MATH 195: Gödel, Escher, and Bach (Spring 2001)
Problem Set 14:
To be discussed Tuesday, April 24

Gödel numbering and its implications

1. Arithmetize the "ADD S" rule (p.219). Here's how I translate the IF clause (*r* and *t* represent TNT strings):

   IF **g**(*r*) 111 **g**(*t*) is a TNT-Number

   Or better yet:

   IF $\underbrace{\mathbf{g}(r)\cdot 10^{\#\text{ of digits in }\mathbf{g}(t)+3}}_{\text{part 1}} + \underbrace{111\cdot 10^{\#\text{ of digits in }\mathbf{g}(t)}}_{\text{part 2}} + \underbrace{\mathbf{g}(t)}_{\text{part 3}}$ is a TNT-Number

   You complete the THEN clause. Don't worry too much about the "shortcut" involved in counting the digits (which avoids the use of more complicated inequalities).

2. Show that your answer to Problem 1 works correctly when applied to the input TNT-string:

   (S0+0)=S0

   yielding the output TNT-number:

   123 362 123 666 112 666 323 111 123 123 666

   To do this, you'll need to define what *r* and *t* are and show what the input string works out to. I advise that you split the input string into three parts, as shown above in Problem 1, and show how they come together to make the output string.

3. Using Axiom 3 of TNT as a starting point, it is possible to derive (see p.269):

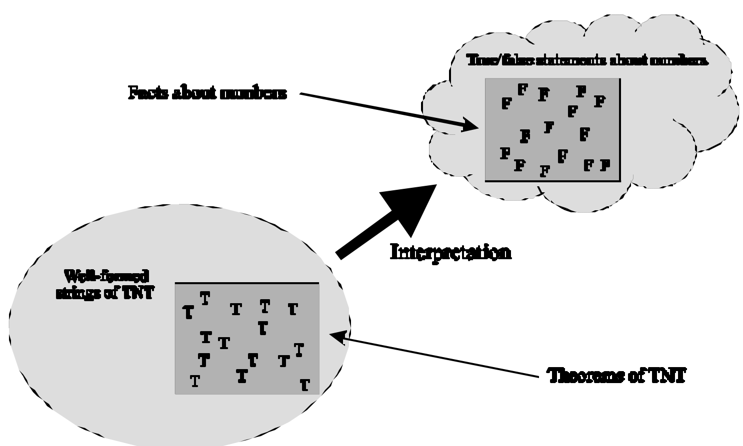
   (S0+S0) = SS0

   Based on this information, write down a TNT-proof-pair. (By the way, note the interplay between arithmetic shortening and arithmetic lengthening rules on p.269)

4. Explain the relationship between Hofstadter's diagram of the Central Dogma of Mathematical Logic and Kerckhove's Jump Out/Jump In diagram (available on the web if you've lost your copy).

5. Now focus on the corners of the Jump Out/Jump In diagram:

   In this "interpretation diagram" we've cast TNT in the role of the FORMAL SYSTEM and Number Theory in the role of REALITY. We wish to examine how well the formal system TNT models the reality of numbers and arithmetic. The key questions involve consistency and completeness.



5a. Suppose the interpretation of TNT is CONSISTENT with the reality of numbers and arithmetic. What would that mean in terms of this interpretation diagram? (How would you represent this in the diagram?)

5b. Suppose the interpretation of TNT gives a COMPLETE model for the reality of numbers and arithmetic. What would that mean in terms of this interpretation diagram? (How would you represent this in the diagram?)

NOTE: You will need a precise understanding of the terms "consistent" and "complete": see p.101, think about sets and subsets, and be sure your answers indicated some difference between these two concepts.

6. How is it that strings of TNT can "talk about" other strings of TNT? Describe your understanding of this phenomenon.

7. The string **G** of TNT is reported say of itself '**G** is not a theorem of TNT". Is **G** a theorem of TNT or not?

8. Is **g**(**G**) a TNT-Number?

9. What does **g**(**G**) say about itself? (another one of those talking numbers!)
Here's some help: You should aim for a short answer AND a long answer:

- We transform the 5 **axioms** of TNT into 5 specific numbers. For example:

  AXIOM 1: **∀a:~Sa=0**

  Arithmetized AXIOM 1: 626262636223123262111666

- We transform the **rules** of TNT into arithmetic rules. For example:

  RULE OF ~~: The string '~~' can be deleted from any theorem.

  Arithmetized RULE OF ~~ (forward direction): If a number of the form

  $k \cdot 10^{m+6} + 223223 \cdot 10^{m} + n$ (where $n < 10^{m}$)

  is a TNT-Number, then so is

  $k \cdot 10^{m} + n$

- We transform questions of theoremhood within TNT into questions of existence of solutions of certain systems of equations. For example:

  Question of theoremhood: Is **∃a:**~~**a=a** is a theorem within TNT?

  Question of existence: Is it true that

  ∃*k*: ∃*m*: ∃ *n*: $k \cdot 10^{m} + n$ (where $n < 10^{m}$) is a TNT-Number

  and $k \cdot 10^{m+6} + 223223 \cdot 10^{m} + n$ is well-formed?

10. Often, the statement of Godel's Theorem is written as follows: "If a formal system is sufficiently powerful, then that formal system is incomplete".

   10a. What does the phrase "sufficiently powerful" mean?

   10b. Give an example of a truth whose corresponding theorem ought to be derivable within the pq system in order for it to be considered "sufficiently powerful".

   10c. By decree, the TNT-system qualifies as "sufficiently powerful". Illustrate the fact that TNT is more powerful than PQ by giving an example of a truth whose corresponding theorem can be derived within the TNT-system, but not within the PQ-system.

   10d. What does Godel's theorem say about the insufficiently powerful **pq**-system? ANSWER: Nothing. Explain.

   10e. What is the analogy to Crab's phonographs?

11. Read the first paragraph of "Ganto's Ax in Metamathematics" (p.407). Rephrase the key point as a statement of Propositional Calculus.

12. Suppose that it were possible to transfer all of number theory to TNT **and** that one could work out a method to mechanically determine whether any TNT string were a theorem. If this were the case, then any question of number theory could be decided yes or no, and so would any question that could be translated into TNT, which we've seen covers an awful lot of ground. We've seen that any formal process that can be decided in a predictable number of steps (i.e. a BlooP program) can be represented in TNT. **Can we decide <u>theoremhood</u> in this way???** If so, then TNT itself can decide whether strings are theorems! Taking a simpler case:

   12a. Is determining <u>theoremhood for a particular string</u> of MIU a "predictably terminating" process within MIU? Explain.

   12b. Is determining theoremhood for a particular string of MIU a "predictably terminating" process <u>within Number Theory</u>? Explain.

   12c. Is determining <u>validity of an MIU derivation</u> a "predictably terminating" process? Explain. In what realm does this determination take place?

   12d. Is there any difference between determining theoremhood of a given string and determining the validity of a given derivation?

   12e. (Rhetorical) Is there any difference between a "predictably terminating process" and a "finite decision procedure"?

BlooP, FlooP, and Quining

13. How would you go about figuring out whether a number is or is not a prime?

   13a. Answer in the form of a flow diagram or RTN.

   13b. Is the search guaranteed to terminate?

   13c. How many numbers would you have to test before deciding with confidence whether 1741 is or is not a prime?

   13d. Can the statement "1741 is a prime" be expressed within TNT?

14. How would you go about figuring out whether a string is a theorem within **MIU**?

14a. Answer in the form of a flow diagram or RTN.

14b. Is the search guaranteed to terminate?

14c. How many iterations would you have to go through before deciding with confidence that **MU** is or is derivable from **MI** within the **MIU**-system?

14d. Can the statement "**MU** is derivable from **MI** in the **MIU**-system" be expressed within TNT?

15. Let sentence **P** (p.436) be:

" '_______' can be represented within TNT."

Write out sentences **P** and **Q**, if the phrase to be inserted in the blank is:

15a. 'Is a theorem within **MIU**'

15b. 'Can be represented within TNT'

16. Quine each of the following phrases and judge whether the resulting statement is true, false, or undecidable.

16a. 'Is a sentence without an adjective'

16b. 'Contains a misspelled word'

16c. 'Yields falsehood when preceded by its quotation'

17. Which of the following strikes you as closest to the TNT equivalent of quining?

A. Taking a string and repeating it
Example: **S0=0** goes to **S0=0S0=0**

B. Replacing a variable within a string with the entire string
Example: **$a: a=S0** goes to **$a: ($a: a=S0)=S0**

C. Replacing a variable within a string with the Gödel number of the entire string
Example: **$a: a=S0** goes to **S…S0=S0** (333262636262111123666 **S**'s)

Gödel's Proof

18. Consider the two pairs of numbers reproduced below from p.440. Which is the real proof-pair?

(1) *m* = 626 262 636 223 123 262 111 666 611 223 123 666 111 666
n = 123 666 111 666

(2) *m* = 626 262 636 223 123 262 111 666 611 223 333 262 636 123 262 111 666
n = 223 333 262 636 123 262 111 666

19. Suppose that TNT-PROOF-PAIR is the name of a BlooP program that takes two numbers, **a** and **a'** as input. Since its loop is bounded, it is representable in TNT. Rather than write out the humongous TNT statement that represents it, we'll refer to it simply as TNT-PROOF-PAIR (**a**,**a'**). Use this notation to translate the seven statements given on p.442-443.

20. Arithmoquine **~a=S0**

21. Consider the "unutterably gigantic number" in the middle of p.446.

21a. Why does it begin "123 123 123…123" (262,111,123,666 copies of '123')?

21b. Why does it end "666 111 123 666"?

22. Read and reread the instructions of how to interpret **G** (p.447-448). Presume each statement Hofstadter makes to be FALSE, and see what comes of that. For example, early on he says:

*Now there certainly is a number **a**' that is the arithmoquinification of u*

Let's say he's wrong. Then you can't find an example of such a number **a**'. Find one.

23. Is G's uncle TRUE or FALSE?

23a. Is G's uncle a theorem of TNT? What if it were?

23b. What if ~(G's uncle) were a theorem?

24. On p.449, the section "Godel's Second Theorem" begins

Since G's interpretation is true, the interpretation of its negation ~G is false. And we know that no false statements are derivable in TNT.

24a. What is G's interpretation as referred to in the sentence above?

24b. G's interpretation is a statement of Meta-TNT because _____

24c. We concluded that G's interpretation is TRUE because _____

24d. We conclude from this that the interpretation of ~G is FALSE because _____ (I'd say "because we assume the Law of the Excluded Middle holds for statements of Number Theory." Though if you want to use this, you need to say what that Law means AND how it is related to the conclusion about ~G.)

24e. We know that no false statements are derivable in TNT because _____