rather than $G$ to denote the more general type of investment function that (ii)′ allows. The next corollary identifies an important difference between the smooth and non-smooth cases.

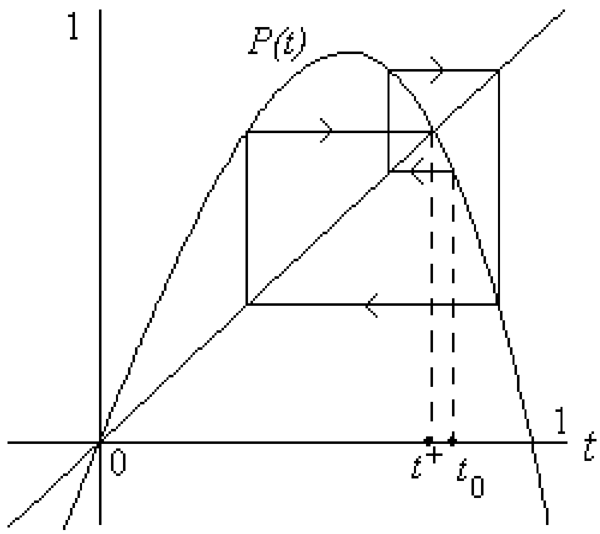


Figure 2: A snap-back repeller in Puu's investment

**Corollary 6**. (Economic ruin) *Assume that* $A_0 + C_0 + G_0 = 0$. *Then every solution of*

$$y_n = cy_{n-1} + H(y_{n-1} - y_{n-2}) \tag{16}$$

*converges to zero, eventually monotonically. Moreover, the following is true:*

(a) *If* $H \in C^1(\mathbb{R})$, *then* $H'(0) = 0$ *and the origin is locally asymptotically stable. Thus, the income trajectory stays near the origin if the initial income difference is sufficiently small.*

(b) *If* $H(t) = bt$ *on an interval* $(0, r)$ *for some* $r > 0$ *and*

$$b \geq \left(1 + \sqrt{1-c}\right)^2 \tag{17}$$

*then the origin is not stable. If* $0 = y_{-1} < y_0 < r$ *then the income trajectory* $\{y_n\}$ *is increasing, moving away from the origin until* $y_n - y_{n-1} > r$, *no matter how close* $y_0$ *is to zero.*

*Proof.* If $A_0 + C_0 + G_0 = 0$, then $H(t) = 0$ for $t \leq 0$. Now, there are two possible cases: (I) Some solution $\{y_n\}$ of (16) is strictly increasing as $n \to \infty$, or (II) For every solution there is $k \geq 1$ such that $y_{k-1} \geq y_k$. Case (I) cannot occur for positive solutions, since by Theorem 2 the increasing trajectory has a bounded limit $\tilde{y}$ with

$$\tilde{y} = \lim_{n\to\infty} [cy_{n-1} + H(y_{n-1} - y_{n-2})] = c\tilde{y} + H(0) = c\tilde{y},$$

which implies that $\tilde{y} = 0$. For $y_{-1}, y_0 < 0$ the sequence $\{y_n\}$ is increasing since

$$y_{n+1} = cy_n + H(y_{n-1} - y_{n-2}) \geq cy_n > y_n$$

as long as $y_n$ remains negative. Thus either $y_n \to 0$ as $n \to \infty$, or $y_n$ must become positive, in which case the preceding argument applies. In case (II), we find that

$$y_{k+1} = cy_k < y_k$$

so that, proceeding inductively, $\{y_n\}$ is strictly decreasing for $n \geq k$. Since the origin is the only fixed point of (16), it follows that $y_n \to 0$ as $n \to \infty$.

Next, suppose that (a) holds. If $H(t)$ is constant for $t \leq 0$, then $H'(t) = 0$ for $t < 0$, and thus, $H'(0) = 0$ if $H'$ is continuous. Thus the linearization of (16) at the origin has eigenvalues 0 and $c$, both with magnitude less than 1.

Now assume that (b) is true. On the interval $(0, r)$, a little algebraic manipulation shows that due to condition (17), the eigenvalues $\lambda_1$ and $\lambda_2$ of the linear equation

$$y_{n+1} = cy_n + b(y_n - y_{n-1}) = (b + c)y_n - by_{n-1} \tag{18}$$

are real and that

$$0 < \lambda_1 = \frac{b + c - \sqrt{(b + c)^2 - 4b}}{2} < 1 < \lambda_2 = \frac{b + c + \sqrt{(b + c)^2 - 4b}}{2}.$$

With initial values $y_{-1} = 0$ and $y_0 \in (0, r)$, the corresponding solution of the linear equation (18) is

$$y_n = \frac{y_0}{\sqrt{(b + c)^2 - 4b}} \left(\lambda_2^{n+1} - \lambda_1^{n+1}\right)$$

which is clearly increasing exponentially away from the origin, at least until $y_n$ (hence also the difference $y_n - y_{n-1}$ since $y_0 - y_{-1} = y_0$) exceeds $r$ and $H$ assumes a possibly different form. The instability of the origin is now clear.

**Remark**. (*Unstable global attractors*) Under conditions of Corollary 6(b), the origin is evidently a globally attracting equilibrium which however, is not stable. This is a consequence of the non-smoothness of the Hicks-Goodwin map at the origin, since in Part (a), where the map $H$ is smooth, the origin is indeed stable.

**Remark**. (*Off-equilibrium oscillations*) Note that the mapping $H(t)$ of Corollary 6(b) has its minimum value at the origin, which is also the unique fixed point or equilibrium of the system. If the mapping $f$ in (1) is characterized by this property, then solutions of (1) are in general capable of exhibiting other types of strange behavior that do not occur with non-decreasing maps of type $H$.

Suppose that $f$ has a global (though not necessarily unique) minimum at the origin and without loss of generality, assume that $f(0) = 0$. Then the origin is the unique fixed point of (1). Clearly, if (1) exhibits oscillatory behavior in this case, then such oscillations occur *off-equilibrium*, i.e., they do not occur about the equilibrium or fixed point. In particular, if $x_n$ is a solution exhibiting such oscillations, then its limit superior is distinct from its limit inferior. As a very simple example of this sort of oscillation, it is easy to verify that with

$$f(t) = \min\{|t|, 1\}, \quad c = 0$$

(1) has a periodic solution $\{0, 1, 1\}$ exhibiting off-equilibrium oscillations with limit superior 1 and limit inferior 0. However, off-equilibrium oscillations can generally be quite complicated (and include chaotic behavior) with non-monotonic $f$; see the example of non-monotonic convergence after Conjecture 1 below. For examples not involving convergence see Sedaghat [13].

# 4 Global Attractivity

In this section we consider various conditions that imply the global attractivity of the unique fixed point of (1), namely, $\bar{x} = f(0)/(1-c)$. Throughout this section it is assumed that $0 \leq c < 1$. We begin with a condition on $f$ under which the origin is globally asymptotically stable. We need a result from Sedaghat [12] which we quote here as a lemma.

**Lemma 3**. *Let $g : \mathbb{R}^m \to \mathbb{R}$ be continuous and let $\bar{x}$ be an isolated fixed point of*

$$x_{n+1} = g(x_n, x_{n-1}, \ldots, x_{n-m}).$$

*Also, assume that for some $\alpha \in (0,1)$ the set*

$$A_\alpha = \{(u_1, \ldots, u_m) : |g(u_1, \ldots, u_m) - \bar{x}| \leq \alpha \max\{|u_1 - \bar{x}|, \ldots, |u_m - \bar{x}|\}$$

*has a nonempty interior (i.e., $g$ is not very steep near $\bar{x}$) and let $r$ be the largest positive number such that $[\bar{x}-r, \bar{x}+r]^m \subset A_\alpha$. Then $\bar{x}$ is exponentially stable relative to the interval $[\bar{x} - r, \bar{x} + r]$.*

The function $g$ in Lemma 3 is said to be a *weak contraction* on the set $A_\alpha$; see Sedaghat [13] for further details about weak contractions and weak expansions. As a corollary to Lemma 3 we have the following simple, yet general fact about equation (1).

**Theorem 7**. *If $|f(t)| \leq a|t|$ for all $t$ and $0 < a < (1-c)/2$ then the origin is globally attracting in* (1).

*Proof.* The inequality involving $f$ in particular implies that $f(0) = 0$, so that the origin is the unique fixed point of (1). Define $g(x,y) = cx + f(x-y)$ and notice that

$$\begin{aligned}|g(x,y)| &\leq c|x| + a|x-y| \\ &\leq (c+a)|x| + a|y| \\ &\leq (c+2a)\max\{|x|,|y|\}.\end{aligned}$$

Since $c + 2a < 1$ by assumption, it follows that $g$ is a weak contraction on the entire plane and therefore, Lemma 3 implies that the origin is globally attracting (in fact, exponentially stable) in (1).

**Remark**. If $f(0) = 0$ and $f$ is continuously differentiable with derivative bounded in magnitude by $a$ or more generally, if $f$ satisfies the Lipschitz inequality

$$|f(t) - f(s)| \leq a|t-s|$$

then in particular (with $s = 0$), $|f(t)| \leq a|t|$ for all $t$. However, if $f$ satisfies the conditions of Theorem 7 then it need not satisfy a Lipschitz inequality. Theorems 8 and 9 below improve the range of values for $a$ in Theorem 7 with the help of extra hypotheses.

## 4.1 When $f$ is minimized at the origin

In this sub-section we look at the case where $f$ has a global minimum (not necessarily unique) at the origin. These types of maps, which include *even* functions that are minimized at the origin, were noted in the previous section when remarking on off-equilibrium oscillations. It will not be any loss of generality to assume that $f(0) = 0$ in the sequel, so that $\bar{x} = 0$. This will simplify the notation. We begin with a simple result about the non-positive solutions.

**Lemma 4**. *If $f(t) \geq 0$ for all $t$ and $f(0) = 0$ then every non-positive solution of* (1) *is nondecreasing and converges to zero.*

Given Lemma 4 and the fact that if $x_k > 0$ for some $k \geq 0$ then $x_n > 0$ for all $n \geq k$, it is necessary to consider only the positive solutions. Before stating the main result of this section, we need another version of Lemma 3 above which we quote here as a lemma. See Sedaghat [12] or Sedaghat [13] for a proof.

**Lemma 5**. *Let $g : \mathbb{R}^m \to \mathbb{R}$ be continuous and let $\bar{x}$ be an isolated fixed point of*

$$x_{n+1} = g(x_n, x_{n-1}, \ldots, x_{n-m}).$$

*Let $V_g(u_1, \ldots, u_m) = (g(u_1, \ldots, u_m), u_1, \ldots, u_{m-1})$ and for $\alpha \in (0, 1)$ define the set*

$$A_\alpha = \{(u_1, \ldots, u_m) : |g(u_1, \ldots, u_m) - \bar{x}| \leq \alpha \max\{|u_1 - \bar{x}|, \ldots, |u_m - \bar{x}|\}$$

*If $S$ is a subset of $A_\alpha$ such that $V_g(S) \subset S$ and $(\bar{x}, \ldots, \bar{x}) \in S$, then $(\bar{x}, \ldots, \bar{x})$ is asymptotically (in fact, exponentially) stable relative to $S$.*

The next theorem is from Sedaghat [14].

**Theorem 8**. *Let $0 \leq f(t) \leq a|t|$ for all $t$.*

(a) *If $a < 1 - c$, then every positive solution $\{x_n\}$ of* (1) *converges to zero.*

(b) *If $a < c$ then every positive solution $\{x_n\}$ of* (1) *eventually decreases monotonically to zero.*

(c) *If $a < \max\{c, 1 - c\}$ then the origin is globally attracting.*

*Proof.* (a) Assume that $a < 1 - c$. Define $g(x, y) = cx + f(x - y)$ and for $x, y \geq 0$ notice that

$$\begin{aligned} g(x, y) &\leq cx + a|x - y| \\ &\leq cx + a \max\{x,\, y\} \\ &\leq (c + a) \max\{x\,, y\}. \end{aligned}$$

Since $c + a < 1$ by assumption, it follows that $g$ is a weak contraction on the non-negative quadrant, i.e.,

$$[0, \infty)^2 \subset A_{a+c}.$$

Since $[0, \infty)^2$ is invariant under $g$, Lemma 5 implies that the origin is exponentially stable relative to $[0, \infty)^2$. Thus every positive solution $\{x_n\}$ of (1) converges to zero.

(b) Let $\{x_n\}$ be a positive solution of (1). Then the ratios

$$r_n = \frac{x_n}{x_{n-1}}, \quad n \geq 0$$

are well defined and satisfy

$$r_{n+1} = c + \frac{f(x_n - x_{n-1})}{x_n} \leq c + \frac{a|x_n - x_{n-1}|}{x_n} = c + a\left|1 - \frac{1}{r_n}\right|.$$

Since it is also true that $r_{n+1} = c + f(x_n - x_{n-1})/x_n \geq c$ we have

$$c \leq r_{n+1} \leq c + a\left|1 - \frac{1}{r_n}\right|, \quad n \geq 0.$$

If $r_1 \leq 1$ then since $r_1 \geq c$, we have

$$c \leq r_2 \leq c - a + \frac{a}{r_1} \leq c - a + \frac{a}{c} < 1$$

where the last inequality holds because $a < c < 1$. Inductively, if for $k \geq 2$,

$$c \leq r_n < 1, \quad n < k$$

then

$$c \leq r_k \leq c - a + \frac{a}{c} < 1$$

so that

$$r_1 \leq 1 \Rightarrow r_n < 1 \quad \text{for all} \quad n > 1. \tag{19}$$

Now suppose that $r_1 > 1$. Then

$$c \leq r_2 \leq c + a - \frac{a}{r_1} < c + a.$$

If $c + a \leq 1$, then $r_2 < 1$ and (19) holds. Assume that $a + c > 1$ and $r_2 > 1$. Then

$$r_3 \leq c + a - \frac{a}{r_2} < r_2.$$

The last inequality holds because for every $r > 1$, $c + a - a/r < r$ if and only if

$$r^2 - (c + a)r + a > 0. \tag{20}$$

Inequality (20) is true because the quadratic on its left side can have zeros only for $r \leq 1$. Now, if $r_3 < 1$, then (19) holds for $n > 2$. Otherwise, using (20) we can show inductively that

$$r_1 > r_2 > r_3 > \cdots$$

so there is $k \geq 1$ such that $r_k \leq 1$ and (19) applies with $n > k$. Hence, we have shown that for any choice of $r_0$, the sequence $r_n$ is eventually less than 1; i.e., $x_n < x_{n-1}$ for all $n$ sufficiently large and the proof is complete.

(c) Immediate from Parts (a) and (b) above and Lemma 4.

Figure 3 shows the parts (shaded) of the unit square in the $(c, a)$ parameter space for which global attractivity is established so far.

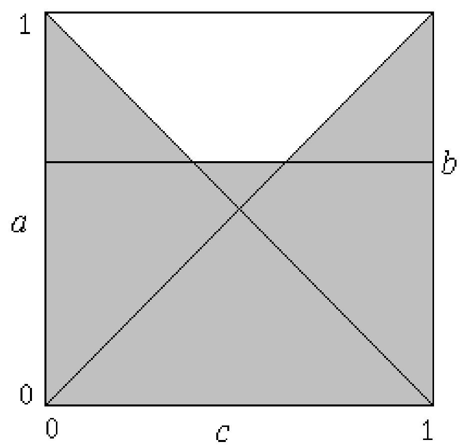


Figure 3: Global attractivity regions when $f$ is minimized at 0

The diagonal lines represent $a = 1 - c$ and $a = c$. The horizontal line $a = b$ in the middle of the diagram comes from Sedaghat [14] where it is shown that if

$$1 - b < a, c < b, \quad b = 2/(\sqrt{5} + 1),$$

then the origin is globally asymptotically stable. Numerical simulations indicate that the origin is possibly attracting for points in the unshaded region of Figure 3 also. In fact, the following seems to be true:

**Conjecture 1**. *If* $0 \leq f(t) \leq a|t|$ *for all* $t$, *then the origin is globally attracting if* $0 \leq a < 1$.

The next example shows that convergence in Part (a) of Theorem 8 need not be monotonic.

**Example**. Let $c < 1/2$ and let $f(t) = a|t|$ with $c < a < 1 - c$. By Theorem 8 every solution $\{x_n\}$ of (1) converges to zero. Let $r_n$ be the ratio defined in the proof of Theorem 8(b) and define the mapping

$$\phi(r) = c + a\left|1 - \frac{1}{r}\right|, \quad r > 0.$$

Then $r_{n+1} = \phi(r_n)$ for all $n \geq 0$ and $\phi$ has a unique positive fixed point

$$\bar{r} = \frac{1}{2}\left[\sqrt{(a-c)^2 + 4a} - (a-c)\right] < \sqrt{a}$$

which is unstable because $|\phi'(\bar{r})| = a/\bar{r}^2 > 1$. Suppose that $r_0 < 1$, i.e., $x_0 < x_{-1}$ but $r_0 \neq \bar{r}$. Then $r_1 = \phi(r_0) = \phi_1(r_0)$ where $\phi_1$ is the decreasing function

$$\phi_1(r) = c - a + \frac{a}{r}.$$

Since $\phi_1(r) > 1$ for $r \in (0, r^*)$ where $r^* = a/(1 + a - c)$, it follows that either $r_1 > 1$ or some iterate $r_k = \phi_1^k(r_0) > 1$. This means that $x_k > x_{k-1}$ while $x_1 > x_2 > \cdots > x_{k-1}$. Next, $r_{k+1} = \phi_2(r_k)$ where $\phi_2$ is the increasing function

$$\phi_2(r) = a + c - \frac{a}{r}.$$

Since $\phi_2(r) < a + c < 1$ for all $r$ we see that $r_{k+1} < 1$ and so the preceding process repeats itself ensuring that there are infinitely many terms $x_{k_j}$, $j = 1, 2, \ldots$ where the inequality $x_{k_j+1} > x_{k_j}$ holds.

The magnitude of the up-jump depends on the parameters; since $c$ is the absolute minimum value of $\phi$ for $r > 0$, we see that

$$r_n \leq \phi(c) = \phi_1(c) = \frac{a}{c} - (a-c)$$

for all $n \geq 1$. Thus

$$x_{k_j} < x_{k_j+1} < \left[\frac{a}{c} - (a-c)\right] x_{k_j}, \quad j = 1, 2, \ldots$$

### 4.2 When $tf(t) \geq 0$

The situation we discuss in this sub-section is, in a sense, complementary to the one we considered in the preceding sub-section. For instance, the condition $tf(t) \geq 0$ is not satsfied by even functions but rather, by certain *odd* functions. However, as seen below there are some interesting parallels between the two cases. The condition $tf(t) \geq 0$ encountered in the section Other Oscillatory Behavior above, has significant consequences in the case of convergence too. We note that if $tf(t) \geq 0$, then by continuity $f(0) = 0$ so the origin is the unique fixed point of (1) in this case. The following is proved in Kent and Sedaghat [4].

**Theorem 9**. (a) *Assume that there is $a > 0$ such that $|f(t)| \leq a|t|$ and that $tf(t) \geq 0$ for all $t$. If*

$$a < \frac{2-c}{3-c} \quad \text{or} \quad a \leq 1-c$$

*then every solution of* (1) *converges to zero; i.e., the origin is globally attracting.*

(b) *Let $b = \left(1 - \sqrt{1-c}\right)^2$. If $a \leq b$ in Part (a), then every solution of* (1) *is eventually monotonic and converges to zero.*

The conditions of Theorem 9 specify the shaded region of the $(c, a)$ parameter space shown in Figure 4 below.

The line $a = 1 - c$ and the curve $a = (2 - c)/(3 - c)$ are readily identified (the latter clearly by its intercepts with $c = 0$ and $c = 1$). The third curve represents $a = b$ where $b$ is defined in Theorem 9(b). Figure 4 is analogous to Figure 3 and it leads to the following conjecture which is analogous to Conjecture 1.

**Conjecture 2**. *If $f(t) \leq a|t|$ and $tf(t) \geq 0$ for all $t$, then the origin is globally attracting if $0 \leq a < 1$.*

# 5 Concluding Remarks and Open Problems

The preceding sections shed some light on the problem of classifying the solutions of (1). However, there are also many unresolved issues, some of which are listed below as open problems and conjectures (they all refer to the case $0 \leq c < 1$):

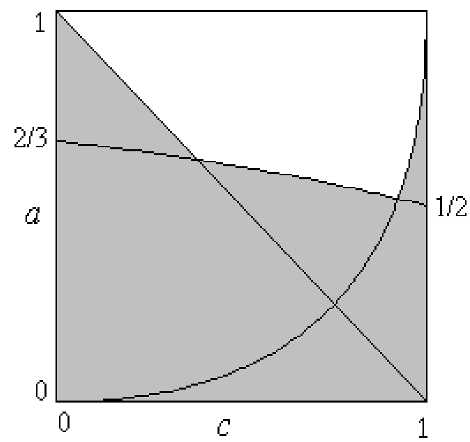


Figure 4: Global attractivity regions when $tf(t) \geq 0$

**Conjecture 3**. All solutions of (1) are bounded if and only if (1) has an absorbing interval.
**Problem 1**. Find sufficient conditions on $f$ that imply (1) has an absorbing interval when $f$ is minimized at the origin.
**Problem 2**. Find sufficient conditions on $f$ for the fixed point $\bar{x} = f(0)/(1-c)$ to be a snap-back repeller. Together with Theorem 3, this establishes the occurrence of chaotic behavior in a compact set.
**Problem 3**. Find either sufficient conditions on $f$, or specify some classes of functions $f$ for which every solution of (1) is eventually periodic or every solution approaches a periodic solution for a range of values of $c$.
**Problem 4**. Extend the results of this paper, where possible, to the more general difference equation

$$x_{n+1} = cx_n + dx_{n-1} + f(x_n - x_{n-1}), \quad c, d \in [0, 1].$$

Puu's general model is a special case of this equation with $c + d \leq 1$.

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