MATH 195: Gödel, Escher, and Bach (Spring 2001)

Notes and Study Questions for Tuesday, March 27

Reading: Chapter VII–*Typographical Number Theory*
pp.215–219 (*The Five Axioms…* to (including) *Rules of Equality and Successorship*)
pp.221-225 (*Something is Missing* to (including) *The Last Rule*)

There is a LOT in this chapter, enough for the better part of the semester, if we had time to smell the roses. We don't. Given this constraint, it is not reasonable to expect to become proficient in the use of TNT, but that's OK. The reason this chapter is in the book so that you will see how a few symbols and a few rules are powerful enough to prove weighty matters of number theory. If you can appreciate why mathematicians in the early 1900's pinned their hopes on this sort of system, then you will feel the earth move when Gödel shows us the deficiencies in TNT.

So, our goal is to learn enough about how TNT works to <u>follow</u> derivations in it and to produce very simple derivations. After doing that, we'll step back and examine TNT from above, asking what are the limits on the power of the system.

The Five Axioms and First Rules of TNT

SQ1. Translate each of the five axioms into intelligible English

SQ2. The title of this section refers to the first rules of TNT. Where are they? Name some rules.

The Five Peano Postulates

SQ3. Given that "*djinn*" is meant to represent the natural numbers (i.e. zero plus the positive integers), translate the five disguised Peano postulates into English, making up reasonable equivalences for "*Genie*" and "*meta*". The fifth postulate is by far the most obscure of the lot. Don't be discouraged if you don't get it.

New Rules of TNT, The Existential Quantifier, and Rules of Equality and Successorship

When we reads sentences, we tend to sit back and let the meaning explode in our heads. For example, "*The giraffe ate the jewel-encrusted unicycle.*" There's little question WHAT the sentence says, even though you may have no idea WHY it says it. Sentences about formal systems are different: "*If the string* **∀*u*: *x*** *is a theorem, then so is* ***x****, and so are…*" No explosions here, at least on my end. It is tempting to note the absence of pyrotechnics and conclude ruefully, "*Well, I'm just no good at math.*" If THIS is your requirement, then NO ONE is good at math! Such sentences require you to <u>actively</u> seek out the meaning. If you do so, you'll find the meaning and may gain some valuable lesson along with it. If you don't, you'll gain only the useless excuse that you're no good at math.

If you find yourself at a loss with a symbolic expression, which is a natural human response, then the time has come to translate it into terms that are more tangible to you. Take the expression, "*If the string* **∀*u*: *x*** *is a theorem, then so is* ***x****, and so are any strings made from* ***x*** *by replacing* ***u****…*", and plug in some values. For "**∀*u*: *x***" I'll try, "*For every car* ***u****, car* ***u*** *has four wheels.*" Not very inspiring, but plausible. Now back to the full sentence, "*If that sentence about cars is true, then so is `car* ***u*** *has four wheels'…*". Is this reasonable? If all cars have four wheels

-- never mind if this really is true -- IF this is true, then certainly car **u**, whatever it is, also has four wheels. Furthermore, "*…and so are any strings made from **x** by replacing **u**…*" translates to "*…and so are statements concerning any specific car…*" like "*Car Toyota Corolla has four wheels.*"

You don't need me to analyze symbolic sentences in this way, just the confidence that there IS sense there to be discovered and that you can discover it.

SQ4. Make up a sentence in TNT with two variables governed by **∀**. Start off **∀a: ∃b:…** and choose the sentence so that it happens to be true**.** Replace **a** with the number "1". Is the sentence still true? Replace **a** with the variable **c**. Still true? Now, replace **a** with the variable **b**. Still true? It may or may not be. Find a sentence where the first two replacements preserve truth but the last one does not.

SQ5. Translate the Rule of Generalization into English and provide an example within TNT that uses the rule.

SQ6. Provide an example in English of the use of the Rule of Interchange.

SQ7. Provide an example in English of the use of the Rule of Existence. You might do this by translating the TNT example provided with the rule.

SQ8. Provide arithmetic formulas that illustrate the Rules of Equality and Successorship.

SQ9. Go through the first derivation (7 lines) and understand the interpreted meaning of each line and the justification for it.

Something is missing

SQ10. Examine the pyramid of formulas on p.221, beginning with **(0+0) = 0.** Translate them into everyday arithmetic. What English sentence can you conceive of that describes all the formulas taken together?

The point of this section is that the very reasonable English sentence you made up in the previous study question is NOT a valid conclusion from the list of formulas. Each separate formula is a theorem in TNT. The generalization, however, pops up in your non-TNT mind but not in TNT itself (at least not with the rules we've encountered thus far). Being reasonable is not good enough. There has to be a rule that justifies a conclusion.

SQ11. What's wrong with the proposed Rule of All? What PART of the sentence is impossible to implement?

SQ12. Give another example of a pyramid of formulas that you suspect can be derived in TNT. State a general rule that *ought* to be derivable from the pyramid.

SQ13. The two statements below are well formed formula within TNT. Are they both true? Both false? Something else? What are the implications of this situation?

∀a: (0 + a) = a
~∀a: (0 + a) = a

SQ14. Give an example of ω-inconsistency and plain vanilla inconsistency. Why is one more tolerable than the other?

The Last Rule

The Rule of Induction is a workhorse in proving propositions of number theory. Let's see it in action (in an informal way). I want to show any 3^N is always greater than 2^N for all N greater than 0. To use the Rule of Induction, I need to show two separate things:

I. My proposition is true for the FIRST possible number

II. If I *presume* that my rule is true for *some* number, it is true for the *next* number as well

Here's the (informal) proof.

Ia. 1 is the first natural number greater than 0

Ib. 3^1 (= 3), which is greater than 2^1 (= 2)

Ic. So my proposition is true for the first number greater than 0

IIa. Let's suppose that my rule is true for some number, *x* (premise)

IIb. Then 3^x must be greater than 2^x, according to my supposed rule ($3^x > 2^x$)

IIc. Then $3 \cdot 3^x > 3 \cdot 2^x$ (multiplying both sides by a positive number)

IId. And $2 \cdot 3^x > 2 \cdot 2^x$ (multiplying both sides by a positive number)

IIe. But certainly $3 \cdot 2^x = 2^x + 2 \cdot 2^x$ (factor out 2^x)

IIf. And $2^x + 2 \cdot 2^x > 2 \cdot 2^x$ (since 2^x is a positive number)

IIg. So $3 \cdot 2^x > 2 \cdot 2^x$ (transitivity)

IIh. And $3 \cdot 3^x > 2 \cdot 2^x$ (transitivity)

IIi. And $3^{x+1} > 2^{x+1}$ (definition of exponents)

IIj. So my rule, if true for *x*, is also true for the number after *x*

Since my rule is true for the first possible number and, if true for one, also true for the next number, it's true for ALL numbers, according to the Rule of Induction.

SQ15. What part of the Rule of Induction is fulfilled by the 9-line derivation on p.224?

Hofstadter introduces new symbols *not part of TNT* to help describe the Rule of Induction. At first the symbols may seem strange, but they are nothing that you won't get if you don't let your eyes glaze over.

SQ16. If X{**a**} is the formula **$ b: (Sa + 2) = (b + 1)**, then interpret X{**Sa**/**a**}.

SQ17. Go through the formal description of the Rule of Induction on p.224 phrase by phrase and relate it to the informal description I gave you on the previous page of these notes.