MATH 195: Gödel, Escher, and Bach (Spring 2001)

Problem Set 12: Derivations

To be discussed Thursday, April 5

1. On p.260, Hofstadter gives HIS solution to the MU-puzzle, which (gratifyingly) is about the same as what we found a few weeks ago. In his solution, he appeals to "a simple fact from number theory". Let's treat this statement as a challenge to our budding abilities within TNT.

1a. Complete the outline of the proof of this simple fact, a fact that may be represented semi-symbolically as:

" b: < (3 divides **(2·b)) É (3** divides **b) >**

To make the proof short enough for our purposes, we've introduced an unofficial shorthand notation "(m divides n)" (represented in regular font rather than **bold** to remind you that it is not part of TNT. We'll expand the short hand in problem **1b**. Also, we've given you two theorems (lines 1 and 2), which you may consider as already proven.

(1)	**" a: " b: <(3** divides **(a·b)) É <(3** divides **a) Ú (3** divides **b)>>**	(given)
(2)	**~ (3** divides **2)**	(given)
(3)	**[**	Push
(4)	**(3** divides **2·b)**	premise
(5)	**" a: " b: <(3** divides **(a·b)) É <(3** divides **a)Ú (3** divides **b)>>**	Carry-over
(6)	**" b: < (3** divides **(2·b)) É < (3** divides **2) Ú (3** divides **b)>>**	Specification, with __________
(7)	____	Specification, with **b=b**
(8)	____	Detachment, lines (2) and (__)
(9)	**< ~ (3** divides **2) É __________ >**	Switcheroo, line (__)
(10)	**~ (3** divides **2)**	__________
(11)	**(3** divides **b)**	Detachment
(12)	**]**	Pop
(13)	____	Fantasy rule
(14)	**" b: < (3 divides (2·b)) É (3 divides b) >**	__________

1b. Translate **~ (3** divides **2)** completely into symbols in two ways:

~$a: __________

" a: __________

Preliminaries for problems 2 and 3.

The rules on p.263 refer to 8 different "NUMBER FORMS" (note that Forms 1a and 1b are found in Rule 1, etc.):

Form 1a:	$10 \cdot m+1$	Form 1b:	$10 \cdot (10 \cdot m+1)$
Form 2a:	$3 \cdot 10^{m} + n$	Form 2b:	$10^{m} \cdot (3 \cdot 10^{m} + n) + n$
Form 3a:	$k \cdot 10^{m+3} + 111 \cdot 10^{m} + n$	Form 3b:	$k \cdot 10^{m+1} + n$
Form 4a:	$k \cdot 10^{m+2} + n$	Form 4b:	$k \cdot 10^{m} + n$

2. Using the four rules in the forward direction.

2a. Show that 31 can be written in the NUMBER FORMS 1a and 2a by finding appropriate values for m and n.

2b. Using the values of m and n from your answer to **2a**, what numbers are produced by NUMBER FORMS 1b and 2b?

2c. What numbers can be produced from the four rules using 31 as input?

2d. Does the answer to question **2c** make sense in terms of the isomorphism at the level of symbols between the **MIU**-system and the **310**-system? Explain.

3. Using the four rules in the backwards direction.

3a. Show that 31 can be written in the NUMBER FORM 4b in two different ways by finding appropriate values for k, m, and n (use $n<10^m$). HINT: 0 is a natural number.

3b. With the values of m and n from part a., what two numbers are produced by NUMBER FORM 4a?

3c. What input numbers can produce 31 as output?

3d. Does the answer to question **3c** make sense in terms of the isomorphism at the level of symbols between the **MIU**-system and the **310**-system? Explain.

4. Using the numbering shown on p.261, translate the following strings into derivations within the **MIU** system (“,” means “new line”):

4a. 31,311,31111,311110

4b. 311111111,3011111,30110

5. Translate the following **MIU** strings into 310 numbering:

5a. MIUIUIU

5b. UIMMM

6. Which of the following numerical strings, when translated, are theorems within the **MIU** system?

6a. 30101

6b. 311110110111100111101

(If this last one throws you, then you need to read pp.260-261)

7. Complete the following derivations within the 310 system, using the rules (translated from **MIU**) shown on p.263.

(1) 31	Axiom
(2) ______	Rule 2, **m**=1, **n**=1
(3) ______	Rule 2, **m**=2, **n**=11
(4) ______	Rule 1, **m**=3111
(5) ______	Rule 3, **m**=1, **n**=0, **k**=31
(6) ______	Rule 4, **m**=0, **n**=0, **k**=31

8. Translate the derivation below from the **MIU** system to the 310 system. Be sure to provide the number of the rule and the appropriate values or **m**, **n**, and **k**, as needed.

(1) **MI** Axiom
(2) **MII**
(3) **MIIII**
(4) **MIIIIIIII**
(5) **MIUIIII**
(6) **MIUUI**
(7) **MII**