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RESEARCH ARTICLE

Boolean Lattices of Function Algebras on Rectangular Semigroups

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Consider the following characterization of the uniform convergence of a sequence of functions in terms of the uniform continuity of a single associated function. Let $\{g_n\}$ denote a sequence of bounded, uniformly continuous, complex-valued functions on a topological group G (assume the right uniform structure). Suppose that $\{g_n\}$ has a pointwise limit g and define a function on the direct product $\mathbf{N}_\infty \times G$ as $f(n,u) = g_n(u)$ and $f(\infty,u) = g(u)$, $u \in G$ ($\mathbf{N}_\infty$ denotes the usual one-point compactification of the positive integers $\mathbf{N}$). Note that if $\mathbf{N}_\infty$ is given the left zero multiplication (i.e., ab=a), then $\mathbf{N}_\infty \times G$ is a topological left group. Corollary 5.7 below shows that if, e.g., the group G is locally compact or complete metric, then $g_n \to g$ uniformly if and only if f is uniformly continuous on $\mathbf{N}_\infty \times G$ (with respect to the product uniformity). However, on certain other groups (such as the group $(\mathbf{Q},+)$ of the usual additive rationals) the uniform continuity of f on $\mathbf{N}_\infty \times G$ is no longer necessary for the uniform convergence of the sequence $\{g_n\}$. A weaker form of continuity is needed which coincides with uniform continuity in groups but not in left groups (2.6, 2.10). This type of continuity, which we call *left local continuity* is the main subject studied in this paper. The above characterization of uniform convergence on topological groups extends to other semitopological semigroups (and, as a degenerate case, to general topological spaces) upon suitable generalizations (Section 5). The focus here will be on the special class of semitopological semigroups known as rectangular semigroups, since such semigroups (which include left groups) possess highly non-trivial left locally continuous structures that are nevertheless amenable to substantial (though not exhaustive) analysis within a single paper.

Left local continuity is responsible for (and characterized by) the emergence of many new C*-algebras of continuous functions as one passes from group structures to more general types of semigroups. Indeed, consider the topological left group S of 3×3 matrices of the form:

$$\begin{pmatrix} 1 & 0 & 0 \\ x & 0 & 0 \\ 0 & 0 & y \end{pmatrix}, \qquad x \in K,\ y \in \mathbf{R},\ y \neq 0$$

under matrix multiplication, where K is any compact subset of the reals $\mathbf{R}$ with an infinite number of limit points (e.g., $K=[0,1]$). We show that there exists an *uncountable Boolean lattice of translation invariant* C*-*algebras of bounded, continuous functions on* S, *where each algebra contains all bounded, uniformly*

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continuous functions on S (see 2.10, 4.1 and the comments before 5.7 below). As a consequence, the class of all *universal* semigroup compactifications of S which extend the $LUC(S)$-compactification is uncountable (with cardinal number 2^c if, e.g., $K = [0,1]$, where c is the cardinal number of the continuum; see 4.5. The universal mapping property of these compactifications is discussed in [9]).

Such multitudes of C*-algebras of continuous functions on the above close relatives of topological groups simply do not materialize on the groups themselves. The closest things to these algebras could already be found in [8] (see also [2, p.175]), but it was in [9] where the above C*-algebras made their debut. Specifically, we showed in [9] that the size and structure of the class of locally continuous algebras for lattices of groups mimics the algebraic structure of the semigroup (as determined by the natural semilattice congruence), provided that the topological structure permits the application of the Joint Continuity Theorem in [6]. Boolean lattices of locally continuous algebras then correspond to similar lattices of idempotents and principal ideals in the semigroup.

In this paper we show that the algebraic determinant of the left locally continuous structure in a rectangular semigroup is the Green's relation $\mathcal{R}$ (under nominal topological hypotheses). Thus the class of locally continuous algebras forms a lattice which corresponds to the lattice of right ideals (see Sections 2 and 3 below). Before proceeding with the discussion of left local continuity however, we recall a few preliminary concepts.

A *semitopological semigroup* S is a semigroup and a (Hausdorff) topological space where the semigroup operation (often called multiplication) is separately continuous (if the multiplication is jointly continuous on $S \times S$, then we say that S is *topological*) [2]. For any semitopological semigroup S, we use the notation $C(S)$ for the set of all bounded, continuous, complex-valued functions on S. Recall that $C(S)$ is a translation invariant C*-algebra with respect to the sup-norm topology. We need here two translation invariant C*-subalgebras of $C(S)$: The algebra $LUC(S)$ of all *left uniformly continuous functions* (i.e., functions $f \in C(S)$ for which the mapping $s \mapsto L_s f : S \mapsto C(S)$ is norm continuous) and the algebra $LMC(S)$ of all $f \in C(S)$ for which the mapping $s \mapsto \mu(L_s f) : S \mapsto \mathbf{C}$ is continuous for all μ in the spectrum of $C(S)$. $LMC(S)$ is usually called the algebra of *left multiplicatively continuous functions* on S [2]. In these definitions, $L_s f$ denotes the left translation of f by s; i.e., $L_s f(t) = f(st)$ for all $t \in S$. Likewise, we define the right translation as $R_s f(t) = f(ts)$. Observe that $LUC(S) \subset LMC(S)$, and that if S is a topological group, then $LUC(S)$ coincides with the set of all functions in $C(S)$ that are uniformly continuous with respect to the right uniform structure. Detailed information on the algebras $LUC(S)$ and $LMC(S)$ may be found in the existing literature, but we need relatively little of that knowledge for our purposes here (we do mention, however, the double limit criterion for $LMC(S)$, a discussion of which may be found in [2], Chapter Four). Closer to the subject of this paper, in [3] a comparative study of the algebras LUC and LMC, along with the algebras of almost periodic functions and weakly almost periodic functions is presented on semitopological left groups.

In this paper, we will be primarily concerned with a class of algebraically simple semigroups which includes such familiar structures as the left groups and the right groups. Following conventions similar to those adopted in [3] for semitopological left groups, we define S to be a *semitopological rectangular semigroup* if it is isomorphic to a direct product $Z_l \times G \times Z_r$, where G is a semitopological group and Z_l, Z_r are, respectively, a left zero semigroup and a right zero semigroup [2]. With this convention, the topology of S is always determined by the topologies of G, Z_l and Z_r via the product topology and, as in [3], we avoid some of the technical complications that may otherwise arise in topologizing S. Also, left groups may be represented as $Z_l \times G$, right groups as $G \times Z_r$ and rectangular bands as $Z_l \times Z_r$.

1. The Lattice of Left Locally Continuous Algebras

The aim of this section is to establish the basic facts about left local continuity in semitopological semigroups in general. In particular, we generalize the results of Section 2 in [9] concerning the lattice structure. A function $f \in LMC(S)$ (S is a semitopological semigroup) is said to be *left locally continuous at* $a \in S$ if the mapping

$$s \mapsto L_s f : S \mapsto C(S)$$

is continuous at the point a relative to the uniform topology on $C(S)$. The set of all functions of this type is denoted by $LLC(S, a)$. Further, if A is a non-empty subset of S, we define

$$LLC(S, A) = \bigcap\{LLC(S, a) : \ a \in A\}.$$

It is clear that $LUC(S) = LLC(S, S) \subset LLC(S, a)$ for every $a \in S$, and if $A \subset B$, then $LLC(S, B) \subset LLC(S, A)$. It is thus reasonable to define $LLC(S, \emptyset) = LMC(S)$ where $\emptyset$ represents the empty set.

Remark. The restriction in the above definition that every left locally continuous function be a member of $LMC(S)$ is adopted from [9], where the use of semigroup compactifications required the restriction. By continuing to adhere to the original definition here, in addition to preserving consistency, with a little extra effort we succeed in demonstrating the existence of large lattices of function algebras within $LMC(S)$ and not just within $C(S)$. The loss of generality that results from such a restriction is inconsequential at this stage, where little is known about LLC structures, while at the same time it may be desirable to learn more about $LMC(S)$. In fact, rather than seeking the most general hypotheses possible, we aim to establish throughout much of this paper that highly non-trivial lattices of subalgebras arise within $LMC(S)$ on fairly "commonplace" semigroups.

The routine proof of the following basic lemma is ommited.

Lemma 1.1. (a) *For every non-empty family $\mathcal{F}$ of subsets of S,*

$$LLC(S, \cup\mathcal{F}) = \bigcap\{LLC(S, A) : \ A \in \mathcal{F}\},$$

$$LLC(S, \cap\mathcal{F}) \supset \bigcup\{LLC(S, A) : \ A \in \mathcal{F})\}.$$

(b) *For every subset $A \subset S$, $LLC(S, A)$ is a right translation invariant C*-subalgebra of $C(S)$.*

(c) *For semitopological semigroups S and T, let $\theta : S \mapsto T$ be a continuous homomorphism. If θ^* denotes the dual of θ, then $\theta^* LLC(T, \theta A) \subset LLC(S, A)$.* ∎

In the light of 1.1(b) above, we may refer to each $LLC(S,A)$, $A \subset S$, as a *left locally continuous algebra* on the semigroup S.

Let S be a semitopological semigroup, and for each pair of subsets A and B of S, define a relation ρ in the power set $\mathcal{P}(S)$ by

$$A\rho B \quad \text{if and only if} \quad LLC(S,A) = LLC(S,B).$$

Clearly ρ is an equivalence relation. Let $[A]$ represent the ρ-cell (or equivalence class) of the subset $A \subset S$. Lemma 1.1(a) implies that $\cup[A] \in [A]$, so *we have a unique representation of each ρ-cell by its maximum element.* Let $\mathcal{E}(S)$ denote the set of all such maximal representatives for ρ, and note that $\mathcal{E}(S)$ is non-empty since it always contains S. Further, let $\mathcal{F}$ be any non-empty subset of $\mathcal{E}(S)$, and note that if $A \in \mathcal{F}$ then $\cup[A] = A$. From 1.1(a)

$$\bigcup\{LLC(S,A) : A \in \mathcal{F}\} \subset LLC(S, \cap\mathcal{F}) = LLC(S, \cup[\cap\mathcal{F}])$$

so that $\cup[\cap\mathcal{F}] \subset A$ for every $A \in \mathcal{F}$. It follows that $\cap\mathcal{F} = \cup[\cap\mathcal{F}] \in \mathcal{E}(S)$. Hence, $\mathcal{E}(S)$ is closed under intersections and the next theorem follows (via [1, p. 489]).

Theorem 1.2. *If S is a semitopological semigroup, then $\mathcal{E}(S)$ is a complete lattice.* ■

Theorem 1.2 shows that what we called "semilattices" in Section 2 of [9] are in fact much more (but also see 1.3 below). In $\mathcal{E}(S)$ the join of a family $\mathcal{F}$ defined as $\vee\mathcal{F} = \cap\{B \in \mathcal{E}(S) : \cup\mathcal{F} \subset B\}$ can easily be shown to equal $\cup[\cup\mathcal{F}]$; i.e., the join of a subset of $\mathcal{E}(S)$ is the maximal element of the ρ-cell of its union.

We sometimes use the notation $(\mathcal{E}(S), \cap, \vee)$ below for the lattice of Theorem 1.2. Notation-wise, the order in which $\cap$ and $\vee$ appear will be important in later comparisons. Notice that

$$\{LLC(S,A) : A \subset S\} = \{LLC(S,B) : B \in \mathcal{E}(S)\}.$$

We denote the above class of all left locally continuous algebras on S by $LLC(S)$, and note that distinct members of $LLC(S)$ are in one-to-one correspondence with the members of $\mathcal{E}(S)$. For any family $\mathcal{F}$ of sets in $\mathcal{E}(S)$, we have

$$\bigcap\{LLC(S,A) : A \in \mathcal{F}\} = LLC(S, \cup\mathcal{F}) = LLC(S, \vee\mathcal{F}).$$

Also, it is easy to check that the join of LLC algebras defined by

$$\bigvee\{LLC(S,A) : A \in \mathcal{F}\} = LLC(S, \cap\mathcal{F})$$

indeed represents the supremum (in $LLC(S)$) of the family $\{LLC(S,A) : A \in \mathcal{F}\}$. It follows that *$(LLC(S), \vee, \cap)$ is a complete lattice of C^*-subalgebras of $C(S)$, which is isomorphic to the lattice $(\mathcal{E}(S), \cap, \vee)$ of subsets of S.* We will use the term *LLC-lattice*, or the *lattice of left local continuity*, in refering to either one of the isomorphic lattices $(\mathcal{E}(S), \cap, \vee)$ or $(LLC(S), \vee, \cap)$.

At this point it may be relevant to take advantage of the equality in the first statement of 1.1(a) and prove a stronger result for $LLC(S)$. The next corollary shows, in particular, that every *semilattice* $(LLC(S), \cap)$ (or equivalently, $(\mathcal{E}(S), \vee)$) is a homomorphic image of the power set semilattice $(\mathcal{P}(S), \cup)$.

Corollary 1.3. *Let S be a semitopological semigroup. Then ρ is a congruence in the semilattice $(\mathcal{P}(S), \cup)$, and the quotient semigroup $\mathcal{P}(S)/\rho$ is isomorphic to the semilattice $(LLC(S), \cap)$.*

Proof. That ρ is a congruence follows from Lemma 1.1(a). Since multiplication in $\mathcal{P}(S)/\rho$ is given by $[A][B] = [A \cup B]$, the mapping $[A] \mapsto LLC(S, A) : \mathcal{P}(S)/\rho \mapsto LLC(S)$ is the required isomorphism. ∎

Evidently S in $\mathcal{E}(S)$ corresponds to the C*-algebra $LUC(S)$ in $LLC(S)$. Similarly, the maximal algebra $LMC(S)$ in $LLC(S)$ corresponds to $\cup[\emptyset]$, namely, the maximal element of the ρ-cell of the empty set. The set $\cup[\emptyset]$, which may be non-empty (see, e.g., 3.3 below) is the largest subset of S having the property that *every* member of $LMC(S)$ is left locally continuous at each of its points. The following definitions are useful for classification purposes.

1.4 Definitions and Remarks. Let S be a semitopological semigroup.

(a) If $LLC(S)$ is a singleton (equivalently, if $LMC(S) = LUC(S)$ or if $\cup[\emptyset] = S$), then we say that S is *LLC-trivial.*

(b) If $LLC(S) = \{LUC(S), LMC(S)\}$ (equivalently, if $\mathcal{E}(S) = \{\cup[\emptyset], S\}$), then we say that S is *LLC-simple.* Note that every LLC-trivial semigroup is LLC-simple.

Examples of LLC-trivial and LLC-simple semigroups include many elementary or familiar cases, such as semitopological groups or compact topological semigroups. Some examples of semigroups which are not LLC-simple (including compact semitopological semigroups) appear in [9]. Also many examples of rectangular semigroups below are locally compact topological semigroups that are not LLC-simple.

(c) For each pair of elements $a, b \in S$, we define $a\rho_0 b$ if $LLC(S, a) = LLC(S, b)$. Then ρ_0 is an equivalence relation in S (rather than in the power set $\mathcal{P}(S)$), and for each $s \in S$, the ρ_0-cell $[s]$ is precisely the union of all the singletons in the ρ-cell $[\{s\}]$.

By 1.1(a) $[s] \in [\{s\}]$, implying $[s] \subset \cup[\{s\}]$. This containment is usually strict: In fact, it is easy to see that for each $s, t \in S$, we have

$t \in \cup[\{s\}]$ *if and only if* $LLC(S, s) \subset LLC(S, t)$

whereas, by definition,

$t \in [s]$ *if and only if* $LLC(S, s) = LLC(S, t)$.

Unlike its sister relation ρ whose cells have a natural representation by unions, in general there is no explicit way of selecting a well-defined representative from each ρ_0-cell. Still, for certain types of semigroups ρ_0 has very interesting properties. In the case of rectangular semigroups, as we will see later ρ_0 is often identified with the Green's relation $\mathcal{R}$.

The next lemma is a general result concerning the transformations of semitopological semigroups and their LLC structures.

Lemma 1.5. *Let S and T be semitopological semigroups, and let $\theta : S \to T$ be a continuous homomorphism. Suppose that there is a subsemigroup $S_1 \subset S$ which is topologically isomorphic to T under θ. Then for every non-empty $A_1 \subset S_1$ and every $A \subset S$, $LLC(S, A) \subset LLC(S, A_1)$ implies $LLC(T, \theta A) \subset LLC(T, \theta A_1)$.*

Proof. Let $f \in LLC(T, \theta A)$, let a be any point of A_1 and suppose that $\{t_\eta\}$ is a net in T that converges to $\theta(a)$. Since $\theta^* LLC(T, \theta A) \subset LLC(S, A)$, it follows that $\theta^* f \in LLC(S,a)$. Now if $\theta' : T \to S_1$ is the inverse of $\theta|_{S_1}$, then

$$\begin{aligned}\|L_{t_\eta} f - L_{\theta(a)} f\| &= \sup_{y\in T} \left|L_{\theta'(t_\eta)}\theta^* f(\theta'(y)) - L_a \theta^* f(\theta'(y))\right| \\ &\le \|L_{\theta'(t_\eta)}\theta^* f - L_a\theta^* f\|.\end{aligned}$$

Hence, $f \in LLC(T,\theta(a))$, and due to the arbitrary nature of $a \in A_1$, $f \in LLC(T,\theta A_1)$. ∎

Lemma 1.5 applies naturally to the projections of certain subsemigroups of direct products (e.g., subdirect products) onto various subspaces, and also to the projection onto one of the coordinate semigroups in a semidirect product. Since rectangular semigroups are expressed as direct products, the following special case of 1.5 will be very useful to us in this paper.

Corollary 1.6. *Let $S = H \times T$ be a direct product of the semitopological semigroups H and T. If $A_1, A_2 \subset H$, $A_2 \neq \emptyset$ and $B_1, B_2 \subset T$, such that $LLC(S, A_1 \times B_1) \subset LLC(S, A_2 \times B_2)$, then $LLC(T, B_1) \subset LLC(T, B_2)$.* ∎

Of course, if $S = H \times T$ is a direct product, then H and T play symmetric roles in 1.6. This situation occurs naturally in rectangular semigroups which are defined as direct products, and explains the usefulness of 1.6 in the sequel.

2. The *LLC* Structure of Rectangular Semigroups

In this section we investigate the structure of the LLC-lattice for semitopological rectangular semigroups. In particular, we show that right groups are always LLC-simple whereas with even the most common topologies, rectangular semigroups (including left groups and rectangular bands) can have large power sets as their LLC-lattice. It is a curious fact that such elaborate local structure can exist in simple semigroups like general rectangular semigroups, given that (Corollary 2.2 below) the *right* simple right groups display no such structure. In spite of its simple proof, the next lemma is of central importance and highlights the significance of the right ideals.

Lemma 2.1. *Let $S = Z_l \times G \times Z_r$ be a rectangular semigroup. Then $LLC(S,A) = LLC(S, A_1 \times G \times Z_r)$, for $A \subset S$ with A_1 the projection of A into Z_l.*

Proof. The stated equality clearly holds if A is empty. For non-empty A, let $(a,b,c) \in A_1 \times G \times Z_r$ and observe that for any point $(a,u,v) \in A$ and for every $(x,y,z) \in S$ and $f \in LLC(S,A)$,

$$\|L_{(x,y,z)}f - L_{(a,b,c)}f\| = \|L_{(x,y,v)}f - L_{(a,b,v)}f\| = \|L_{(x,yb^{-1}u,v)}f - L_{(a,u,v)}f\|.$$

It follows that $f \in LLC(S, A_1 \times G \times Z_r)$. Thus, $LLC(S,A) \subset LLC(S, A_1 \times G \times Z_r)$. The reverse inclusion follows from the inclusion $A \subset A_1 \times G \times Z_r$. ∎

Notation: As in the above lemma, for each $A \subset S$, we use the subscript "1" to denote the projection A_1 of A into Z_l. Also throughout the rest of this paper we abbreviate $LLC(S, A_1 \times G \times Z_r)$ to $L(A_1)$ for every subset $A_1 \subset Z_l$, and if $A_1 = \{a\}$ is a singleton, we write $L(a)$.

We may consider a right group $R = G \times Z_r$ to be topologically isomorphic to the rectangular semigroup $S = \{z\} \times R$ where $\{z\}$ is any singleton set. Thus the following corollary is an immediate consequence of 2.1 and 1.6.

Corollary 2.2. *Let $R = G \times Z_r$ be a semitopological right group. Then R is LLC-simple. Also, if R is LLC-trivial then the group G is LLC-trivial.* ∎

If a semitopological right group R is not LLC-trivial, then by Lemma 2.1 $\cup[\emptyset] = \emptyset$. According to the above corollary, an important case in which this can happen is when the group G is not LLC-trivial. On the other hand, if R is LLC-trivial then, obviously, $\cup[\emptyset] = R$. By [2; 4.5.10] a locally compact or complete metric topological right group R is LLC-trivial.

Corollary 2.2 fails to hold for left groups, and we demonstrate this in 2.6 below. First we need some lemmas.

Lemma 2.3. *Let $S = Z_l \times G$ be a semitopological left group where G is not LLC-trivial. If A_1 is a proper, non-empty open and closed subset of Z_l, then $L(x)\backslash L(a) \neq \emptyset$ for each $x \in Z_l\backslash A_1$, $a \in A_1$.*

Proof. Let I_{A_1} denote the indicator (or characteristic) function of A_1 on Z_l, and observe that due to the nature of A_1, $I_{A_1} \in C(Z_l) = LMC(Z_l)$. Since $LUC(G) \neq LMC(G)$, let $g \in LMC(G)\backslash LUC(G)$, and define f on S by

$$f(x,y) = I_{A_1}(x)g(y), \qquad x \in Z_l,\ y \in G.$$

Clearly $f \in C(S)$. Further, the product $Z_l^{LMC} \times G^{LMC}$ of LMC-compactifications is itself a semigroup compactification of S and therefore, a factor of S^{LMC}. It follows that $f \in LMC(S)$ [2; 3.1.9 and 4.5.3].

Next, let $\{y_\alpha\}$ be a net in G which converges to the identity e of G such that $\|L_{y_\alpha}g - g\|$ does not converge to zero, and let $a \in A_1$. Since $\|L_{(a,y_\alpha)}f - L_{(a,e)}f\| = \|L_{y_\alpha}g - g\|$, we conclude that $f \notin L(a)$. However, if $x \in Z_l\backslash A_1$ and $\{(x_\alpha, u_\alpha)\}$ is any net in S converging to (x,u) where u is any point of G, then as $Z_l\backslash A_1$ is open, there is α_0 such that $\|L_{(x_\alpha,u_\alpha)}f - L_{(x,u)}f\| = 0$ for all $\alpha \geq \alpha_0$. Therefore, $f \in L(x)$. ∎

Lemma 2.4. *Let S be an LLC-trivial semitopological semigroup. If z is an isolated point of a left zero semigroup Z_l, then $LLC(Z_l \times S, \{z\} \times S) = LMC(Z_l \times S)$.*

Proof. Define q_z to be the embedding $y \mapsto (z,y) : S \mapsto Z_l \times S$ and let $f \in LMC(Z_l \times S)$. Since $q_z^* f \in LMC(S) = LUC(S)$, and the equality

$$\|L_{(z,y)}f - L_{(z,v)}f\| = \|L_y q_z^* f - L_v q_z^* f\|$$

holds for every $y, v \in S$, it follows that $f \in LLC(Z_l \times S, \{z\} \times S)$. ∎

The next lemma is a general result on rectangular semigroups akin to Lemma 2.1.

Lemma 2.5. *Let $S = Z_l \times G \times Z_r$ be a rectangular semigroup and let L denote the mapping $A_1 \mapsto LLC(S, A_1 \times G \times Z_r) : \mathcal{P}(Z_l) \mapsto LLC(S)$. Then L is a lattice isomorphism of $(\mathcal{P}(Z_l), \cup, \cap)$ onto $(LLC(S), \cap, \vee)$ if and only if L is injective.*

Proof. Suppose that L is injective. Then Lemma 2.1 implies that L is a bijection, and if $\cup[A_1 \times G \times Z_r] = A_1' \times G \times Z_r$ for some $A_1' \subset Z_l$, then $L(A_1') = L(A_1)$. Thus $A_1' = A_1$, and it follows that $A_1 \times G \times Z_r \in \mathcal{E}(S)$ for all $A_1 \subset Z_l$. Since for $A_1, B_1 \subset Z_l$ we have

$$(A_1 \times G \times Z_r) \vee (B_1 \times G \times Z_r) = \cup[(A_1 \cup B_1) \times G \times Z_r] = (A_1 \cup B_1) \times G \times Z_r$$

$$(A_1 \times G \times Z_r) \cap (B_1 \times G \times Z_r) = (A_1 \cap B_1) \times G \times Z_r,$$

we conclude that $(\mathcal{P}(Z_l), \cup, \cap)$ is lattice isomorphic to $(\mathcal{E}(S), \vee, \cap)$, and hence to $(LLC(S), \cap, \vee)$. The converse requires no proof. ■

Theorem 2.6. *Let $S = Z_l \times G \times Z_r$ be a rectangular semigroup with Z_l discrete.*

(a) *If G is not LLC-trivial, then $LLC(S)$ is isomorphic to the Boolean lattice $(\mathcal{P}(Z_l), \cup, \cap)$.*

(b) *S is LLC-trivial if and only if $G \times Z_r$ is LLC-trivial.*

Proof. In order that this theorem not reduce to Corollary 2.2, we assume that Z_l contains at least two elements. To prove Part (a), let L be the mapping $A_1 \mapsto L(A_1) : \mathcal{P}(Z_l) \mapsto LLC(S)$, as in Lemma 2.5, and let $A_1, B_1 \subset Z_l$, $A_1 \neq B_1$. If $a \in A_1 \backslash B_1$, then (since $\{a\}$ is open) by Lemma 2.3 there is a function $f \in L_1(x) \backslash L_1(a)$ for $x \neq a$, $x \in Z_l$, where $L_1(C_1) = LLC(Z_l \times G, C_1 \times G)$ for $C_1 \subset Z_l$. Hence, $f \in L_1(B_1) \backslash L_1(A_1)$; i.e., $L_1(A_1) \neq L_1(B_1)$. But then 1.6 implies that $L(A_1) \neq L(B_1)$. It follows that L is injective.

As for Part (b), if S is LLC-trivial, then 1.6 implies that $G \times Z_r$ is LLC-trivial. Conversely, assume $LUC(G \times Z_r) = LMC(G \times Z_r)$. Then by Lemma 2.4, $L(z) = LMC(S)$ for all $z \in Z_l$; i.e., $LUC(S) = LMC(S)$. ■

In Theorem 2.6 little or no restrictions are assumed on G or Z_r. This is due to the fact that Z_l is topologically restricted in that theorem. Naturally, if we do not wish to distinguish between *all* subsets of Z_l then we may not need to require that Z_l be discrete. Alternately, we may replace or supplement Lemma 2.3, or as in the next lemma and the rest of this section, put some restrictions on G or Z_r.

Lemma 2.8. *Let $S = Z_l \times Z_r$ be a rectangular band, where Z_l is metrizable and Z_r is first countable and not psuedo-compact. If a is a limit point in Z_l, then $L(x) \backslash L(a) \neq \emptyset$ for all $x \in Z_l$, $x \neq a$.*

Proof. Since Z_r is not pseudo-compact, there is a continuous function $f : Z_r \mapsto [0, \infty)$ and a sequence $\{y_n\}$ in Z_r such that $f(y_n) \to \infty$ as $n \to \infty$. Without loss of generality, we may assume that for all $n \geq 1$, $f(y_n) < f(y_{n+1})$. Let f_1 be a continuous strictly increasing function on $[0, \infty)$ such that $f_1(f(y_n)) = n$, $n = 1, 2, 3, \ldots$, and define $g = f_1 \circ f$. Suppose d is a metric for Z_l and define the function $F_a : S \mapsto [0, 1]$ as follows:

$$F_a(x, y) = \frac{1}{1 + \left|d(x, a)^{-1} - g(y)\right|} \text{ if } x \neq a, \quad F_a(a, y) = 0, \quad x \in Z_l,\ y \in Z_r.$$

Clearly $F_a \in C(S)$. To show that $F_a \in LMC(S)$, let $\{s_i\} = \{(x_i, y_i)\}$ and $\{t_j\} = \{(u_j, v_j)\}$ be sequences in S with the range of the former sequence relatively compact such that all the limits involved in the following two quantities exist:

$$Q = \lim_{j \to \infty} \lim_{i \to \infty} F_a(s_i t_j), \qquad Q' = \lim_{i \to \infty} \lim_{j \to \infty} F_a(s_i t_j).$$

We need to show that $Q = Q'$ (first countability of Z_r permits the application of 4.5.6 in [2]). Note that $s_i t_j = (x_i, v_j)$. If $g(v_j) \to \infty$, then $Q' = 0$, and likewise, $Q = \lim_{j \to \infty} F_a(x_0, v_j) = 0$ where $x_0 \in Z_l$ is a limit point of the relatively compact sequence $\{x_i\}$. If $\{g(v_j)\}$ is bounded, then as a sequence of real numbers, it has a limit point y_0. So $Q' = \lim_{i\to\infty}\left(1+\left|d(x_i,a)^{-1}-y_0\right|\right)^{-1}$. In this case, let x_0 be a limit point of $\{x_i\}$. If $x_0 = a$ then $Q' = 0$, and also $Q = \lim_{j\to\infty} F_a(a, v_j) = 0$. If $x_0 \neq a$ then $Q = \left(1+\left|d(x_0,a)^{-1}-y_0\right|\right)^{-1} = Q'$. It follows that $F_a \in LMC(S)$.

Now let $x \in Z_l$, $x \neq a$, and take a sequence $\{(x_n, z_n)\}$ in S which converges to (x,z), where z is a point of Z_r. Without loss of generality, we may assume that $x_n \neq a$ for all n. Then

$$\begin{aligned}\|L_{(x_n,z_n)}F_a - L_{(x,z)}F_a\| &= \sup_{y\in Z_r} |F_a(x_n,y) - F_a(x,y)| \\ &\leq \sup_{y\in Z_r}\left|\left|\frac{1}{d(x,a)} - g(y)\right| - \left|\frac{1}{d(x_n,a)} - g(y)\right|\right| \\ &\leq \left|\frac{1}{d(x_n,a)} - \frac{1}{d(x,a)}\right|.\end{aligned}$$

Since the last quantity above converges to zero as $n \to \infty$, it follows that $F_a \in L(x)$. On the other hand, if $\{a_m\}$ is a sequence in Z_l that converges to a (assume $a_m \neq a$ for all $m \geq 1$), then for any $b \in Z_r$ and for each $m \geq 1$,

$$\|L_{(a_m,b)}F_a - L_{(a,b)}F_a\| \geq \sup_{n\geq 1}\frac{1}{1+\left|d(a_m,a)^{-1} - g(y_n)\right|} \geq \frac{1}{2}$$

where $\{y_n\}$ is the sequence defined above. Thus $F_a \notin L(a)$. ∎

Remark. In metrizable spaces compactness is equivalent to psuedo-compactness [10]. Thus 2.8 *applies to all metrizable rectangular bands $Z_l \times Z_r$ in which Z_r is not compact.*

We also have the following lemma on left groups.

Lemma 2.9. *Let $S = Z_l \times G$ be a left group where Z_l is metrizable and G is first countable and homomorphic (as a topological group) to a non-trivial subgroup of $(\mathbf{R}, +)$. If a is a limit point in Z_l, then $L(x)\backslash L(a) \neq \emptyset$ for all $x \in Z_l$, $x \neq a$.*

Proof. Let $g : G \to \mathbf{R}$ be a continuous homomorphism of topological groups, and note that by the hypothesis, the range $g(G)$ is unbounded. Define $F_a : S \mapsto [0,1]$ as in the proof of Lemma 2.8, and arguing as we did there (with only minor modifications), it is evident that $F_a \in LMC(S)$. Now, let $\{a_i\}$ be a sequence in Z_l which converges to a and $a_i \neq a$ for all $i \geq 1$. Let b be any element of G that is not in the kernel of g, and note that $g(b^n) = ng(b)$ for every integer n. Then for each $i \geq 1$:

$$\|L_{(a_i,1)}F_a - L_{(a,1)}F_a\| \geq \sup_{n\in\mathbf{Z}}\frac{1}{1+\left|d(a_i,a)^{-1} - ng(b)\right|} \geq \frac{1}{1+|g(b)|}.$$

Hence, $F_a \notin L(a)$. On the other hand, if $x \neq a$, and $\{(x_j, y_j)\}$ is any sequence in S which converges to $(x, 1)$, then from the inequalities below it is easy to see that $F_a \in L(x)$:

$$\begin{aligned}\|L_{(x_j,y_j)}F_a - L_{(x,1)}F_a\| &\leq \sup_{y\in G}\left|\left|\frac{1}{d(x,a)} - g(y)\right| - \left|\frac{1}{d(x_j,a)} - (g(y_j)+g(y))\right|\right| \\ &\leq |g(y_j)| + \left|\frac{1}{d(x_j,a)} - \frac{1}{d(x,a)}\right|.\end{aligned}$$

∎

Examples and Remarks. 1. The condition on G in 2.9 implies (but is not implied by) the existence of non-trivial continuous *characters* for G (e.g., functions $e^{ih(y)}$, $y \in G$, where $h : G \to (\mathbf{R},+)$ is a continuous non-zero homomorphism). The weaker condition that G just have non-trivial characters is not sufficient for the validity of 2.9 (e.g., if G is compact; see 2.12 below). On the other hand, in 2.9 G need be neither abelian nor locally compact (e.g., the group G of all invertible 2×2 matrices with rational entries under matrix multiplication; see the next paragraph).

2. Many familiar groups satisfy the condition of 2.9. Obvious examples are $(\mathbf{R}^n,+)$ and the group of units of $(\mathbf{R}^n,\cdot)$. Another specific example is the general linear group $GL(n,F)$ where $F = \mathbf{R}$ or $\mathbf{C}$. The required homomorphisms in each case are easy to construct using the coordinate projections, the logarithm, the determinant and the modulus, as appropriate. Those subgroups of the above three groups that are not contained in the kernel of the homomorphism in each case can also be used in 2.9. For $(\mathbf{R}^n,+)$ this means every non-trivial subgroup, while for $GL(n,F)$ it means those subgroups that are not contained in the special linear group $SL(n,F)$.

Theorem 2.10. *Let $S = Z_l \times G \times Z_r$ be a rectangular semigroup, where Z_l is metrizable. Denote the set of all limit points in Z_l by Z_l' and suppose that one of the following two conditions holds:*

(i) *Z_r is first countable and not pseudo-compact;*

(ii) *G is first countable and homomorphic to a non-trivial subgroup of $(\mathbf{R},+)$.*

Then $LLC(S)$ is isomorphic to the lattice $(\mathcal{P}(Z_l'),\cup,\cap)$ if $G\times Z_r$ is LLC-trivial, and to the lattice $(\mathcal{P}(Z_l),\cup,\cap)$ if G is not LLC-trivial.

Proof. If Z_l' is empty, then this theorem is a special case of Theorem 2.6. Thus suppose that $Z_l' \neq \emptyset$. We first assume that condition (i) holds. Let L be the mapping $A_1 \mapsto L(A_1) : \mathcal{P}(Z_l) \mapsto LLC(S)$, $A_1 \subset Z_l$ of Lemma 2.5. If G is not LLC-trivial and $A_1 \neq B_1$, let $a \in A_1 \backslash B_1$. If a is isolated, then Lemma 2.3 implies that $L(x)\backslash L(a) \neq \emptyset$ for $x \neq a$, so that $L(B_1) \neq L(A_1)$. On the other hand, if a is a limit point, then by Lemma 2.8 $L_1(x)\backslash L_1(a) \neq \emptyset$, where $L_1(C_1) = LLC(Z_l \times Z_r, C_1 \times Z_r)$ for each $C_1 \subset Z_l$. Hence $L_1(A_1) \neq L_1(B_1)$ and therefore, by 1.6, $L(A_1) \neq L(B_1)$. Thus L is injective and consequently, a lattice isomorphism.

If $G \times Z_r$ is LLC-trivial, then for every $A_1 \subset Z_l$, Lemma 2.4 implies that $L(A_1) = L(A_1 \cap Z_l')$. Let L' be the restriction of L to $\mathcal{P}(Z_l')$. In order that $L' : \mathcal{P}(Z_l') \mapsto LLC(S)$ be a lattice isomorphism, it is sufficient that L' be injective. Let $A_1, B_1 \subset Z_l'$, $A_1 \neq B_1$. Then, using Lemma 2.8 as before, $L'(A_1) = L(A_1) \neq L(B_1) = L'(B_1)$. It follows that L' is a lattice isomorphism.

If condition (ii) holds, then using Lemma 2.9 instead of 2.8 the theorem is proved in the same manner as above. ∎

The next corollary is now immediate from 2.10 and the Remark following 2.8.

Corollary 2.11. *Let $S = Z_l \times G \times Z_r$ be a rectangular semigroup where Z_l and Z_r are metrizable and Z_r is not compact. If either G is not LLC-trivial or Z_l is dense-in-itself (e.g., if Z_l is connected), then $LLC(S)$ is isomorphic to the lattice $(\mathcal{P}(Z_l), \cup, \cap)$.* ∎

As the following general theorem demonstrates, some unboundedness hypotheses are required on Z_r or G in Theorem 2.10.

Theorem 2.12. *Let $S = Z_l \times T$, where T is a compact topological semigroup. Then S is LLC-trivial.*

Proof. Let $f \in C(S)$, and observe that if $(z,t) \in S$ and $\{(z_\alpha, t_\alpha)\}$ is any net in S converging to (z,t), then for each α there is $v_\alpha \in T$ such that

$$\|L_{(z_\alpha,t_\alpha)}f - L_{(z,t)}f\| = \left|f(z_\alpha, t_\alpha v_\alpha) - f(z, tv_\alpha)\right|.$$

Denote the norm quantity above by Q_α and assume, as we may, that the net $\{v_\alpha\}$ converges to a point $v_0 \in T$. Then for each α,

$$Q_\alpha \leq \left|f(z_\alpha, t_\alpha v_\alpha) - f(z, tv_0)\right| + \left|f(z, tv_0) - f(z, tv_\alpha)\right|,$$

so that $Q_\alpha \to 0$ for all $z \in Z_l$, $t \in T$. Hence, $f \in LUC(S)$. ∎

3. Left Local Continuity and the Green's Relation

Let S be a semigroup. Adhering to the somewhat customary notations, we recall the following three of the four familiar relations of Green: [4]

$a\mathcal{R}b$ if $aS \cup \{a\} = bS \cup \{b\}$, $(a, b \in S)$;
$a\mathcal{L}b$ if $Sa \cup \{a\} = Sb \cup \{b\}$, $(a, b \in S)$;
and: $\mathcal{H} = \mathcal{R} \cap \mathcal{L}$.

If S is a rectangular semigroup, then it can be shown by a straightforward argument that each of the above relations is a congruence. Supposing $S = Z_l \times G \times Z_r$, note that for each $s \in S$, $sS = \{s_1\} \times G \times Z_r$, where s_1 is the projection of s into Z_l. Therefore, $s\mathcal{R}t$ if and only if $s_1 = t_1$, and the $\mathcal{R}$-cell of s is the set $\{s_1\} \times G \times Z_r$. Thus, in view of Lemma 2.1, it is reasonable to expect a relationship between $\mathcal{R}$ and the LLC relation ρ_0. This is indeed the case, and more. We begin with a corollary to Lemma 2.1.

Corollary 3.1. *Let $S = Z_l \times G \times Z_r$ be a semitopological rectangular semigroup. Then ρ_0 is a congruence and $\mathcal{R} \subset \rho_0$. Hence $\rho_0 \cap \mathcal{L}$ is also a congruence which contains the relation $\mathcal{H}$.*

Proof. Note that by Lemma 2.1 $LLC(S, st) = LLC(S, s)$, $s, t \in S$, which readily implies that ρ_0 is a congruence in S. Also, since $s_1 = t_1$ implies $LLC(S, s) = LLC(S, t)$, it follows from the remarks prior to the corollary that $\mathcal{R} \subset \rho_0$. The rest of the corollary follows immediately. ∎

The inclusions of Corollary 3.1 may easily be strict (e.g., when S is LLC-trivial and therefore, ρ_0 is the universal relation $S \times S$). However, under certain conditions $\rho_0 = \mathcal{R}$. If for each $s, t \in S$, it is true that $L(s_1) = L(t_1)$ if and only if $s_1 = t_1$ then $s\rho_0 t$ if and only if $s\mathcal{R}t$. Several sufficient conditions based on 2.6 and 2.10 for the equality between ρ_0 and $\mathcal{R}$ are listed in the next corollary.

Corollary 3.2. *Let $S = Z_l \times G \times Z_r$ be a rectangular semigroup such that at least one of the following conditions hold:*

(i) *Z_l is discrete and G is not LLC-trivial;*

(ii) *Z_l is metrizable and G is non-LLC-trivial, first countable and homomorphic to a non-trivial subgroup of $(\mathbf{R}, +)$;*

(iii) *Z_l is metrizable and dense-in-itself (e.g., connected), and G is first countable and homomorphic to a non-trivial subgroup of $(\mathbf{R}, +)$;*

(iv) *Z_l is metrizable and dense-in-itself (e.g., connected), and Z_r is first countable and not psuedo-compact;*

(v) *Z_l is metrizable, Z_r is first countable and not pseudo-compact, and G is not LLC-trivial;*

(vi) *Z_l and Z_r are metrizable with Z_r not compact, and either G is not LLC-trivial or Z_l is dense-in-itself.*

Then $\rho_0 = \mathcal{R}$ and $\rho_0 \cap \mathcal{L} = \mathcal{H}$. Hence, in this case, the ρ_0-cells are just the principal right ideals, and if S is a left group, then each different right identity represents a different ρ_0-cell. ∎

Examples 3.3. (Rectangular semigroups of real or complex matrices)

Recall that $GL(n, F)$ is the (General Linear) group of invertible $n \times n$ matrices with entries in the field F (here $F = \mathbf{R}$ or $\mathbf{C}$). Let G denote a subgroup of $GL(n, F)$, and let Z_l and Z_r denote subspaces of $\mathbf{R}$ with left zero and right zero multiplication, respectively. Define $S = Z_l \times G \times Z_r \subset \mathbf{R}^{m+2}$, where $m = n^2$ if $F = \mathbf{R}$ and $m = 2n^2$ if $F = \mathbf{C}$. Then the rectangular semigroup S with the relative usual topology of $\mathbf{R}^{m+2}$ is topologically isomorphic to the semigroup of block diagonal matrices of the type

$$M = \begin{pmatrix} M_1 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & M_3 \end{pmatrix}$$

under matrix multiplication, where $M_2 \in G$ and

$$M_1 = \begin{pmatrix} 1 & 0 \\ a & 0 \end{pmatrix} \qquad M_3 = \begin{pmatrix} 1 & b \\ 0 & 0 \end{pmatrix},$$

with $a, b \in \mathbf{R}$. Note that if Z_r is a singleton, then S reduces to a left group (equivalently, M_3 is dropped from M). In discussing the left locally continuous structure of S, the following cases (among others) arise:

(a) *Z_l is discrete* (e.g., $a \in \mathbf{Z}$ in M_1). In this case, if $G \times Z_r$ is LLC-trivial (e.g., if the subgroup G and the subspace Z_r are both locally compact), then $LLC(S) = \{LUC(S)\}$ is also LLC-trivial by Theorem 2.6, ρ_0 is the universal relation in S, and $\cup[\emptyset] = S$ (claims about $\cup[\emptyset]$ in these examples are easy to prove; or see Corollary 3.4 below). Also by Theorem 2.6, if G is not LLC-trivial, then $LLC(S)$ is isomorphic to the lattice $(\mathcal{P}(Z_l), \cup, \cap)$, $\rho_0 = \mathcal{R}$ by Corollary 3.2, and $\cup[\emptyset] = \emptyset$.

(b) *Z_r is not compact* (i.e., Z_r is not closed or not bounded). In this case, if Z_l is dense in $\mathbf{R}$ (e.g., $a \in \mathbf{Q}$ in M_1) or if Z_l is a connected subset of $\mathbf{R}$, then by 2.11 $LLC(S)$ is isomorphic to the lattice $(\mathcal{P}(Z_l), \cup, \cap)$ for *any* subgroup G of $GL(n, F)$. On the other hand, if G is not LLC-trivial in $GL(n, F)$, then by 2.11 again, $LLC(S)$ is isomorphic to the lattice

$(\mathcal{P}(Z_l), \cup, \cap)$ for *any* subset Z_l of $\mathbf{R}$. In either case, $\cup[\emptyset] = \emptyset$ and by 3.2 $\rho_0 = \mathcal{R}$ in S.

(c) *Z_r is compact and G is not a subgroup of $SL(n, F)$.* In this case, if $G \times Z_r$ is LLC-trivial, then Theorem 2.10 implies that $LLC(S)$ is isomorphic to $(\mathcal{P}(Z_l'), \cup, \cap)$, while if G is not LLC-trivial, then $LLC(S)$ is isomorphic to $(\mathcal{P}(Z_l), \cup, \cap)$. In the first case, $\cup[\emptyset] = (Z_l \backslash Z_l') \times G \times Z_r$ and in the second, $\cup[\emptyset] = \emptyset$. In either case, when $Z_l = Z_l'$, $\rho_0 = \mathcal{R}$.

Now we consider the set relation ρ and its cells. For each subset $A \subset S$, Lemma 2.1 implies that $\cup[A] = A_1' \times G \times Z_r$ and $A_1 \times G \times Z_r \subset \cup[A]$ (A_1' is some subset of Z_l which, of course, contains the projection A_1). The inclusion may be strict if, e.g., S is LLC-trivial. The next corollary lists some non-trivial cases in which we can determine the representative maximal element of each ρ-cell exactly.

Corollary 3.4. *Let $S = Z_l \times G \times Z_r$ be a rectangular semigroup.*

(a) *If any one of conditions* (i)-(vi) *in Corollary* 3.2 *holds, then* $\cup[A] = A_1 \times G \times Z_r$ *for every subset* $A \subset S$.

(b) *If $G \times Z_r$ is LLC-trivial and either of the two conditions* (i) *or* (ii) *in* 2.10 *hold, then* $\cup[A] = ((Z_l \backslash Z_l') \cup A_1) \times G \times Z_r$ *for every subset* $A \subset S$.

Proof. (a) Let $\cup[A] = A_1' \times G \times Z_r$, as above. Then

$$L(A_1') = LLC(S, \cup[A]) = LLC(S, A) = L(A_1),$$

so that by the injectivity of the LLC mapping L, $A_1' = A_1$.

(b) For each $A \subset S$, let $\cup[A] = A_1' \times G \times Z_r$ as in Part (a) above, and note that by Lemma 2.4 and maximality of $\cup[A]$, $(Z_l \backslash Z_l') \subset A_1'$. Let L' denote the restriction of the LLC mapping L to $\mathcal{P}(Z_l')$ (see the proof of Theorem 2.10) and observe that

$$L'(A_1' \cap Z_l') = L(A_1') = LLC(S, A) = L(A_1) = L'(A_1 \cap Z_l').$$

As was shown in Theorem 2.10, L' is injective. Thus $A_1' \cap Z_l' = A_1 \cap Z_l'$. Now since

$$(Z_l \backslash Z_l') \cup A_1 = (Z_l \backslash Z_l') \cup (A_1' \cap Z_l') = (Z_l \backslash Z_l') \cup A_1' = A_1'$$

we may conclude that $\cup[A] = ((Z_l \backslash Z_l') \cup A_1) \times G \times Z_r$. ∎

Part (a) of the above corollary in conjunction with Corollary 3.2 may lead to the conjecture that "if $\rho_0 = \mathcal{R}$, then $\cup[A] = A_1 \times G \times Z_r$ for all $A \subset S$". We further remark that the *converse* of this conjecture is true, since then for each $s \in S$,

$$[s] \subset \cup[\{s\}] = \{s_1\} \times G \times Z_r \subset [s].$$

However, as shown in the next example, the conjecture itself is false. Hence, in general, the equality $\rho_0 = \mathcal{R}$ is not sufficient to determine the relation ρ and the LLC-lattice completely. Stronger hypotheses like those in 3.4(a) are needed.

Example. Let $Z_l = [0, 1] \cup \{2\}$ with the relative usual topology of $\mathbf{R}$, and let $G = (\mathbf{R}, +)$ denote the usual additive reals. Then $Z_l' = [0, 1]$, and denote by S the topological left group $Z_l \times G$. Note that $\rho_0 = \mathcal{R}$ for this semigroup S; however, if $A = [0, 1] \times G$, then $LUC(S) = LLC(S, A)$ so that $\cup[A] = S$ while $A_1 \times G = A$.

4. Translation Invariance

When translation invariant, the left locally continuous algebras in $LMC(S)$ result in semigroup compactifications of S that are universal with respect to a local joint continuity property [9]. In rectangular semigroups, left translation invariance is also intimately related to the *lattice* structure of the LLC-lattice, and has some interesting consequences (see 4.5 below). If $S = Z_l \times Z_r$ is a rectangular *band*, then a straight-forward computation shows that $L_SC(S) \subset LUC(S)$, so that $LLC(S,A)$ is left translation invariant for every $A \subset S$. At the other extreme, in certain *left groups* S, the only left translation invariant members of $LLC(S)$ are $LUC(S)$ and $LMC(S)$ ([9; 2.4]; also see 4.3 and 4.4 below). From this observation, one might infer (perhaps ironically) that the prominence of the group part in a rectangular semigroup prevents some members of $LLC(S)$ from being left translation invariant. Our results below will support and clarify this assertion. We begin with a general sufficient condition for (left) translation invariance.

Theorem 4.1. *Let $S = Z_l \times T$, where T is an LLC-trivial semitopological semigroup. Then every* C*-*algebra in $LLC(S)$ is left translation invariant.*

Proof. Since $LUC(T) = LMC(T)$, if for each fixed $z \in Z_l$, q_z is the canonical injection $y \mapsto (z,y) : T \mapsto S$, we have $q_z^*LMC(S) \subset LUC(T)$. Let $f \in LMC(S)$ and note that

$$\|L_{(x,y)}(L_{(z,v)}f) - L_{(s,t)}(L_{(z,v)}f)\| = \|L_{(z,vy)}f - L_{(z,vt)}f\| = \|L_{vy}q_z^*f - L_{vt}q_z^*f\|$$

for every $(z,v),(x,y),(s,t) \in S$. Thus $L_{(z,v)}f \in LUC(S)$, implying that $L_SLMC(S) \subset LUC(S)$. In particular, $LLC(S,A)$ is left translation invariant for every subset $A \subset S$. ∎

According to 2.10 and 3.3, there are rectangular semigroups $S = Z_l \times R$ where the right group R is LLC-trivial and where $LLC(S)$ is a large Boolean lattice whose members are translation invariant by 4.1. In the next corollary, the LLC-triviality of the group G is a *necessary* condition for left translation invariance.

Corollary 4.2. *Let $S = Z_l \times G \times Z_r$ be a rectangular semigroup in which Z_l is disconnected. If every member of $LLC(S)$ is left translation invariant, then G is LLC-trivial.*

Proof. Suppose that G is not LLC-trivial. Let A_1 be a proper non-empty open and closed subset of Z_l, and define the functions f, g and L_1 as in the proofs of Lemma 2.3 and Theorem 2.6. Then $f \in L_1(x)\backslash L_1(a)$ for each $a \in A_1$ and $x \in Z_l\backslash A_1$. If p_{12} is the projection of S onto the left group $Z_l \times G$, then

$$p_{12}^*f \in p_{12}^*LLC(Z_l \times G, \{x\} \times G) \subset LLC(S, \{x\} \times G \times Z_r) = L(x).$$

On the other hand, if e is the identity of G then there is a net $\{y_\alpha\}$ in G which converges to e, but $\{L_{y_\alpha}g\}$ does not converge to g in norm. Since for any given $z \in Z_r$ we have

$$\begin{aligned}\|L_{(x,y_\alpha,z)}(L_{(a,e,z)}p_{12}^*f) - L_{(x,e,z)}(L_{(a,e,z)}p_{12}^*f)\| &= \|L_{(a,y_\alpha,z)}p_{12}^*f - L_{(a,e,z)}p_{12}^*f\| \\ &= \sup_{y\in G}\left|f(a,y_\alpha y) - f(a,y)\right| \\ &= \|L_{y_\alpha}g - g\|,\end{aligned}$$

it follows that $L_{(a,e,z)}p_{12}^*f \notin LLC(S,(x,e,z)) = L(x)$. Hence $L(x)$ is not left translation invariant. ∎

Corollary 4.3. *Let $S = Z_l \times G$ be a semitopological left group in which Z_l is disconnected. Then every member of $LLC(S)$ is left translation invariant if and only if G is LLC-trivial.* ∎

We now show what can happen if G is not LLC-trivial in 4.2 and 4.3.

Corollary 4.4. *Let $S = Z_l \times G \times Z_r$ be a rectangular semigroup in which Z_l is discrete and G is not LLC-trivial. Then the only left translation invariant algebras in $LLC(S)$ are the extreme members $LUC(S)$ and $LMC(S)$.*

Proof. Let $A \subset S$ such that $A_1 \neq \emptyset, Z_l$, and let $a \in A_1$, $z \in Z_l \backslash A_1$. Choose $f \in L(A_1) \backslash L(z) = LLC(S, A) \backslash L(z)$ and a net $\{y_\alpha\}$ in $G \times Z_r$ converging to an idempotent e such that $\{L_{(z,y_\alpha)} f\}$ does not converge in norm to $L_{(z,e)} f$. Since

$$\|L_{(a,y_\alpha)}(L_{(z,e)} f) - L_{(a,e)}(L_{(z,e)} f)\| = \|L_{(z,y_\alpha)} f - L_{(z,e)} f\|,$$

we may conclude that $L_{(z,e)} f \notin LLC(S,(a,e)) = L(a)$. Thus $L_{(z,e)} f \notin LLC(S, A)$, implying that $LLC(S, A)$ is not left translation invariant. ∎

Combining our results in Section 2 and in this section with those of M. H. Stone [11], we obtain a dramatic illustration of how large and structured the class of left locally continuous algebras on a rectangular semigroup can be.

Corollary 4.5. *Let $\mathbf{B}$ be a Boolean algebra and let $G = (\mathbf{Z}, +)$ be the discrete group of all integers. Then there is a Banach space $\mathbf{V}$ and a subspace $Z_l \subset \mathbf{V}$ such that $\mathbf{B}$ can be isomorphically imbedded in the class of all universal semigroup compactifications of the left group $S = Z_l \times G$ that extend the compactification S^{LUC}.*

Proof. Using [2; 3.1.9] and [9; 1.3, 1.6] (with 4.1 above implying translation invariance), it suffices to show that $LLC(S)$ contains an isomorphic copy of $\mathbf{B}$. According to Stone's Theorem in [11], $\mathbf{B}$ is isomorphic to a field of sets, whose union we label X. Let $\mathbf{V}$ be the Banach space of all bounded, real-valued functions on X under the sup-norm (if X is finite with n elements, then $\mathbf{V} = \mathbf{R}^n$). Now, for each $x \in X$, let I_x denote the indicator (or characteristic) function of the set $\{x\}$ and define

$$Z_l = \bigcup_{n=1}^{\infty} \{(1 - \frac{1}{n}) I_x : x \in X\} \cup \{I_x : x \in X\}.$$

Then $Z_l' = \{I_x : x \in X\}$ has the same cardinal number as X, and with the left zero multiplication on Z_l, Theorem 2.10 may be applied to conclude the proof. ∎

It should be clear that instead of $(\mathbf{Z}, +)$ in 4.5, we could use any LLC-trivial semitopological right group that satisfies at least one of the two conditions 2.10(i) or 2.10(ii). Thus another simple choice might be the discrete space $\mathbf{N}$ of all positive integers under the right zero multiplication.

5. On Representing Uniform Convergence

The main results of this section, namely, Theorem 5.5 and its corollaries, present a rather curious application of left local continuity. We begin with a few preliminaries to establish the required framework. The next definition resembles its almost periodic analog in [5].

Definition 5.1. Let S a semitopological semigroup and let Z be a topological space endowed with the left zero multiplication. We define a function $f \in C(Z \times S)$ to be *Z-left uniformly continuous* or *left uniformly continuous on S with respect to parameters in Z* if for each fixed $z \in Z$ and each $y \in S$, given $\epsilon > 0$, there is a neighborhood $N(z, \epsilon)$ of y in S such that

$$\|L_{(z,x)}f - L_{(z,y)}f\| < \epsilon, \qquad x \in N(z, \epsilon).$$

We denote the set of all such functions f by $\mathcal{U}(Z, S)$. We also refer to the direct product $Z \times S$ as *a left semigroup of S* (recall 2.4, 2.12 and 4.1). It is evident that given S, all left semigroups of S having the same cardinality are algebraically isomorphic. Further, every left semigroup of a group is a left group and every left semigroup of a right group is a rectangular semigroup.

When $S = (\mathbf{R}, +)$ is the group of all additive real numbers, the subset of $\mathcal{U}(Z, S)$ consisting of all functions in $C(Z \times S)$ for which the neighborhoods $N(z, \epsilon) = N(\epsilon)$ are independent of z for all $z \in Z$, is the class of all "Ω-uniformly continuous" functions in [5]. These latter functions are the uniformly continuous analogs of the functions that are "almost periodic uniformly with respect to parameters" on $(\mathbf{R}, +)$ (this uniform dependence with respect to parameters relates to H. Bohr's "analytic almost periodic functions", and the discussions of almost periodic solutions of differential equations). In [3] it was observed that the parametrized almost periodic functions in [5] correspond exactly to the almost periodic functions on an associated left group whenever the set of parameters is compact. Adopting a somewhat similar point of view, we consider parametrizing uniformly continuous functions on right groups with the aim of getting to Theorem 5.5 as quickly as possible.

Lemma 5.2. *Let S be a semitopological semigroup and let $Z \times S$ be a left semigroup of S. Denote by q_z the canonical injection $y \mapsto (z, y) : S \mapsto Z \times S$. Then* (a) *is equivalent to* (b), *and if S has a left identity e, then* (a), (b) *and* (c) *are equivalent:*

(a) $f \in \mathcal{U}(Z, S)$;
(b) $q_z^* f \in LUC(S)$ *for all* $z \in Z$;
(c) $L_{(z,e)} f \in LUC(Z \times S)$ *for all* $z \in Z$.

Proof. Since $\|L_{(z,x)}f - L_{(z,y)}f\| = \|L_x q_z^* f - L_y q_z^* f\|$, it follows that (a) and (b) are equivalent. The equivalence of (b) and (c) is an immediate consequence of the identities $q_z^* f = q_z^* L_{(z,e)} f$, $L_{(z,e)} f = p_2^* q_z^* f$ ($p_2 : Z \times S \mapsto S$ is the projection map) and the inclusions

$$q_z^* LUC(Z \times S) \subset LUC(S), \qquad p_2^* LUC(S) \subset LUC(Z \times S). \qquad \blacksquare$$

Theorem 5.3. *Let $Z \times S$ be a left semigroup of a semitopological semigroup S.*

(a) *If S has a left identity e, then $\mathcal{U}(Z, S)$ is a translation invariant* C*-*subalgebra of $C(Z \times S)$ containing $LUC(Z \times S)$.*
(b) *If Z is discrete, then $\mathcal{U}(Z, S) = LUC(Z \times S)$.*
(c) *If S is LLC-trivial, then $LMC(Z \times S) \subset \mathcal{U}(Z, S)$.*

Proof. (a) Proving that $\mathcal{U}(Z, S)$ is a right translation invariant C*-algebra is routine. The inclusion $LUC(Z \times S) \subset \mathcal{U}(Z, S)$ is a consequence of Lemmas 1.1(c) and 5.2(b). To prove the left translation invariance, let $(z,y) \in Z \times S, f \in \mathcal{U}(Z,S)$ and note that $L_{(z,y)}f = L_{(z,y)}L_{(z,e)}f$. Since by 5.2(c) $L_{(z,e)}f \in LUC(Z \times S)$ and $LUC(Z \times S)$ is left translation invariant, we have $L_{(z,y)}f \in LUC(Z \times S) \subset \mathcal{U}(Z,S)$. Hence $\mathcal{U}(Z,S)$ is left translation invariant.

(b) If $f \in \mathcal{U}(Z,S)$, then from 5.2(b) and the equality $\|L_{(z,y)}f - L_{(z,t)}f\| = \|L_y q_z^* f - L_t q_z^* f\|$, for all $z \in Z$, $y,t \in S$, we conclude that if Z is discrete, then $f \in LUC(Z \times S)$.

(c) This is clear from $q_z^* LMC(Z \times S) \subset LMC(S) = LUC(S)$ for all $z \in Z$. ∎

Definition 5.4. Let $\{g_n : n \geq 1\}$ be a sequence of functions in $C(S)$, where S is a semitopological semigroup, and assume that $\{g_n\}$ has a point-wise limit g (for example, if $\{g_n\}$ has a uniformly bounded subsequence). If the one-point compactification $\mathbf{N}_\infty$ of the set $\mathbf{N}$ of positive integers is endowed with the left zero multiplication, then the direct product $\mathbf{N}_\infty \times S$ is a left semigroup of S, which we denote by ${}^*S$. Define the function f on ${}^*S$ by

$$f(n,s) = g_n(s), \qquad f(\infty, s) = g(s), \qquad s \in S.$$

We say that f *represents the sequence* $\{g_n\}$ *on* ${}^*S$, or that f is the *left representation of* $\{g_n\}$ and g. Note that $g = q_\infty^* f$ and for $n \geq 1$, $g_n = q_n^* f$.

Theorem 5.5. *Let S be a semitopological semigroup with a left identity e, and let $\{g_n\}$ be a sequence of functions in $LUC(S)$. Suppose that $\{g_n\}$ has a point-wise limit g and let f be the left representation of $\{g_n\}$ and g. Then $\{g_n\}$ converges uniformly to g if and only if $f \in LLC({}^*S, (\infty, e))$.*

Proof. Suppose that $f \in LLC({}^*S, (\infty, e))$. Since $\|L_{(n,e)}f - L_{(\infty,e)}f\| = \|g_n - g\|$, it follows that $g_n \to g$ uniformly.

Conversely, assume that the sequence $\{g_n\}$ converges to g uniformly. Then $g \in LUC(S)$, implying that $q_z^* f \in LUC(S)$ for all $z \in \mathbf{N}_\infty$. Also for each $y, v \in S$,

$$|f(n,y) - f(\infty,v)| = |g_n(y) - g(v)| \leq \|g_n - g\| + |g(y) - g(v)|.$$

It follows that $f \in C({}^*S)$, and hence, $f \in \mathcal{U}(\mathbf{N}_\infty, S)$. Next, we prove that $f \in LMC({}^*S)$ by showing that the mapping $s \mapsto \mu(L_s f) : {}^*S \mapsto \mathbf{C}$ is continuous for each μ in the spectrum β^*S. First note that the mapping $z \mapsto q_z^* f : \mathbf{N}_\infty \mapsto LUC(S)$ is (norm-)continuous at $z = \infty$ since for $n \geq 1$, $\|q_n^* f - q_\infty^* f\| = \|g_n - g\|$. Now for $(z,y), (\infty, v) \in {}^*S$ we have

$$\begin{aligned}
|\mu(L_{(z,y)}f) - \mu(L_{(\infty,v)}f)| &\leq \|L_{(z,y)}f - L_{(\infty,v)}f\| \\
&= \|L_{(z,y)}(L_{(z,e)}f) - L_{(\infty,v)}(L_{(\infty,e)}f)\| \\
&\leq \|L_{(z,y)}(L_{(z,e)}f) - L_{(z,y)}(L_{(\infty,e)}f)\| \\
&\quad + \|L_{(z,y)}(L_{(\infty,e)}f) - L_{(\infty,v)}(L_{(\infty,e)}f)\| \\
&\leq \|q_z^* f - q_\infty^* f\| + \|L_{(z,y)}h_0 - L_{(\infty,v)}h_0\|,
\end{aligned}$$

where $h_0 = L_{(\infty,e)}f \in LUC({}^*S)$ (since $f \in \mathcal{U}(\mathbf{N}_\infty, S)$). Also for each $n \geq 1$,

$$|\mu(L_{(n,y)}f) - \mu(L_{(n,v)}f)| \leq \|L_{(n,y)}f - L_{(n,v)}f\| = \|L_y q_n^* f - L_v q_n^* f\|.$$

It follows that $f \in LMC(^*S)$. Finally, since

$$\begin{aligned}\|L_{(n,y)}f - L_{(\infty,e)}f\| &= \|L_y g_n - g\| \\ &\leq \|L_y g_n - L_y g\| + \|L_y g - g\| \\ &\leq \|g_n - g\| + \|L_y g - g\|\end{aligned}$$

we conclude that $f \in LLC(^*S, (\infty, e))$. ∎

If S is a right group in 5.5, then $^*S$ is a rectangular semigroup and the results of Sections 2 and 3 may be used to analyze the LLC structure of $^*S$. This proves to be an important special case as the next two corollaries show. For instance, when S is a right zero semigroup an application of Theorem 5.5 yields the following result for topological spaces.

Corollary 5.6. *Let X be a topological space and let $\{g_n\}$ be a sequence of functions in $C(X)$ which has a point-wise limit g. Give X the right zero multiplication so that the left semigroup $^*X$ is the rectangular band $\mathbf{N}_\infty \times X$. Then the following statements are equivalent:*

(a) *$\{g_n\}$ converges uniformly to g;*
(b) *The left representation f is left locally continuous at some point (∞, x) of $^*X$;*
(c) *$f \in LUC(^*X)$.*

Proof. In view of Theorem 5.5, we need only show that (b) implies (c). Since $\mathbf{N}_\infty$ has only one limit point, ∞, 2.4 implies that $LLC(^*X, \mathbf{N} \times X) = LMC(^*X)$. Hence, by 2.1 $LLC(^*X, (\infty, x)) = LLC(^*X, \{\infty\} \times X) = LUC(^*X)$ for every $x \in X$. ∎

Another application of 5.5 in the context of rectangular semigroups involves groups. Recall that if G is a topological group, then G has a natural uniformity (the right uniform structure) such that $LUC(G)$ coincides with the set of all bounded, complex-valued functions that are uniformly continuous with respect to this uniformity. Also, if Z is a compact, Hausdorff topological space then there is a uniformity that uniquely induces the topology of Z. Thus by [3; 2.1] $LUC(Z \times G)$ coincides with the set of all bounded complex-valued functions that are uniformly continuous with respect to the product uniformity of $Z \times G$. These facts lead to the following corollary.

Corollary 5.7. *Let G be an LLC-trivial (e.g., locally compact or complete metrizable) topological group, and let $\{g_n\}$ be a sequence of bounded, uniformly continuous complex-valued functions on G. Suppose $\{g_n\}$ has a point-wise limit g and let f be their left representation. Then $\{g_n\}$ converges uniformly to g if and only if f is uniformly continuous on the left group $^*G$.*

Proof. As in the proof of 5.6, it is easy to see that $LLC(^*G, (\infty, 1)) = LUC(^*G)$. Since $\mathbf{N}_\infty$ is compact, by the remarks preceeding this corollary and by Theorem 5.5 we arrive at the conclusion. ∎

Remark. Evidently, Corollary 5.7 establishes a one-to-one correspondence between the set of all uniformly continuous (bounded and complex valued) functions on the left group $^*G$ and the family of all uniformly convergent sequences of similar functions on G. However, if the group G above is not LLC-trivial, then the uniform continuity of f is no longer a necessary condition for the uniform convergence of the sequence $\{g_n\}$ and the one-to-one correspondence above no longer holds. In this case, proper local continuity is involved (as described in Sections 2 and 3 and as specified in Theorem 5.5). Indeed, if the group G of 5.7 is not LLC-trivial, then Theorem 2.10 implies the existence of infinite chains of distinct function algebras such as, e.g.,

$$LLC(^*G, \{\infty\} \times G) \supset \cdots \supset LLC(^*G, J_n \times G) \supset \cdots \supset LUC(^*G)$$

where $J_n = \{\infty, 1, 2, \ldots n\}$, for $n = 1, 2, 3, \ldots$. Since the class $LUC(^*G)$ of bounded, uniformly continuous functions on $^*G$ in this case is so completely separated from $LLC(^*G, (\infty, 1))$ which is required in 5.5, it is clear that the representations of uniform convergence discussed in this section involve left *local* continuity in a very fundamental way.

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