MATH 195: Gödel, Escher, and Bach (Spring 2000)

Problem Set 2: *The MU-Puzzle*

SOLUTIONS TO SELECTED PROBLEMS:

2.2. Starting with **MI**, derive **MIUUI**.

MI goes to MII by Rule 2

MII goes to MIIII by Rule 2

MIIII goes to MI(III)(III)I by Rule 2 with parentheses added to indicate next step

MI(III)(III)I goes to MI(III)(U)I by Rule 3

MI(III)(U)I goes to MI(U)(U)I by Rule 3 again

MI(U)(U)I goes to MIUUI by erasing parentheses

Note: The parentheses are not part of the MIU system but were added just to aid readability.

2.6. Prove that if **MI** is the sole axiom, the *MIU-system* cannot produce **MU** in less than five steps.

This was the hardest problem. Several students exhibited ONE sequence of 5 strings,

1) each of which was legally derived from the previous one

AND

2) that started with MI

AND

3) that did not end with MU.

Those students have solved the problem

"Prove that if **MI** is the sole axiom, the *MIU-system* CAN produce a sequence of five legal steps that does NOT end with MU."

I hope you see that this problem is different from problem 2.6 as stated above. The solution of problem 2.6 requires us to prove that

"there is NO sequence of 5 strings legally derived from MI that DOES end with MU".

Do you see that you might rephrase
"there is NO sequence of 5 strings legally derived from MI that DOES end with MU" as
"EVERY sequence of 5 strings legally derived from MI DOES NOT end in MU"?

This rephrasing makes Problem 2.6 much harder. Conceptually, what you need to do is best described with reference to that generating tree for the *MIU-system* on page 40. That generating tree contains ALL sequence of 4 strings legally derived by starting with MI. What you needed to do was to fill in the bottom of each branch in order to establish that "EVERY sequence of 5 strings legally derived from MI DOES NOT end in MU". Tedious I know, but that's what was asked.

Negations of statements can be quite tricky. We will come back to them.

2.8. Consider the *MIU-system* with **MII** as the sole axiom. Is **MI** a theorem?
YES.

MII goes to MIIII via rule 2,
MIIII goes to MIIIIU via rule 1,
MIIIIU goes to MIUU via rule 3,
MIUU goes to MI via rule 4.

2.9. Consider the *MIU-system* with **MU** as the sole axiom. Is **MI** a theorem?
NO.

Student solutions to this problem are related to the next chapter "Figure and Ground". Several students wrote something like "Starting with MU, there is no way to produce any strings containing any occurrences of the symbol I". This leads to

Metatheorem 2.9.1: No theorem of the *MIU-system* with **MU** as the sole axiom contains the symbol I.

That's a nice characterization of a huge collection of well-formed strings in the *MIU-system* with **MU** as the sole axiom that are NOT theorems. This is part of the "negative space" of the *MIU-system* with **MU** as the sole axiom.

Another solution to this problem would involve

Metatheorem 2.9.2: The theorems of the *MIU-system* with **MU** as the sole axiom are exactly the strings of the form Mx, where x is a string composed of U's only and the length of x is a power of 2 (this time including the possibility that x is the empty string).

That's a characterization of the "positive space" for the *MIU-system* with **MU** as the sole axiom.