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Effects of temporal heterogeniety in the Baumol-Wolff productivity growth model*

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Summary. In their utilization of R&D (information) output, different sectors of a heterogeneous industry display different reaction times. This paper analyzes the effects of this temporal heterogeniety on output and productivity for an extended version of the Baumol-Wolff model. Results include conditions implying persistent, non-decaying oscillations in the output and hence also in the productivity rate.

Keywords and Phrases: Heterogeneous sectors, Reaction times, Productivity rate, Higher order difference equation, Instability, Oscillation.

JEL Classification Numbers: O47, O41, O32, C62.

1 Introduction

In a 1992 paper [2], Baumol and Wolff argue that the existence of a feedback loop between the output of the R&D (research and development) sector, namely, information, and the outside or non-R&D industries' demand for this product can be an endogeneous source of instability in the aggregate productivity growth rate of the economy. They capture this feedback loop in a mathematical model that relates the productivity rate to the relative rate of change in the price of information and then to the relative rate of change in the output of the R&D sector.

In this paper, we propose and mathematically analyze a multisector extension of the Baumol-Wolff feedback model in which a heterogeneous non-R&D industry utilizes information in a temporally non-uniform fashion. Suppose that there are m sectors outside the R&D sector and that each sector has its own

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productivity growth rate r_t^i in period t, where $i = 1, 2, \ldots, m$. We assume, as is done in the BW model, that r_t^i is an increasing function of the R&D output level in a past time period; however, here this past period need not be the immediately preceding one, but rather $t - k_i$. The positive integer k_i, which we may call the *reaction time* of sector i, represents the time period it *typically* takes sector i to properly integrate the new information (its share of the R&D output) in its production processes to the extent that affects its productivity rate. It is assumed for simplicity that each k_i is independent of time and is a characteristic feature of the corresponding sector.

We show that this temporally heterogeneous version of the BW model will have a unique nonzero equilibrium whose stability characteristics are generally different from (and often less stable than) the first order or temporally homogeneous BW. This identifies another inherent source, namely, temporal heterogeniety among the reaction times of various sectors, that is capable of significantly influencing the growth rate of (aggregate) productivity. We also derive conditions on the various parameters that imply disequilibrium and bounded, persistent oscillations in the productivity rate in the case of two or more temporally heterogeneous sectors. These results are based on some recent developments in the theory of nonlinear higher order difference equations.

2 The extended BW model

Following [2] and the above Introduction, we consider the following equations, only the first of which is new:

$$\begin{array}{rcl} r_t & = & \sum_{i=1}^m f_i(y_{t-k_i}), \quad f_i' \geq 0, \quad \sum_{i=1}^m f_i' > 0 \\ (p_t - p_{t-1})/p_{t-1} & = & h(r_t), \quad h' > 0, \; h(0) = 0 \\ (y_t - y_{t-1})/y_{t-1} & = & g[(p_t - p_{t-1})/p_{t-1}], \quad g' < 0, \; g(0) = 0 \end{array} \tag{1}$$

When $k_i = 1$ for all i, then all sectors are *temporally* homogeneous; in this case, defining

$$f = \sum_{i=1}^m f_i \tag{2}$$

we obtain the original BW equations. The quantity r_t is the aggregate productivity rate, defined here as the sum of productivity rates $r_t^i = f_i(y_{t-k_i})$ of the m R&D client sectors, with y_t denoting the output level of the R&D sector in each period t. As in [2], p_t denotes the price of information in period t.

Making the obvious substitutions into the bottom demand equation, shows system (1) to be equivalent to the single scalar equation

$$y_t = y_{t-1}(1 + g[h[\sum_{i=1}^m f_i(y_{t-k_i})]]) \tag{3}$$

whose nonzero equilibria of (3) must satisfy the equation

$$g[h[\sum_{i=1}^{m} f_i(y)]] = 0. \tag{4}$$

Now, using (2) and the monotonicity hypotheses signified by the derivative inequalities in (1), we obtain a unique solution for (4)

$$\bar{y} = f^{-1}(h^{-1}(g^{-1}(0))) = f^{-1}(0).$$

Since f is increasing, we see that $\bar{y} > 0$ if and only if $f(0) < 0$. The number $f(0) = \alpha$ is refered to as the "autonomous rate" in [2], *which we assume to be negative in the sequel*. Note that this does not imply $f_i(0) < 0$ for all i, which allows for the possibility that some industries (evidently, not the high technology ones) may have a non-negative autonomous rate.

Next, to assure that y_t is always non-negative, we assume throughout the sequel that

$$g[h[\sum_{i=1}^{m} f_i(x_i)]] > -1, \qquad (x_1, \ldots, x_m) \in [0, \infty)^m.$$

At this stage, we recall some standard facts concerning the linearization of (3) at $\bar{y}$. For simplicity, let $k_i = i$;[1] define

$$F(x_1, \ldots, x_m) = x_1(1 + g[h[\sum_{i=1}^{m} f_i(x_i)]]) \tag{5}$$

and let

$$V_F(x_1, \ldots, x_m) = [F(x_1, \ldots, x_m), x_1, x_2, \ldots, x_{m-1}]$$

be the usual vectorization of F. The standard theory ([3], [5], [8]) gives the characteristic polynomial of the linearization of V_F as

$$P(\lambda) = \lambda^m - [1 - c\bar{y}f_1'(\bar{y})]\lambda^{m-1} + c\bar{y}\sum_{i=2}^{m} f_i'(\bar{y})\lambda^{m-i}$$

where

$$c = -g'(0)h'(0) > 0. \tag{6}$$

In the Mathematical Appendix of [4] it is proved (essentially[2]) that the following inequality on the sum-norm of the gradient of F,

$$\|\nabla F(\bar{y}, \ldots, \bar{y})\| = |1 - c\bar{y}f_1'(\bar{y})| + c\bar{y}\sum_{i=2}^{m} f_i'(\bar{y}) < 1 \tag{7}$$

[1] This assumption can be dropped if we wish to consider *all* mathematically possible cases, e.g., different possible forms for the characteristic polynomial $P(\lambda)$.

[2] Although Hicks only worked with linear maps, certainly his induction argument may be applied to the linearization of a nonlinear map; moreover, as shown in [12], this condition extends to mappings F which need not be differentiable. Hicks's linearization inequality (7) was rediscovered some years later in the mathematical and scientific literature with a shorter proof than Hicks's that is a simple consequence of Rouché's theorem; see, e.g., [3], [5] and [6].

implies that all roots of the characteristic polynomial $P(\lambda)$ are inside the unit disk in the complex plane; i.e., $\bar{y}$ is locally asymptotically stable. We may refer to (7) as the *Hicks condition*. The next result applies Hicks's observation to the BW model.

Proposition 1. *The positive equilibrium* $\bar{y}$ *of* (3) *is locally asymptotically stable if*:

$$f'(\bar{y}) < 2\min\{1/c\bar{y},\, f_1'(\bar{y})\}. \tag{8}$$

Proof. Using (7), we obtain

$$c\bar{y}\sum_{i=2}^{m} f_i'(\bar{y}) < 1 - |1 - c\bar{y}f_1'(\bar{y})| \tag{9}$$

which may be re-written as

$$c\bar{y}\sum_{i=1}^{m} f_i'(\bar{y}) = -c\bar{y}f'(\bar{y}) < 1 + c\bar{y}f_1'(\bar{y}) - |1 - c\bar{y}f_1'(\bar{y})|$$

Since $(\alpha+\beta) - |\alpha - \beta| = 2\min\{\alpha, \beta\}$, the last inequality is the same as (8). □

To compare the stability profiles of the higher order model with that of the temporally homogeneous model, define the *equivalent first order BW model* corresponding to (3) as

$$y_t = y_{t-1}[1 + g(h(f(y_{t-1}))] \tag{10}$$

where f is defined by (2) - this occurs if the reaction-time differences between sectors are ignored. Note that this equivalent first order BW *structurally* or *spatially* heterogeneous, i.e., different sectors can have different rate characteristics f_i. Equation (10) has precisely the same equilibria as (3), and with F now defined as $F(x) = x[1 + g(h(f(x))]$, the standard result $|F'(\bar{y})| < 1$ yields the sufficient condition

$$f'(\bar{y}) < 2/c\bar{y} \tag{11}$$

for the local stability of $\bar{y}$. Also, the reverse inequality in (11) implies that $\bar{y}$ is unstable. It is clear that *this discrepancy in stability profiles is due solely to the temporal heterogeniety of equation* (3). A comparison of inequalities (11) and (8) proves the following.

Corollary 1. *If* $f_1'(\bar{y}) \geq 1/c\bar{y}$, *i.e., if Sector 1 reacts at a sufficiently high rate, then the stability of* $\bar{y}$ *in the first order case implies the stability of* $\bar{y}$ *in the higher order cases for all* m. □

Additional results concerning local stability of the positive equilibrium can be obtained from some new results in the literature. For instance, a very detailed stability profile can be obtained in the case of two temporally distinguished sectors, the slower of which may lag behind the other by k periods, $k \geq 2$; see [6] and [10]. Also, some new stability and instability conditions of a *nonlocal* and *nonlinear* nature have recently been published that apply to certain special cases of (3); see [12].

Next, define an equilibrium point to be *strongly unstable* (or *repelling*) if *all* roots of the characteristic polynomial are outside the unit disk. The zero equilibrium of (3) is unstable, though not strongly unstable when $m > 1$, as may be infered from the characteristic polynomial. On the other hand, the positive equilibrium $\bar{y}$ can be strongly unstable as in the next result.

Proposition 2. *The positive equilibrium* $\bar{y}$ *of* (3) *is strongly unstable if*:

$$c\bar{y}f_m'(\bar{y}) > 1 + |1 - c\bar{y}f_1'(\bar{y})| + c\bar{y}\sum_{i=2}^{m-1} f_i'(\bar{y}). \tag{12}$$

Proof. Note that all roots of $P(\lambda)$ are outside the unit disk if all roots of the polynomial

$$Q(\lambda) = c\bar{y}f_m'(\bar{y})\lambda^m P(1/\lambda)$$

are inside the disk. Now (12) is obtained by applying the Hicks condition (7) to the coefficients of Q. □

3 Instability and oscillations

There are a number of results in the first order theory that imply persistent oscillation of trajectories and chaotic behavior; perhaps the most often quoted is the now familiar result of [9]. However, such results have no bearing on higher order equations and an important question in the BW model beyond stability and instability is whether bounded trajectories will oscillate without converging. Conditions on the various parameters that imply such behavior for (almost) all trajectories establish possible endogenous sources of complex behavior.

We define a trajectory $\{y_t\}$ as *persistently oscillating* if, as a sequence of points, it is bounded and has more than one limit point. Let $Y_0 = (y_0, y_{-1}, \ldots, y_{1-m})$ represent a set of initial values, i.e., a point on a trajectory of (3) from which we start measuring the output and other quantities. Equation (3) is said to be *positively permanent* if there are *positive* real numbers $\mu < \omega$ such that for every Y_0 (with $y_0 > 0$) there is a positive integer $t_0 = t_0(Y_0)$ such that $\mu \leq y_t \leq \omega$ for all $t \geq t_0$.

It is clear that if positively permanent, then all positive solutions of (3) are eventually bounded within the *same* interval regardless of initial values, and that no subsequence of the output sequence can approach 0. In the next result, $X = (x_1, \ldots, x_m)$ denotes a vector in $[0, \infty)^m$ and $\|X\| = \max\{x_1, \ldots, x_m\}$.

Theorem 1. (a) *If the following inequality holds (see Corollary 2 below)*:

$$\liminf_{||X||\to\infty} \sum_{i=1}^{m} f_i(x_i) > 0 \tag{13}$$

then Eq.(3) *is positively permanent exempting zero solutions; that is, if* $y_0 > 0$ *then* $y_t \in [\mu, \omega]$ *for all sufficiently large* t, *where* $\omega \geq \bar{y} \geq \mu > 0$.

(b) *If* (13) *holds and if* $\bar{y}$ *is strongly unstable, then every nontrivial solution of* (3) *oscillates persistently in a bounded interval* $[\mu, \omega]$ *where* $\omega > \bar{y} > \mu > 0$.

Proof. (a) Under our hypotheses on g, h, inequality (13) is equivalent to

$$\limsup_{||X||\to\infty} g(h(\sum_{i=1}^{m} f_i(x_i))) < 0$$

which is essentially the hypothesis of Corollary 1 in [11]; thus there exists $\omega \geq \bar{y}$ such that every trajectory of (3) is eventually within the interval $[0, \omega]$.

Next, define $\gamma = 1 + g(h(f(\omega + 1)))$ and note that $0 < \gamma < 1$. Let t_0 be the (least) positive integer such that $t > t_0$ implies $y_t \leq \omega < \omega + 1$. For $t > t_0 + m$, we have $\sum_{i=1}^{m} f_i(y_{t-i}) < f(\omega + 1)$, so that

$$g(h(\sum_{i=1}^{m} f_i(y_{t-i}))) > g(h(f(\omega + 1))).$$

which implies that $y_t > \gamma y_{t-1}$ for all $t > t_0 + m$. Now, either (i) there is $k \geq t_0 + m$ such that $y_k \geq \bar{y}$, or (ii) no such k exists. In case (i),

$$y_{k+1} > \gamma y_k \geq \gamma \bar{y}$$

and by induction, $y_{k+m} > \gamma^m \bar{y}$. For $t > k + m$, if $y_{t-i} < \bar{y}$ for all $i = 1, \ldots, m$ then $g(h(\sum_{i=1}^{m} f_i(y_{t-i}))) > 0$ so that $y_t > y_{t-1}$. This implies that $y_t \geq \gamma^m \bar{y}$ for all $t > t_0 + m$, so we may let $\mu = \gamma^m \bar{y}$ to conclude the proof of (a).

If case (ii) above holds, then $y_t < \bar{y}$ for all large t so that $\sum_{i=1}^{m} f_i(y_{t-i}) < f(\bar{y}) = 0$. Therefore, $g(h(\sum_{i=1}^{m} f_i(y_{t-i}))) > 0$ which shows that $y_t > y_{t-1}$ for all large t. Since the increasing sequence $\{y_t\}$ must then converge to $\bar{y}$, it is clear that eventually $y_t \geq \mu$ for all large t.

(b) Because every trajectory is eventually in the compact set $[\mu, \omega]$ as argued in Part (a), there is at least one limit point $y^* > 0$ for each trajectory through Y_0 with $y_0 > 0$. Suppose that for *some* such trajectory $\{y_t\}$, y^* is the *only* limit point, so that $y^* = \bar{y}$. Given the unstable nature of $\bar{y}$, we now show that this is impossible for nontrivial trajectories. First note that

$$F(\bar{y}, \ldots, \bar{y}, y) = \bar{y} \tag{14}$$

if and only if $y = \bar{y}$. To prove this assertion, observe that (14) is equivalent to

$$f_m(y) = -\sum_{i=1}^{m-1} f_i(\bar{y}). \tag{15}$$

From the strong instability of $\bar{y}$, it follows that $f_m'(\bar{y}) \neq 0$ so it must be that $f_m'(\bar{y}) > 0$. Since f_m is a nondecreasing function everywhere, this proves that $\bar{y}$ is the unique solution of (15) and hence also of (14). Next, if there is a positive integer n such that $y_t = \bar{y}$ for all $t \geq n$, then in particular,

$$\bar{y} = y_{n+m-1} = F(y_{n+m-2}, \ldots, y_{n-1}) = F(\bar{y}, \ldots, \bar{y}, y_{n-1})$$

from which it follows that $y_{n-1} = \bar{y}$. Continuing in this way, we see that $y_t = \bar{y}$ for $t < n$ as well, so nontrivial trajectories cannot reach $\bar{y}$ in a finite number of steps.

Next, since $\partial F/\partial x_m(\bar{y}, \ldots, \bar{y}) = -c\bar{y}f_m'(\bar{y}) \neq 0$, the implicit function theorem (see [7]) implies that there is an open neighborhood $U \subset (0, \infty)^m$ of $(\bar{y}, \ldots, \bar{y})$ on which the vectorization V_F is a C^1-diffeomorphism. Defining $Y_t = (y_t, \ldots, y_{t-m+1})$ for each $t \geq 1$, the convergence of $\{y_t\}$ to $\bar{y}$ implies that the vector sequence $\{Y_t\} = \{V_F^t(Y_0)\}$ is eventually in U and $Y_t \to (\bar{y}, \ldots, \bar{y})$ as $t \to \infty$. But $(\bar{y}, \ldots, \bar{y})$ is strongly unstable, and since by the Hartman-Grobman theorem (see, e.g., [1]) the behavior of $\{Y_t\}$ matches that of its linearization near $(\bar{y}, \ldots, \bar{y})$, we have reached a contradiction. This contradiction can be avoided only by assuming that nontrivial trajectories have more than one limit point; i.e., all such trajectories must persistently oscillate.

Finally, we note that $\omega > \bar{y}$ when $\bar{y}$ is unstable, since if $\omega = \bar{y}$ then as argued in (a), every nonzero sequence $\{y_t\}$ is nondecreasing and converges to $\bar{y}$ from below, which is impossible. □

Corollary 2. *Any one of the following conditions implies* (13), *hence positive permamence and if* $\bar{y}$ *is strongly unstable, also persistent oscillations*:

(a) F *in* (5) *is bounded*;

(b) *For some* $i = 1, \ldots, m$, *the function* f_i *is unbounded on* $[0, \infty)$;

(c) *For every* $i = 1, \ldots, m$ *there is* $x_i^* \geq 0$ *such that* $f_i(x_i^*) = 0$. □

Interestingly, Theorem 1 (or its variant in [13]) is not true in the first order case, especially for the bounded unimodal or hill-shaped mappings of interest in [2], since the stable set of $\bar{y}$ contains points other than $\bar{y}$ itself even when $\bar{y}$ is strongly unstable. Indeed, it can be rigorously shown that for mappings F that are topologically conjugate to the well-known logistic map $4x(1-x)$ (which has chaotic dynamics over its invariant interval $[0, 1]$) the stable set of $\bar{y}$ is dense in the invariant interval of F.

References

1. Arrowsmith, D.K., Place, C.M.: An introduction to dynamical systems. Cambridge: Cambridge University Press 1990
2. Baumol, W.J., Wolff, E.N.: Feedback between R&D and productivity growth: A chaos model. In: Benhabib, J. (ed.) Cycles and chaos in economic equilibrium, pp. 355-373. Princeton: Princeton University Press 1992
3. Elaydi, S.N.: An introduction to difference equations. New York: Springer 1996
4. Hicks, J.R.: A contribution to the theory of the trade cycle. Oxford: Clarendon Press 1965
5. Kocic, V.L., Ladas, G.: Global behavior of nonlinear difference equations of higher order with applications. Dordrecht: Kluwer Academic 1993
6. Kuruklis, S.A.: The asymptotic stability of $x_{n+1} - ax_n + bx_{n-k} = 0$. Journal of Mathematical Analysis and Applications **188**, 719–731 (1994)
7. Lang, S.: Analysis I. Menlo Park: Addison-Wesley 1968
8. LaSalle, J.P.: The stability and control of discrete processes. New York: Springer 1986
9. Li, T-Y, Yorke, J.: Period 3 implies chaos. American Mathematical Monthly **82**, 985–992 (1975)
10. Papanicolaou, V.G.: On the asymptotic stability of a class of linear difference equations. Mathematics Magazine **69**, 34–43 (1996)
11. Sedaghat, H.: General permanence conditions for nonlinear difference equations of higher order. Journal of Mathematical Analysis and Applications **213**, 496–510 (1997)
12. Sedaghat, H.: Geometric stability conditions for higher order difference equations. Journal of Mathematical Analysis and Applications **224**, 255–272 (1998)
13. Sedaghat, H.: Bounded oscillations in the Hicks business cycle model and other delay equations. Journal of Difference equations and Applications **4**, 325–341 (1998)