MATH 195: Gödel, Escher, and Bach (Spring 2001)

Problem Set 13: Gödel numbering - Hint for #5

Problem 5 seemed difficult for everyone. On the face of it, the double-tilde rule seems to be the simplest rule possible and it is similar to the rule of MIU that removes two occurences of U. That is why it was assigned.

As we got into the problem, I found "0" to be important in several different ways. Perhaps the fact that "0" shifted roles was what made the problem difficult to fathom. Here then are the uses made of "0".

First, 0 is a codon for Hofstadter's Godel-Numbering of the MIU-system; the codon corresponding to the original symbol U.

Second, 0 is used as a place-holder in base 10 arithmetic. In formulating Arithmetized Rule 4 of MIU, we cast the input string in the form $k*10^{(m+2)} + n$. You see that $k*10^{(m+2)}$ has at least two zeros at the right end. Moreover, since $n<10^m$, left two leftmost zeros coming from $10^{(m+2)}$ are not effected by adding n. Thus, our use of powers of ten guarantees that the a)-form input string for our Arithmetized Rule 4 has at least two consecutive digits that are zeros. The output of Arithmetized Rule 4 has form $k*10^m + n$. Thus the two zeros are removed. For example, $4007 = 4*10^{(1+2)} + 7$ goes to $47 = 4*10^1 + 7$ by means of Arithmetized Rule 4. Similarly, 3001 goes to 31, mirroring MUUI goes to MI.

Third, 0 is used as a symbol of TNT. Hofstadter's codon for this symbol is 666. This use is not very important to problem 5.

It is really the place-holder use of 0 that is important in trying to arithmetize the rules of a system. Here's how it all works in the case of the Double-Tilde Rule.

For Double-Tilde, we might express the rule as x~~y goes to xy. The input side becomes, after passing through the Godel-Numbering dictionary for TNT, g(x)223223g(y). In the spirit of what Hofstadter does (recall the Central Proposition of p.264), I'd like to express that string of digits as a sum, using powers of 10 to create zeros as place-holders. I write, in a mixture of arithmetic and English,

> " g(x) times 10^(6+number of digits in g(y))
> + 223223*10^(number of digits in g(y))
> + g(y) "

Counting the number of digits in g(y) gives me a way to know how many place-holders to include via those powers of 10. With a bit more work, I write the a)-form (input form) for the rule as

> $k* 10^{(6+m)} + 223223*10^m + n$
> where k is arbitrary and $10^{(m-1)} \le n < 10^m$.

Remember that I'm going to think eventually about whether certain equations have solutions – the rule can be applied to a given number exactly when there exist values of k, m, and n such that the given number can be written in this form.

The output form of the Arithmetized Double-Tilde Rule should delete the two tildes. In other words, the output form is

k*10^m + n, where as before k is arbitrary and 10^(m-1) <= n < 10^m.

There, I've provided you with my thought process and solution for Problem 5. I encourage you to try the formulas with actual strings of TNT containing ~~ in order to check out k, m, and n values before and after the ~~ is deleted.

Just for fun, here's how I would treat the Switcheroo Rule. In TNT, (half of) the rule is

< x ⊃ y > goes to < ~x ∨ y >

Via the dictionary, this becomes

212,g(x),633,g(y),213 goes to 212,223,g(x),616,g(y),213

To "arithmetize" the rule, I'd write the input form as the "mixed" expression

212*10^(#digits in g(x)+3+#digits in g(y)+3)
+g(x)*10^(3+#digits in g(y)+3)
+633*10^(#digits in g(y)+3)
+g(y)*10^3
+213

and the output form as the "mixed" expression

212223*10^(#digits in g(x)+3+#digits in g(y)+3)
+ g(x)*10^(3+#digits in g(y)+3)
+ 616*10^(#digits in g(y)+3)
+g(y)*10^3
+213

The rest involves substituting, say
k1 for g(x)
m1 for #digits in g(x)
with inequality 10^(m1-1) <= k1 < 10^m1 and
k2 for g(y)
m2 for #digits in g(y)
with inequality 10^(m2-1) <= k2 < 10^m2, using these inequalities to "count the digits".

Once again, the question of whether or not a given number is possibly an output number from this rule is the question of whether or not there exist values for the variables (the k's and the m's) that satisfy the equations and inequalities.